



Time scale boundary value problems on infinite intervals

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Abstract

Two new existence results are presented for time scale boundary value problems on infinite intervals. The first is based on a growth argument and the second on an upper and lower solution idea. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let \mathbb{T} (time scale) be a closed subset of \mathbb{R} . Define the forward (respectively, backward) jump operator at t for $t < \sup \mathbb{T}$ (respectively, for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \tau \in \mathbb{T}\} \quad (\text{respectively, } \rho(t) = \sup\{\tau < t : \tau \in \mathbb{T}\})$$

for all $t \in \mathbb{T}$. We assume throughout that \mathbb{T} has the topology that it inherits from the standard topology on \mathbb{R} .

If $a < b$ are points in \mathbb{T} , then we let

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Throughout this paper $a \in \mathbb{T}$ is fixed, and we assume that there exists $t_n \in \mathbb{T}$, $n \in \{1, 2, \dots\} \equiv \mathbb{N}$, with

$$a < t_1 < t_2 < \dots < t_n < \dots \quad \text{with } t_n \uparrow \infty \text{ as } n \rightarrow \infty.$$

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Let

$$[a, \infty) = \bigcup_{n=1}^{\infty} [a, t_n].$$

In this paper, we are interested in establishing the existence of solutions to the time scale boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) &= 0 \quad \text{for } t \in [a, \infty), \\ y(a) &= 0, \\ y(t) \text{ is bounded for } t &\in [a, \infty), \end{aligned} \tag{1.1}$$

where $a \in \mathbb{T}$ is fixed. To understand (1.1) we need to recall some standard definitions (see [1,2,7–11] for an introduction to this subject).

Definition 1.1. Fix $t \in \mathbb{T}$. Let $y: \mathbb{T} \rightarrow \mathbb{R}$. Then we define $y^{\Delta}(t)$ to be the number (if it exists) with the property that given $\varepsilon > 0$ there is a neighborhood U of t with

$$|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$. We call $y^{\Delta}(t)$ the derivative of $y(t)$.

Definition 1.2. If $F^{\Delta}(t) = f(t)$ then we define the integral by

$$\int_a^t f(\tau)\Delta\tau = F(t) - F(a).$$

Time scale boundary value problems on the finite interval have received a lot of attention in the literature (see [2,4–6,8,9]). However, to our knowledge, this is the first paper discussing time scale boundary value problems on the infinite interval. Two existence results will be presented, one based on a nonlinear alternative of Leray–Schauder type [3] and the other on Schauder’s fixed point theorem [3]. For the convenience of the reader, we state these two results here.

Theorem 1.1. *Let C be a convex subset of a Banach space E , and let U be an open subset of C with $0 \in U$. Then every compact, continuous map $N: \bar{U} \rightarrow C$ has at least one of the following two properties:*

- (A1) N has a fixed point in \bar{U} ; or
- (A2) there is an $x \in \partial U$ and $\lambda \in (0, 1)$ with $x = \lambda Nx$.

Theorem 1.2. *Let K be a convex subset of a Banach space E and $N: K \rightarrow K$ a compact, continuous map. Then N has a fixed point in K .*

2. Existence

Let $a \in \mathbb{T}$. Consider the time scale boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) &= 0 \quad \text{for } t \in [a, \infty), \\ y(a) &= 0, \\ y(t) \text{ is bounded for } t &\in [a, \infty). \end{aligned} \tag{2.1}$$

Throughout this section we assume that there exists $t_n \in \mathbb{T}$, $n \in \mathbb{N}$, with

$$a < t_1 < t_2 < \dots < t_n < \dots \quad \text{with } t_n \uparrow \infty \text{ as } n \rightarrow \infty.$$

Two existence results will be established in this section, one based on a growth argument and the other based on the idea of upper and lower solutions.

