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Time Scale Systems On Infinite Intervals

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Abstract

A fixed point theorem of Leray–Schauder type is used to establish existence results for the time scale system $y^{\triangle}(t) = A(t)\,y(t) + F(t, y(t))$ for $t \in [a, \infty)$, $L\,y = l \in \mathbb{R}^n$.

1 Introduction

Let $\mathbb{T}$ (time scale) be a closed subset of $\mathbb{R}$. Define the forward (respectively, backward) jump operator at t for $t < \sup \mathbb{T}$ (respectively, for $t > \inf \mathbb{T}$) by

$$\sigma(t) = \inf\{\tau > t : \ \tau \in \mathbb{T}\} \quad \text{(respectively, } \rho(t) = \sup\{\tau < t : \ \tau \in \mathbb{T}\}\text{)}$$

for all $t \in \mathbb{T}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$.

If $a < b$ are points in $\mathbb{T}$, then we let

$$[a, b] = \{t \in \mathbb{T} : \ a \le t \le b\}.$$

Throughout this paper $a \in \mathbb{T}$ is fixed, and we assume that there exists $t_n \in \mathbb{T}$, $n \in \{1, 2, ...\} \equiv \mathbb{N}$, with

$$a < t_1 < t_2 < \ldots\ldots < t_n < \ldots\ldots \quad \text{with} \quad t_n \uparrow \infty \quad \text{as} \quad n \to \infty.$$

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Let

$$[a, \infty) = \cup_{n=1}^{\infty} [a, t_n].$$

In this paper we consider the system

(1.1) $$y^{\triangle}(t) = A(t)\, y(t) + F(t, y(t)) \quad \text{for} \quad t \in [a, \infty),$$

or

(1.2) $$y^{\triangle}(t) = A(t)\, y(t) + F(t, y(\sigma(t))) \quad \text{for} \quad t \in [a, \infty),$$

where $A : [a, \infty) \to B(\mathbb{R}^n)$ is rd–continuous and regressive [11, 14], and $a \in \mathbb{T}$ is fixed. Throughout this paper $C[a, \infty)$ will denote the space of continuous functions $u : [a, \infty) \to \mathbb{R}^n$, and $B[a, \infty)$ will denote the functions $u \in C[a, \infty)$ with $\|u\|_\infty = \sup_{n \in \mathbb{N}} \|u\|_n < \infty$; here $\|u\|_n = \sup_{t \in [a, t_n]} \|u(t)\|$ and $\|\alpha\| = \sum_{i=1}^{n} |\alpha_i|$ if $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbf{R}^n$. To understand (1.1) we need to recall some standard definitions (see [11, 14] for an introduction to this subject).

Definition 1.1. Fix $t \in \mathbb{T}$. Let $y : \mathbb{T} \to \mathbb{R}$. Then we define $y^{\triangle}(t)$ to be the number (if it exists) with the property that given $\epsilon > 0$ there is a neighborhood U of t with

$$|[y(\sigma(t)) - y(s)] - y^{\triangle}(t)\,[\sigma(t) - s]| < \epsilon\, |\sigma(t) - s|$$

for all $s \in U$. We call $y^{\triangle}(t)$ the derivative of $y(t)$.

Definition 1.2. If $F^{\triangle}(t) = f(t)$ then we define the integral by

$$\int_a^t f(\tau) \triangle \tau = F(t) - F(a).$$

In this paper we study (1.1) (or alternatively (1.2)) subject to the boundary condition

(1.3) $$L\, y = l \in \mathbb{R}^n;$$

here L is a bounded linear operator mapping $B[a, \infty)$ into $\mathbb{R}^n$.

In Section 2 we begin by considering the system (1.1) with $F(t, y) = b(t)$ i.e. the linear system

(1.4) $$y^{\triangle}(t) = A(t)\, y(t) + b(t), \quad \text{for} \quad t \in [a, \infty),$$

where $b : [a, \infty) \to \mathbb{R}^n$ is rd-continuous. We give necessary and sufficient conditions for the existence of a solution to (1.3), (1.4). This result is then used, together with a fixed point result of Leray–Schauder type, to establish existence results for the nonlinear system (1.1), (1.3). For the convenience of the reader we state here the fixed point result [3] we will use in this paper.

