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Trigonometric Transformations of Symplectic Difference Systems

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In this paper we show that any symplectic difference system can be transformed into a trigonometric system, using a transformation that preserves oscillatory properties. Necessary and sufficient conditions for nonoscillation of a certain class of trigonometric systems are given, and this result is applied to oscillation theory of Hamiltonian difference systems.

1. INTRODUCTION

We consider transformations of symplectic difference systems (further SdS)

$$z_{k+1} = \mathcal{S}_k z_k, \tag{S}$$

where $z_k \in \mathbb{R}^{2n}$ and $\mathcal{S}_k$ is a real-valued, symplectic $2n \times 2n$-matrix for all $k \in \mathbb{Z}$, i.e.,

$$\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J} \qquad \text{with} \quad \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \tag{1.1}$$

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where I denotes the identity matrix and ^{T} stands for the transpose of the matrix indicated. Sometimes it will be convenient to write (S) in the form

$$\begin{pmatrix} x_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} \tag{S}$$

with $x, u \in \mathbb{R}^n$ and real-valued $n \times n$-matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ (we will occasionally suppress the subscript k in some of the results that follow, where the meaning will remain clear; e.g., if we write $\mathcal{S}^T \mathcal{J} \mathcal{S} = \mathcal{J}$ we understand that this means $\mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}$ for all $k \in \mathbb{Z}$). Using (1.1), it is easy to see that the matrix $\mathcal{S}$ in (S) is symplectic iff

$$\mathcal{A}^T \mathcal{D} - \mathcal{C}^T \mathcal{B} = I, \qquad \mathcal{A}^T \mathcal{C} = \mathcal{C}^T \mathcal{A}, \qquad \mathcal{B}^T \mathcal{D} = \mathcal{D}^T \mathcal{B}. \tag{1.2}$$

Since (1.1) is equivalent to $\mathcal{S}_k \mathcal{J} \mathcal{S}_k^T = \mathcal{J}$, (1.2) is equivalent to

$$\mathcal{A} \mathcal{D}^T - \mathcal{B} \mathcal{C}^T = I, \qquad \mathcal{A} \mathcal{B}^T = \mathcal{B} \mathcal{A}^T, \qquad \mathcal{C} \mathcal{D}^T = \mathcal{D} \mathcal{C}^T. \tag{1.3}$$

Oscillation properties and transformations of SdS have been investigated in [6] (see also [1, 3, 5, 7, 11, 13, 14]), where the so-called Reid Roundabout Theorem for (S) is established. This theorem relates oscillation properties of (S) to positivity of the corresponding quadratic functional and to the solvability of associated Riccati-type matrix difference equation.

A trigonometric SdS is a system (S) where the matrix $\mathcal{S}$ satisfies the additional condition

$$\mathcal{J}^T \mathcal{S}_k \mathcal{J} = \mathcal{S}_k, \tag{1.4}$$

which implies that if $\binom{x}{u}$ is a solution of (S), then $\binom{-u}{x}$ is a solution as well. Using (1.4), we have that (S) is a trigonometric SdS iff $\mathcal{A} = \mathcal{D}$ and $\mathcal{B} = -\mathcal{C}$. Consequently, combining this with (1.2) and (1.3), we see that a trigonometric SdS is a system of the form

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \qquad u_{k+1} = -\mathcal{B}_k x_k + \mathcal{A}_k u_k, \tag{1.5}$$

where

$$\mathcal{A}^T \mathcal{A} + \mathcal{B}^T \mathcal{B} = I, \qquad \mathcal{A}^T \mathcal{B} = \mathcal{B}^T \mathcal{A} \tag{1.6}$$

which is equivalent to $\mathcal{A} \mathcal{A}^T + \mathcal{B} \mathcal{B}^T = I$, $\mathcal{A} \mathcal{B}^T = \mathcal{B} \mathcal{A}^T$. The terminology "trigonometric system" is justified by the fact that in the scalar case $n = 1$, (1.6) implies that $\mathcal{A}_k = \cos \varphi_k$, $\mathcal{B}_k = \sin \varphi_k$ for some φ, and then solutions of (1.5) are

$$\left(\cos \left(\sum^{k-1} \varphi_j \right), \sin \left(\sum^{k-1} \varphi_j \right) \right) \quad \text{and} \quad \left(-\sin \left(\sum^{k-1} \varphi_j \right), \cos \left(\sum^{k-1} \varphi_j \right) \right).$$

Basic properties of solutions of trigonometric SdS with nonsingular B are established in the recent paper of Anderson [2].

In the continuous case, the concept of trigonometric systems was introduced by Barrett [4] in connection with the extension of the classical Prüfer transformation to matrix differential systems. A differential trigonometric system is a system

$$x' = Q(t)\,u, \qquad u' = -Q(t)\,x \tag{1.7}$$

with a symmetric $n \times n$-matrix-valued function Q. The main result of our paper, Theorem 3.1 below, can be viewed as a discrete version of [9, Theorem 1] where it is shown that any differential Hamiltonian system can be transformed, using a transformation preserving oscillatory properties of the transformed system, into a trigonometric system (1.7). More details are given in the next section.

The paper is organized as follows. In the next section we recall basic facts concerning oscillation properties and transformations of SdS (S). We also formulate, for the sake of comparison, the main results of transformation theory for differential Hamiltonian systems. In the third section we present the main result of our paper—the trigonometric transformation of SdS (S). In Section 4 we give a necessary and sufficient condition for nonoscillation of trigonometric systems with positive definite $\mathcal{Q}$, and the last section is devoted to some applications of trigonometric transformations in oscillation theory for discrete Hamiltonian systems.