Theorem 2.1. *Suppose the following conditions are satisfied:*

$$f : [a, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous,} \tag{2.2}$$

$$\begin{aligned} |f(t, u)| &\leq \phi(t)w(|u|) \quad \text{on } [a, \infty) \times \mathbb{R} \text{ with } w \geq 0 \text{ continuous} \\ &\text{and nondecreasing on } [0, \infty) \text{ and } \phi : [a, \infty) \rightarrow [0, \infty) \text{ continuous} \end{aligned} \tag{2.3}$$

and

$$\exists r > 0 \text{ with } \frac{r}{w(r)\sup_{n \in \mathbb{N}}[\sup_{t \in [a, \sigma^2(t_n)]} \int_a^{\sigma(t_n)} \phi(s)|G_n(t, s)|\Delta s]} > 1; \tag{2.4}$$

here $G_n(t, s)$ is Green's function (see [8, 9]) for

$$\begin{aligned} y^{\Delta\Delta}(t) &= 0 \quad \text{for } t \in [a, t_n], \\ y(a) &= 0, \\ y(\sigma^2(t_n)) &= 0. \end{aligned} \tag{2.5}$$

Then (2.1) has a solution $y \in C[a, \infty)$ with $|y(t)| \leq r$ for $t \in [a, \infty)$.

Remark 2.1. By $C[a, \infty)$ we mean the space of continuous functions $u : [a, \infty) \rightarrow \mathbb{R}$.

Proof. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) &= 0 \quad \text{for } t \in [a, t_n], \\ y(a) &= 0, \\ y(\sigma^2(t_n)) &= 0. \end{aligned} \tag{2.6}^n$$

We begin by showing that (2.6)ⁿ has a solution $y_n \in C[a, \sigma^2(t_n)]$ with $|y_n(t)| < r$ for $t \in [a, \sigma^2(t_n)]$ (note also that $y_n^{\Delta\Delta} \in C_{rd}[a, t_n]$ since $y_n^\sigma = y_n \circ \sigma$ is rd-continuous on $[a, t_n]$).

Remark 2.2. By $C[a, \sigma^2(t_n)]$ we mean the Banach space of continuous functions $y : [a, \sigma^2(t_n)] \rightarrow \mathbb{R}$ equipped with the norm $|y|_n = \sup_{t \in [a, \sigma^2(t_n)]} |y(t)|$. $C_{rd}[a, t_n]$ is the space of rd-continuous functions [11] on $[a, b]$.

To see the above we will use Theorem 1.1, so we consider the boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + \lambda f(t, y(\sigma(t))) &= 0 \quad \text{for } t \in [a, t_n], \\ y(a) &= 0, \\ y(\sigma^2(t_n)) &= 0 \end{aligned} \tag{2.7}'_2$$

for $0 < \lambda < 1$. Solving (2.7)'₂ is equivalent [9] to solving the fixed point problem $y = \lambda Ny$ where $N : C[a, \sigma^2(t_n)] \rightarrow C[a, \sigma^2(t_n)]$ is given by

$$Ny(t) = \int_a^{\sigma(t_n)} G_n(t, s) f(s, y(\sigma(s))) \Delta s.$$

In [4] we showed that

$$N : C[a, \sigma^2(t_n)] \rightarrow C[a, \sigma^2(t_n)] \quad \text{is continuous and completely continuous.} \tag{2.8}$$

Next, let y be any solution of $y = \lambda Ny$ for $0 < \lambda < 1$. Then, for $t \in [a, \sigma^2(t_n)]$ we have from [10, p. 38] or [11, p. 35] that

$$\begin{aligned} |y(t)| &\leq \int_a^{\sigma(t_n)} |G_n(t, s)| \phi(s) w(|y(\sigma(s))|) \Delta s \\ &\leq w(|y|_n) \int_a^{\sigma(t_n)} |G_n(t, s)| \phi(s) \Delta s \\ &\leq w(|y|_n) \sup_{n \in \mathbb{N}} \left[\sup_{t \in [a, \sigma^2(t_n)]} \int_a^{\sigma(t_n)} \phi(s) |G_n(t, s)| \Delta s \right], \end{aligned}$$

where $|y|_n = \sup_{t \in [a, \sigma^2(t_n)]} |y(t)|$. Consequently,

$$\frac{|y|_n}{w(|y|_n) \sup_{n \in \mathbb{N}} \left[\sup_{t \in [a, \sigma^2(t_n)]} \int_a^{\sigma(t_n)} \phi(s) |G_n(t, s)| \Delta s \right]} \leq 1. \tag{2.9}$$