Theorem 1.1. *Let C be a convex subset of a Banach space E, and let U be an open subset of C with $0 \in U$. Then every compact, continuous map $N : \overline{U} \to C$ has at least one of the following two properties:*

(A1). N has a fixed point in $\overline{U}$; or

(A2). there is a $x \in \partial U$ and $\lambda \in (0,1)$ with $x = \lambda N x$.

2 Existence

We begin by considering the linear system

$$\text{(2.1)} \qquad \begin{cases} y^{\triangle}(t) = A(t)\, y(t) + b(t) \quad \text{for} \quad t \in [a, \infty), \\ y(a) = c \in \mathbb{R}^n \end{cases}$$

where $A : [a, \infty) \to B(\mathbb{R}^n)$ is rd–continuous and regressive, $b : [a, \infty) \to \mathbb{R}^n$ is rd–continuous, and $a \in \mathbb{T}$ is fixed. Recall [11, 14] any solution of (2.1) has the form

$$y(t) = \Phi_A(t, a)\, c + \int_a^t \Phi_A(t, \sigma(s))\, b(s) \triangle s$$

where $\Phi_A(t, a) \equiv \Phi_A(t)$ is the fundamental matrix solution of

$$y^{\triangle} = A(t)\, y$$

with $\Phi_A(a, a) = I$ (the identity matrix).

Theorem 2.1. *Let $A : [a, \infty) \to B(\mathbb{R}^n)$ be rd–continuous and regressive, $b : [a, \infty) \to \mathbb{R}^n$ be rd–continuous, and L be a bounded linear operator mapping $B[a, \infty)$ into $\mathbb{R}^n$. Assume the following conditions hold:*

$$\text{(2.2)} \qquad \sup_{n \in \mathbb{N}} \|\Phi_A\|_n < \infty$$

and

$$\text{(2.3)} \qquad \sup_{n \in \mathbb{N}} \sup_{t \in [a, t_n]} \int_a^{t_n} \|\Phi_A(t, \sigma(s))\, b(s)\| \triangle s < \infty;$$

here for the matrix $\Phi_A(t) = (a_{ij}(t))$ we let

$$\|\Phi_A(t)\| = \sup_j \|a_{ij}(t)\| \quad \text{and} \quad \|\Phi_A\|_n = \sup_{t \in [a, t_n]} \|\Phi_A(t)\|.$$

Then a necessary and sufficient condition for the existence of a unique solution to

$$\text{(2.4)} \qquad \begin{cases} y^{\triangle} = A(t)\, y + b(t) \quad \text{for} \quad t \in [a, \infty), \\ L\, y = l \in \mathbb{R}^n \end{cases}$$

is that

$$\text{(2.5)} \qquad G = L[\Phi_A(t)]$$

(constant matrix whose columns are the values of L on the corresponding columns of $\Phi_A(t)$) is nonsingular. Furthermore this solution $y(t)$ can be written as

$$y(t) = \Phi_A(t)\, G^{-1}\, l + \int_a^t \Phi_A(t,\sigma(s))\, b(s) \triangle s$$
$$- \Phi_A(t)\, G^{-1}\, L \left[\int_a^t \Phi_A(t,\sigma(s))\, b(s) \triangle s \right].$$

PROOF. Any solution of $y^{\triangle} = A(t)\, y + b(t)$ for $t \in [a,\infty)$ can be written as

$$y(t) = \Phi_A(t,a)\, c + \int_a^t \Phi_A(t,\sigma(s))\, b(s) \triangle s$$

where c is a constant vector. This solution satisfies $L\,y = l$ if and only if

$$L\,[\Phi_A(t,a)\, c] + L \left[\int_a^t \Phi_A(t,\sigma(s))\, b(s) \triangle s \right] = l.$$

Now since G is nonsingular we get

$$c = G^{-1}\, l - G^{-1}\, L \left[\int_a^t \Phi_A(t,\sigma(s))\, b(s) \triangle s \right].$$

In what follows, for a fixed integer m, let $C[a,t_m]$ denote the space of all real n–valued functions defined and continuous on $[a,t_m]$, and with $C[a,t_m]$ we associate the norm $\|y\|_m = \sup_{t\in[a,t_m]} \|y(t)\|$ (here $y \in C[a,t_m]$). For $y \in C[a,t_m]$ let

$$\overline{y}(t) = \begin{cases} y(t), & t \in [a,t_m] \\ y(t_m), & t \in [t_m,\infty). \end{cases}$$

The set of all such functions will be denoted by $D[a,t_m]$, and with $D[a,t_m]$ we associate the norm

$$\|\overline{y}\|_{D[a,t_m]} = \|y\|_m.$$

Clearly $D[a,t_m]$ is a Banach space.

Next we consider the system

$$\text{(2.6)} \qquad \begin{cases} y^{\triangle}(t) = A(t)\, y(t) + \mu\, f(t,y(t)) & \text{for } t \in [a,\infty), \\ L\,y = l \in \mathbb{R}^n; \end{cases}$$

here $\mu \geq 0$ is a constant.

Theorem 2.2. *Let $A : [a,\infty) \to B(\mathbb{R}^n)$ be rd-continuous and regressive, $f : [a,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, and L be a bounded linear operator mapping $B[a,\infty)$ into $\mathbb{R}^n$. Assume*

(2.2) holds and G (given in (2.5)) is nonsingular. Suppose the following conditions are satisfied:

$$\text{(2.7)} \qquad \begin{cases} \text{there exists a continuous, nondecreasing function } \psi : [0,\infty) \to [0,\infty) \\ \text{with } \|f(s,x(s))\| \le r(s)\,\psi(\|x(s)\|) + q(s) \text{ for all } x \in B[a,\infty) \\ \text{and } s \in [a,\infty) \end{cases}$$

$$\text{(2.8)} \qquad \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} \|\Phi_A(t,\sigma(s))\|\, r(s) \triangle s \equiv W < \infty$$

$$\text{(2.9)} \qquad \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} \|\Phi_A(t,\sigma(s))\|\, q(s) \triangle s \equiv P < \infty$$

and

$$\text{(2.10)} \qquad \mu_0 \ge 0 \ \ \text{satisfies} \ \ \sup_{x\in(0,\infty} \left(\frac{x}{E_0\,\mu_0\,\psi(x) + E_1\,\mu_0 + E_2} \right) > 1;$$

here

$$E_0 = W + W\,Q\,\|G^{-1}\|\,\|L\|,\ E_1 = P + P\,Q\,\|G^{-1}\|\,\|L\|,\ E_3 = Q\,\|G^{-1}\|\,\|l\|$$

and

$$Q = \sup_{n\in\mathbb{N}} \|\Phi_A\|_n.$$

If $0 \le \mu \le \mu_0$ then (2.6) (for every $l \in \mathbb{R}^n$) has a solution in $B[a,\infty)$.

PROOF. Fix $\mu \le \mu_0$. Let $M_0 > 0$ satisfy

$$\text{(2.11)} \qquad \frac{M_0}{E_0\,\mu_0\,\psi(M_0) + E_1\,\mu_0 + E_2} > 1.$$

Fix $m \in \mathbb{N}$. Consider the operator $T_m : D[a,t_m] \to D[a,t_m]$ defined by

$$T_m(\overline{x})(t) = \overline{y}(t) \quad (\text{here } \overline{x} \in D[a,t_m])$$

where for $t \in [a,t_m]$,

$$y(t) = \Phi_A(t)\,G^{-1}\,l + \int_a^t \Phi_A(t,\sigma(s))\,\mu\,f(s,\overline{x}(s)) \triangle s$$
$$- \Phi_A(t)\,G^{-1}\,L\left[\int_a^t \Phi_A(t,\sigma(s))\,\mu\,f(s,\overline{x}(s)) \triangle s\right].$$

We wish to apply Theorem 1.1. First we show $T_m : D[a,t_m] \to D[a,t_m]$ is continuous. To see this let $\overline{x}$, $\overline{x}_n \in D[a,t_m]$ and define $\overline{y}(t) = T_m(\overline{x})(t)$ and $\overline{y}_n(t) = T_m(\overline{x}_n)(t)$. Then

$$\|T_m(\overline{x}) - T_m(\overline{x}_n)\|_{D[a,t_m]} = \|\overline{y} - \overline{y}_n\|_{D[a,t_m]} = \|y - y_n\|_m = \sup_{t\in[a,t_m]} \|y(t) - y_n(t)\|$$
$$\leq \mu \sup_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t,\sigma(s))\| \, \|f(s,\overline{x}(s)) - f(s,\overline{x}_n(s))\| \triangle s$$
$$+\mu\, Q\, \|G^{-1}\|\, \|L\| \sup_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t,\sigma(s))\| \, \|f(s,\overline{x}(s)) - f(s,\overline{x}_n(s))\| \triangle s.$$