Now (2.4) and (2.9) imply $|y|_n \neq r$. Let

$$E = C[a, \sigma^2(t_n)] \quad \text{and} \quad U = \{u \in C[a, \sigma^2(t_n)] : |u|_n < r\}.$$

Now Theorem 1.1 (notice (A2) does not hold since any solution of $y = \lambda Ny$ with $0 < \lambda < 1$ satisfies $|y|_n \neq r$) guarantees that N has a fixed point $y_n \in \bar{U}$, i.e. (2.6)ⁿ has a solution $y_n \in C[a, \sigma^2(t_n)]$ with $|y_n|_n < r$ (note $|y_n|_n \leq r$ by Theorem 1.1 but $|y_n|_n \neq r$ by an argument similar to the one above). We can use this argument for each $n \in \mathbb{N}$.

For $k \in \mathbb{N}$ let

$$u_k(t) = \begin{cases} y_k(t), & t \in [a, \sigma^2(t_k)], \\ 0, & t \in [\sigma^2(t_k), \infty). \end{cases}$$

Let $S = \{u_k\}_{k=1}^\infty$. Notice that

$$|u_k(x)| < r \quad \text{for } x \in [a, \sigma^2(t_1)], \quad k \in \mathbb{N}.$$

Also for $k \in \mathbb{N}$ and $t \in [a, \sigma^2(t_1)]$ we have

$$u_k(t) = \int_a^{\sigma(t_1)} G_1(t, s) f(s, u_k(\sigma(s))) \Delta s + t \frac{u_k(\sigma^2(t_1))}{\sigma^2(t_1)}.$$

Thus, for $k \in \mathbb{N}$ and $t, x \in [a, \sigma^2(t_1)]$ we have

$$u_k(t) - u_k(x) = \int_a^{\sigma(t_1)} [G_1(t, s) - G_1(x, s)] f(s, u_k(\sigma(s))) \Delta s + (t - x) \frac{u_k(\sigma^2(t_1))}{\sigma^2(t_1)}$$

and this together with (2.3) and [11, p. 35] yields

$$|u_k(t) - u_k(x)| \leq w(r) \int_a^{\sigma(t_1)} |G_1(t, s) - G_1(x, s)| \phi(s) \Delta s + |t - x| \frac{r}{\sigma^2(t_1)}.$$

The Arzelà–Ascoli Theorem [12] guarantees that there is a subsequence N_1^\star of \mathbb{N} and a function $z_1 \in C[a, \sigma^2(t_1)]$ with $u_k \rightarrow z_1$ in $C[a, \sigma^2(t_1)]$ as $k \rightarrow \infty$ through N_1^\star . Let $N_1 = N_1^\star \setminus \{1\}$. Notice that

$$|u_k(x)| < r \quad \text{for } x \in [a, \sigma^2(t_2)], \quad k \in N_1.$$

Also for $k \in N_1$ and $t, x \in [a, \sigma^2(t_2)]$ we have

$$|u_k(t) - u_k(x)| \leq w(r) \int_a^{\sigma(t_2)} |G_2(t, s) - G_2(x, s)| \phi(s) \Delta s + |t - x| \frac{r}{\sigma^2(t_2)}.$$

The Arzelà–Ascoli Theorem guarantees that there is a subsequence N_2^\star of N_1 and a function $z_2 \in C[a, \sigma^2(t_2)]$ with $u_k \rightarrow z_2$ in $C[a, \sigma^2(t_2)]$ as $k \rightarrow \infty$ through N_2^\star . Note that $z_1 = z_2$ on $[a, \sigma^2(t_1)]$ since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain for $m \in \{3, 4, \dots\}$ a subsequence N_m^\star of N_{m-1} and a function $z_m \in C[a, \sigma^2(t_m)]$ with $u_k \rightarrow z_m$ in $C[a, \sigma^2(t_m)]$ as $k \rightarrow \infty$ through N_m^\star . Then let $N_m = N_m^\star \setminus \{m\}$.