It is also easy to see that

$$\|\overline{x} - \overline{x}_m\|_{D[a,t_m]} \to 0 \quad \text{implies} \quad \rho_m = \sup_{s\in[a,t_m]} \|f(s,\overline{x}(s)) - f(s,\overline{x}_n(s))\| \to 0$$

as $n \to \infty$. This together with

$$\|T_m(\overline{x}) - T_m(\overline{x}_n)\|_{D[a,t_m]} \leq \mu\, \rho_m \sup_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t,\sigma(s))\| \triangle s$$
$$+ \mu\, Q\, \|G^{-1}\|\, \|L\|\, \rho_m \sup_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t,\sigma(s))\| \triangle s$$

guarantees that $T_m : D[a,t_m] \to D[a,t_m]$ is continuous. Next we show T_m is completely continuous. Let Ω be a bounded subset of $D[a,t_m]$, so there exists $r_0 > 0$ with $\|\overline{x}\|_{D[a,t_m]} \leq r_0$ for each $\overline{x} \in \Omega$. Let $\overline{y}(t) = T_m(\overline{x})(t)$ for $\overline{x} \in \Omega$. Then
(2.12)

$$\|T_m(\overline{x})\|_{D[a,t_m]} = \|\overline{y}\|_{D[a,t_m]} = \|y\|_m$$

$$\leq Q\, \|G^{-1}\|\, \|l\| + \mu \sup\nolimits_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t,\sigma(s))\|\, \|f(s,\overline{x}(s))\| \triangle s$$

$$+\mu\, Q\, \|G^{-1}\|\, \|L\| \sup\nolimits_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t,\sigma(s))\|\, \|f(s,\overline{x}(s))\| \triangle s$$

$$\leq Q\, \|G^{-1}\|\, \|l\| + \mu\, [W\, \psi(r_0) + P] + \mu\, Q\, \|G^{-1}\|\, \|L\|\, [W\, \psi(r_0) + P],$$

and for $z_1,\ z_2 \in [a, t_m]$ we have

$$\|T_m(\overline{x})(z_1) - T_m(\overline{x})(z_2)\| = \|\overline{y}(z_1) - \overline{y}(z_2)\| = \|y(z_1) - y(z_2)\|$$

$$\leq \|[\Phi_A(z_1) - \Phi_A(z_2)]\ G^{-1}\, l\|$$

$$+ \|\textstyle\int_a^{z_1} \Phi_A(z_1, \sigma(s))\, \mu\, f(s, \overline{x}(s)) \triangle s - \int_a^{z_2} \Phi_A(z_2, \sigma(s))\, \mu\, f(s, \overline{x}(s)) \triangle s\|$$

$$+ \left\|[\Phi_A(z_1) - \Phi_A(z_2)]\ G^{-1}\, L \left[\textstyle\int_a^t \Phi_A(t, \sigma(s))\, \mu\, f(s, \overline{x}(s)) \triangle s\right]\right\|$$

$$\leq \|[\Phi_A(z_1) - \Phi_A(z_2)]\ G^{-1}\, l\| \tag{2.13}$$

$$+ \textstyle\int_a^{t_m} \|\Phi_A(z_1, \sigma(s)) - \Phi_A(z_2, \sigma(s))\|\ \mu\ [r(s)\, \psi(r_0) + q(s)] \triangle s$$

$$+ \left|\textstyle\int_{z_1}^{z_2} \|\Phi_A(z_2, \sigma(s))\|\ \mu\ [r(s)\, \psi(r_0) + q(s)] \triangle s\right|$$

$$+ \|\Phi_A(z_1) - \Phi_A(z_2)\|\ \|G^{-1}\|\ \|L\| \times$$

$$\sup\nolimits_{t\in[a,t_m]} \textstyle\int_a^{t_m} \|\Phi_A(t, \sigma(s))\|\ \mu\ [r(s)\, \psi(r_0) + q(s)] \triangle s.$$