Define a function y as follows. Fix $x \in (a, \infty)$ and let $m \in \mathbb{N}$ with $x \leq \sigma^2(t_m)$. Then define $y(x) = z_m(x)$. Then $y \in C[a, \infty)$, $y(a) = 0$ and $|y(x)| \leq r$ for $x \in [a, \infty)$. Again fix $x \in (a, \infty)$ and let $m \in \mathbb{N}$ with $x \leq \sigma^2(t_m)$. Then for $n \in N_m$ we have

$$u_n(x) = \int_a^{\sigma(t_m)} G_m(x, s) f(s, u_n(\sigma(s))) \Delta s + x \frac{u_n(\sigma^2(t_m))}{\sigma^2(t_m)}.$$

Let $n \rightarrow \infty$ through N_m (using [10, p. 38] or [11, p. 35]) to obtain

$$z_m(x) = \int_a^{\sigma(t_m)} G_m(x, s) f(s, z_m(\sigma(s))) \Delta s + x \frac{z_m(\sigma^2(t_m))}{\sigma^2(t_m)},$$

i.e.

$$y(x) = \int_a^{\sigma(t_m)} G_m(x, s) f(s, y(\sigma(s))) \Delta s + x \frac{y(\sigma^2(t_m))}{\sigma^2(t_m)}.$$

We can use this argument for each $x \in [a, \sigma^2(t_m)]$, and for each $m \in \mathbb{N}$. Thus

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_m]$$

for each $m \in \mathbb{N}$. \square

Remark 2.3. It is easy to check whether the requirement that $w : [0, \infty) \rightarrow [0, \infty)$ (as described in (2.3)) be nondecreasing can be omitted, provided (2.4) is replaced by

$$\exists r > 0 \text{ with } \frac{r}{[\sup_{z \in [0, r]} w(z)] \sup_{n \in \mathbb{N}} [\sup_{t \in [a, \sigma^2(t_n)]} \int_a^{\sigma(t_n)} \phi(s) |G_n(t, s)| \Delta s]} > 1.$$

Remark 2.4. We note that it is easy to establish the analogue of Theorem 2.1 for the boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) &= 0, \quad \text{for } t \in [a, \infty), \\ \alpha y(a) - \beta y^{\Delta}(a) &= \gamma, \quad \alpha \geq 0, \beta \geq 0 \quad \text{with } \alpha^2 + \beta^2 > 0, \\ y(t) &\text{ is bounded for } t \in [a, \infty). \end{aligned}$$

In this case, the function $G_n(t, s)$ in the analogue of (2.4) is Green’s function for

$$\begin{aligned} y^{\Delta\Delta}(t) &= 0 \quad \text{for } t \in [a, t_n], \\ \alpha y(a) - \beta y^{\Delta}(a) &= 0, \\ y(\sigma^2(t_n)) &= 0. \end{aligned}$$

Next we present an existence result based on the notion of upper and lower solutions. By an upper solution β to (2.1) we mean a function $\beta \in C[a, \infty)$ with

$$\begin{aligned} \beta^{\Delta\Delta}(t) + f(t, \beta(\sigma(t))) &\leq 0 \quad \text{for } t \in [a, \infty), \\ \beta(a) &\geq 0, \\ \beta(t) &\text{ is bounded for } t \in [a, \infty), \end{aligned} \tag{2.10}$$

and by a lower solution α to (2.1) we mean a function $\alpha \in C[a, \infty)$ with

$$\begin{aligned} \alpha^{\Delta\Delta}(t) + f(t, \alpha(\sigma(t))) &\geq 0 \quad \text{for } t \in [a, \infty), \\ \alpha(a) &\leq 0, \\ \alpha(t) &\text{ is bounded for } t \in [a, \infty). \end{aligned} \tag{2.11}$$

Theorem 2.2. Suppose (2.2) holds and there exists α, β respectively lower and upper solutions to (2.1) with $\alpha(t) \leq \beta(t)$ for $t \in [a, \infty)$. Then (2.1) has a solution $y \in C[a, \infty)$ with $\alpha(t) \leq y(t) \leq \beta(t)$ for $t \in [a, \infty)$.

Proof. Fix $n \in \mathbb{N}$ and consider the boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) &= 0 \quad \text{for } t \in [a, t_n], \\ y(a) &= 0, \\ y(\sigma^2(t_n)) &= \frac{\alpha(\sigma^2(t_n)) + \beta(\sigma^2(t_n))}{2}. \end{aligned} \tag{2.12}^n$$

We first show (2.12)ⁿ has a solution $y_n \in C[a, \sigma^2(t_n)]$ with $\alpha(t) \leq y_n(t) \leq \beta(t)$ for $t \in [a, \sigma^2(t_n)]$. To see this we use Theorem 1.2, so the idea is to look at the boundary value problem

$$\begin{aligned} y^{\Delta\Delta}(t) + f^\star(t, y(\sigma(t))) &= 0 \quad \text{for } t \in [a, t_n], \\ y(a) &= 0, \\ y(\sigma^2(t_n)) &= \frac{\alpha(\sigma^2(t_n)) + \beta(\sigma^2(t_n))}{2}, \end{aligned} \tag{2.13}^n$$

where

$$f^\star(t, u) = \begin{cases} f(t, \beta(\sigma(t))) + \frac{u - \beta(\sigma(t))}{1 + |u|} & \text{if } u \geq \beta(\sigma(t)), \\ f(t, u) & \text{if } \alpha(\sigma(t)) \leq u \leq \beta(\sigma(t)), \\ f(t, \alpha(\sigma(t))) + \frac{u - \alpha(\sigma(t))}{1 + |u|} & \text{if } u \leq \alpha(\sigma(t)). \end{cases}$$

Solving (2.13)ⁿ is equivalent to solving the fixed point problem $y = Ny$ where $N : C[a, \sigma^2(t_n)] \rightarrow C[a, \sigma^2(t_n)]$ is given by

$$Ny(t) = h_n(t) + \int_a^{\sigma(t_n)} G_n(t, s) f(s, y(\sigma(s))) \Delta s,$$

where G_n is as described in (2.5) and h_n is the solution of

$$\begin{aligned} y^{\Delta\Delta}(t) &= 0 \quad \text{for } t \in [a, t_n], \\ y(a) &= 0, \\ y(\sigma^2(t_n)) &= \frac{\alpha(\sigma^2(t_n)) + \beta(\sigma^2(t_n))}{2}. \end{aligned} \tag{2.14}$$

It is easy to see [4] (note f^\star is bounded) that

$$N : C[a, \sigma^2(t_n)] \rightarrow C[a, \sigma^2(t_n)] \quad \text{is continuous and compact.} \tag{2.15}$$

Now, Theorem 1.2 guarantees that (2.13)ⁿ has a solution $y_n \in C[a, \sigma^2(t_n)]$. In fact, the argument in [6] guarantees that

$$\alpha(t) \leq y_n(t) \leq \beta(t) \quad \text{for } t \in [a, \sigma^2(t_n)]. \tag{2.16}$$

Consequently, y_n is a solution of (2.12)ⁿ with (2.16) holding.

For $k \in \mathbb{N}$ let

$$u_k(t) = \begin{cases} y_k(t), & t \in [a, \sigma^2(t_k)], \\ y_k(\sigma^2(t_k)), & t \in [\sigma^2(t_k), \infty). \end{cases}$$

Proceed inductively as in Theorem 2.1 to obtain for $m \in \{1, 2, \dots\}$ a subsequence N_m^\star of N_{m-1} (here $N_0 = \mathbb{N}$) and a function $z_m \in C[a, \sigma^2(t_m)]$ with $u_k \rightarrow z_m$ in $C[a, \sigma^2(t_m)]$ as $k \rightarrow \infty$ through N_m^\star . Then let $N_m = N_m^\star \setminus \{m\}$.

Define a function y as follows. Fix $x \in (a, \infty)$ and let $m \in \mathbb{N}$ with $x \leq \sigma^2(t_m)$. Then define $y(x) = z_m(x)$. Then $y \in C[a, \infty)$, $y(a) = 0$ and $\alpha(x) \leq y(x) \leq \beta(x)$ for $x \in [a, \infty)$. Essentially the same reasoning as in Theorem 2.1 guarantees that

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, t_m]$$

for each $m \in \mathbb{N}$. \square

Remark 2.5. The ideas in Theorem 2.2 extend to the boundary value problem

$$y^{\Delta\Delta}(t) + f(t, y(\sigma(t))) = 0 \quad \text{for } t \in [a, \infty),$$

$$y(a) = \gamma,$$

$$y(t) \text{ is bounded for } t \in [a, \infty).$$

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