The Arzela–Ascoli theorem [15] guarantees that $T_m : D[a, t_m] \to D[a, t_m]$ is completely continuous. Now assume the equation

$$\overline{x}(t) = \lambda\, T_m\, (\overline{x})(t) \tag{2.14$_\lambda$}$$

has a solution in $D[a, t_m]$ for some $0 < \lambda < 1$. Then

$$\begin{aligned}
\|\overline{x}\|_{D[a,t_m]} = \|x\|_m &\leq Q\, \|G^{-1}\|\, \|l\| + \mu \sup_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t, \sigma(s))\|\, \|f(s, \overline{x}(s))\| \triangle s \\
&+ \mu\, Q\, \|G^{-1}\|\, \|L\| \sup_{t\in[a,t_m]} \int_a^{t_m} \|\Phi_A(t, \sigma(s))\|\, \|f(s, \overline{x}(s))\| \triangle s \\
&\leq Q\, \|G^{-1}\|\, \|l\| + \mu \left[W\, \psi\left(\|\overline{x}\|_{D[a,t_m]}\right) + P\right] \\
&+ \mu\, Q\, \|G^{-1}\|\, \|L\| \left[W\, \psi\left(\|\overline{x}\|_{D[a,t_m]}\right) + P\right] \\
&= E_2 + E_1\, \mu + E_0\, \mu\, \psi\left(\|\overline{x}\|_{D[a,t_m]}\right),
\end{aligned}$$

and so

$$\frac{\|\overline{x}\|_{D[a,t_m]}}{E_2 + E_1\,\mu + E_0\,\mu\,\psi\left(\|\overline{x}\|_{D[a,t_m]}\right)} \le 1.$$

This together with (2.11) implies

$$\|\overline{x}\|_{D[a,t_m]} \ne M_0 \tag{2.15}$$

for any solution $\overline{x} \in D[a,t_m]$ to $(2.14)_\lambda$. Apply Theorem 1.1 with

$$U = \left\{\overline{x} \in D[a,t_m] : \ \|\overline{x}\|_{D[a,t_m]} < M_0\right\}, \quad C = E = D[a,t_m]$$

to deduce (note $(A2)$ in Theorem 1.1 cannot occur because of (2.15)) that T_m has a fixed point $\overline{x}_m$ i.e. $\overline{x}_m(t) = T_m(\overline{x}_m)(t)$ with $\|\overline{x}_m\|_{D[a,t_m]} \le M_0$. We can do this for each $m \in \mathbb{N}$. Hence we can find a sequence $\{x_m\}_{m\in\mathbb{N}}$ such that $\overline{x}_m \in D[a,t_m]$ and $\|\overline{x}_m\|_{D[a,t_m]} = \|x_m\|_m \le M_0$, and for $t \in [a,t_m]$ we have

$$\begin{aligned} x_m(t) &= \Phi_A(t)\,G^{-1}\,l + \int_a^t \Phi_A(t,\sigma(s))\,\mu\,f(s,\overline{x}_m(s))\,\triangle\,s \\ &\quad - \Phi_A(t)\,G^{-1}\,L\left[\int_a^t \Phi_A(t,\sigma(s))\,\mu\,f(s,\overline{x}_m(s))\,\triangle\,s\right]. \end{aligned}$$

The Arzela–Ascoli theorem (similar reasoning as in (2.12) and (2.13)) guarantees a subsequence S_1 of $\mathbb{N}$ and a $x^1 \in C[a,t_1]$ with

$$\overline{x}_m \to x^1 \ \text{ in } \ C[a,t_1] \ \text{ as } \ m \to \infty \ \text{ through } \ S_1.$$

Similarly there exists a subsequence S_2 of S_1 and a $x^2 \in C[a,t_2]$ with

$$\overline{x}_m \to x^2 \ \text{ in } \ C[a,t_2] \ \text{ as } \ m \to \infty \ \text{ through } \ S_2.$$

Notice $x^1 = x^2$ on $[a,t_1]$ since $S_2 \subseteq S_1$. Proceed inductively to obtain for $i \in \mathbb{N}$ a subsequence $S_i \subseteq S_{i-1}$ and a $x^i \in C[a,t_i]$ with

$$\overline{x}_m \to x^i \ \text{ in } \ C[a,t_i] \ \text{ as } \ m \to \infty \ \text{ through } \ S_i.$$

Define $x : [a,\infty) \to \mathbb{R}^n$ as follows. Fix $c \in [a,\infty)$ and let $j \in \mathbb{N}$ with $c \le t_j$. Then define $x(c) = x^j(c)$. Now x is well defined and $\sup_{n\in\mathbb{N}} \|x\|_n \le M_0$. Again fix $c \in [a,\infty)$ and choose $j \in \mathbb{N}$ with $c \le t_j$. Then for $m \in S_j$ and $t \in [a,t_j]$ we have

$$\begin{aligned} \overline{x}_m(t) &= \Phi_A(t)\,G^{-1}\,l + \int_a^t \Phi_A(t,\sigma(s))\,\mu\,f(s,\overline{x}_m(s))\,\triangle\,s \\ &\quad - \Phi_A(t)\,G^{-1}\,L\left[\int_a^t \Phi_A(t,\sigma(s))\,\mu\,f(s,\overline{x}_m(s))\,\triangle\,s\right]. \end{aligned}$$

Let $m \to \infty$ through S_j (using [11 pp. 38 or 14 pp. 35]) to obtain

$$x^j(t) = \Phi_A(t)\, G^{-1}\, l + \int_a^t \Phi_A(t,\sigma(s))\, \mu\, f(s, x^j(s)) \triangle s$$
$$- \Phi_A(t)\, G^{-1}\, L \left[\int_a^t \Phi_A(t,\sigma(s))\, \mu\, f(s, x^j(s)) \triangle s \right].$$

That is

$$x(t) = \Phi_A(t)\, G^{-1}\, l + \int_a^t \Phi_A(t,\sigma(s))\, \mu\, f(s, x(s)) \triangle s$$
$$- \Phi_A(t)\, G^{-1}\, L \left[\int_a^t \Phi_A(t,\sigma(s))\, \mu\, f(s, x(s)) \triangle s \right]$$

for $t \in [a, t_j]$ (in particular for $t = c$). Since c is arbitrary the result follows.

If in (2.7), ψ is at most linear growth, then we can obtain another result of Theorem 2.2 type if we use Gronwall's inequality [11 pp. 54 or 14 pp. 61].

Theorem 2.3. *Let $A : [a,\infty) \to B(\mathbb{R}^n)$ be rd-continuous and regressive, $f : [a,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous, and L be a bounded linear operator mapping $B[a,\infty)$ into $\mathbb{R}^n$. Assume (2.2) holds and G (given in (2.5)) is nonsingular. Suppose the following conditions are satisfied:*

$$(2.16) \qquad \begin{cases} \|f(s,x(s))\| \le r(s)\, \|x(s)\| + q(s) \quad \text{for all} \\ x \in B[a,\infty) \quad \text{and} \quad s \in [a,\infty) \end{cases}$$

$$(2.17) \qquad \begin{cases} \exists\, b : [a,\infty) \to [0,\infty) \quad \text{and} \quad K > 0 \quad \text{with for} \quad s \in [a,\infty), \\ \|\Phi_A(t,\sigma(s))\| \le K\, b(s) \quad \text{for all} \quad t \in [a,\infty) \end{cases}$$

$$(2.18) \qquad \sup_{n\in\mathbb{N}} \int_a^{t_n} b(s)\, r(s) \triangle s \equiv W_0 < \infty$$

$$(2.19) \qquad \sup_{n\in\mathbb{N}} \int_a^{t_n} b(s)\, q(s) \triangle s \equiv P_0 < \infty$$

and

$$(2.20) \qquad \mu \ge 0 \quad \text{satisfies} \quad \mu\, E_3 \left(1 + \mu\, K \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} b(s)\, r(s)\, e_c(t,\sigma(s)) \triangle s \right) < 1;$$

here $c = \mu\, K\, b(s)\, r(s)$, e_c is as in [11], and $E_3 = Q\, \|G^{-1}\|\, \|L\|\, K\, W_0$. Then (2.6) (for every $l \in \mathbb{R}^n$) has a solution in $B[a,\infty)$.

PROOF. Fix $m \in \mathbb{N}$ and let $T_m : D[a,t_m] \to D[a,t_m]$ be as in Theorem 2.2. Essentially the same reasoning as in Theorem 2.2 guarantees that $T_m : D[a,t_m] \to D[a,t_m]$ is continuous and completely continuous.

Now assume that the equation

$$(2.21)_\lambda \qquad \overline{x}(t) = \lambda T_m \overline{x}(t)$$

has a solution in $D[a,t_m]$ for some $\lambda \in (0,1)$. Then for $t \in [a,t_m]$ we have

$$\begin{aligned}\|x(t)\| \le & \, Q\,\|G^{-1}\|\,\|l\| + \mu K \int_a^t b(s)\,[r(s)\,\|x(s)\| + q(s)] \triangle s \\ & + \mu Q\,\|G^{-1}\|\,\|L\|\,K\left[W_0\,\|\overline{x}\|_{D[a,t_m]} + P_0\right],\end{aligned}$$

and so

$$\begin{aligned}\|x(t)\| \le & \left[Q\,\|G^{-1}\|\,\|l\| + \mu K P_0 + \mu E_3\,\|\overline{x}\|_{D[a,t_m]} + \mu Q\,\|G^{-1}\|\,\|L\|\,K P_0\right] \\ & + \mu K \int_a^t b(s)\,r(s)\,\|x(s)\| \triangle s.\end{aligned}$$

From Gronwall's inequality [11 pp. 54 or 14 pp. 61] it follows that

$$\|x(t\| \le \left(E_4 + \mu E_5 + \mu E_3\,\|\overline{x}\|_{D[a,t_m]}\right)\left(1 + \mu K \int_a^t b(s)\,r(s)\,e_c(t,\sigma(s)) \triangle s\right)$$

where

$$E_4 = Q\,\|G^{-1}\|\,\|l\| \quad \text{and} \quad E_5 = K P_0 + Q\,\|G^{-1}\|\,\|L\|\,K P_0.$$

As a result

$$\begin{aligned}\|\overline{x}\|_{D[a,t_m]} & \left\{1 - \mu E_3\left(1 + \mu K \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} b(s)\,r(s)\,e_c(t,\sigma(s)) \triangle s\right)\right\} \\ & \le (E_4 + \mu E_5)\left(1 + \mu K \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} b(s)\,r(s)\,e_c(t,\sigma(s)) \triangle s\right).\end{aligned}$$

Thus

$$\begin{aligned}\|\overline{x}\|_{D[a,t_m]} \le & \frac{(E_4 + \mu E_5)\left(1 + \mu K \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} b(s)\,r(s)\,e_c(t,\sigma(s)) \triangle s\right)}{1 - \mu E_3\left(1 + \mu K \sup_{n\in\mathbb{N}} \sup_{t\in[a,t_n]} \int_a^{t_n} b(s)\,r(s)\,e_c(t,\sigma(s)) \triangle s\right)} \\ \equiv & \, M.\end{aligned}$$

Let $M_0 = M + 1$ and apply Theorem 1.1 with

$$U = \left\{\overline{x} \in D[a,t_m] : \ \|\overline{x}\|_{D[a,t_m]} < M_0\right\}, \ \ C = E = D[a,t_m]$$

to deduce that T_m has a fixed point $\overline{x}_m$ i.e. $\overline{x}_m(t) = T_m(\overline{x}_m)(t)$ with $\|\overline{x}_m\|_{D[a,t_m]} \leq M_0$. We can do this for each $m \in \mathbb{N}$. Essentially the same reasoning as in Theorem 2.2 completes the proof of Theorem 2.3.

To conclude this paper we remark that the ideas in this section extend to the problem

$$(2.22) \qquad \begin{cases} y^{\triangle}(t) = A(t)\,y(t) + \mu\, f(t, y(\sigma(t))) \;\; \text{for} \;\; t \in [a, \infty), \\ L\,y = l \in \mathbb{R}^n \end{cases}$$

(i.e. it is easy to obtain the analogue of Theorem 2.2 for (2.22)).

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