2. AUXILIARY RESULTS

We start with some basic properties of solutions of SdS (S). Let (X, U), $(\tilde{X}, \tilde{U})$ be matrix solutions of (S), i.e., X, U, $\tilde{X}$, $\tilde{U}$ are $n \times n$-matrices. Then symplecticity of the matrix $\mathcal{S}$ implies that

$$\Delta\left[\begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}^T \mathcal{J} \begin{pmatrix} X_k & \tilde{X}_k \\ U_k & \tilde{U}_k \end{pmatrix}\right] = 0,$$

where Δ is the usual forward difference operator. Consequently, if $\left(\begin{smallmatrix} X & \tilde{X} \\ U & \tilde{U} \end{smallmatrix}\right)$ is symplectic at some k, then it is symplectic everywhere. If this is the case, then we have the identites

$$X^T\tilde{U} - U^T\tilde{X} = I, \qquad X^T U = U^T X, \qquad \tilde{X}^T\tilde{U} = \tilde{U}^T\tilde{X} \tag{2.1}$$

and

$$X\tilde{U}^T - \tilde{X}U^T = I, \qquad X\tilde{X}^T = \tilde{X}X^T, \qquad U\tilde{U}^T = \tilde{U}U^T. \tag{2.2}$$

A matrix solution (X, U) which satisfies the second identity in (2.1) and $\operatorname{rank}(X^T, U^T) = n$ is said to be a *conjoined basis* of (S). The solutions (X, U), $(\tilde{X}, \tilde{U})$ are called *normalized conjoined bases* of (S).

Following [6], we say that the conjoined basis (X, U) has a *focal point* in an interval $(k, k+1]$ if

$$\operatorname{Ker} X_{k+1} \subset \operatorname{Ker} X_k \qquad \text{and} \qquad X_k X_{k+1}^{\dagger} \mathscr{B}_k \geq 0 \tag{2.3}$$

does not hold. Here, † denotes the Moore–Penrose generalized inverse, and the inequality $\geq$ means nonnegative definiteness, and Ker stands for the kernel of a matrix. Note that if the kernel condition in (2.3) holds, then the matrix $X_k X_{k+1}^{\dagger} \mathscr{B}_k$ is really symmetric, see [6, Sect. 3]. System (S) is said to be *nonoscillatory* if there exists $N \in \mathbb{N}$ and a conjoined basis (X, U) which has no focal point in the interval $[N, \infty)$, in the opposite case (S) is said to be *oscillatory*. For an equivalent definition of oscillation and non-oscillation of symplectic systems using the concept of generalized zeros of vector-valued solutions of (S), see [6, Definition 4]. System (S) is said to be *eventually controllable* if the trivial solution (x, u) is the only solution of (S) for which $x \equiv 0$ eventually.

Now we turn our attention to transformations of SdS. Let $\mathscr{R}_k = \begin{pmatrix} H_k & M_k \\ K_k & N_k \end{pmatrix}$ be a real, symplectic $2n \times 2n$-matrix, i.e.,

$$H^T N - K^T M = I, \qquad H^T K = K^T H, \qquad M^T N = N^T M.$$

Then the transformation

$$\begin{pmatrix} x \\ u \end{pmatrix} = \mathscr{R}_k \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix} \tag{2.4}$$

transforms (S) into another symplectic system

$$\begin{pmatrix} \tilde{x}_{k+1} \\ \tilde{u}_{k+1} \end{pmatrix} = \tilde{\mathscr{S}}_k \begin{pmatrix} \tilde{x}_k \\ \tilde{u}_k \end{pmatrix} \qquad \text{with} \quad \tilde{\mathscr{S}}_k = \begin{pmatrix} \tilde{\mathscr{A}}_k & \tilde{\mathscr{B}}_k \\ \tilde{\mathscr{C}}_k & \tilde{\mathscr{D}}_k \end{pmatrix} = \mathscr{R}_{k+1}^{-1} \mathscr{S}_k \mathscr{R}_k,$$

where in detail,

$$\begin{aligned}
\tilde{\mathscr{A}} &= N_{k+1}^T (\mathscr{A} H + \mathscr{B} K) - M_{k+1}^T (\mathscr{C} H + \mathscr{D} K), \\
\tilde{\mathscr{B}} &= N_{k+1}^T (\mathscr{A} M + \mathscr{B} N) - M_{k+1}^T (\mathscr{C} M + \mathscr{D} N), \\
\tilde{\mathscr{C}} &= H_{k+1}^T (\mathscr{C} H + \mathscr{D} K) - K_{k+1}^T (\mathscr{A} H + \mathscr{B} K), \\
\tilde{\mathscr{D}} &= H_{k+1}^T (\mathscr{C} M + \mathscr{D} N) - K_{k+1}^T (\mathscr{A} M + \mathscr{B} N).
\end{aligned}$$

Moreover, if $M \equiv 0$, then this transformation preserves oscillation behavior of transformed systems, i.e., a conjoined basis (X, U) of (S) has a focal point in $(k, k+1]$ iff $(H^{-1}X, -K^TX+H^TU)$ has there a focal point (see [6, Lemma 7]).

Symplectic difference systems cover a large variety of difference equations and systems. For example, the linear Hamiltonian difference system

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \quad \Delta u_k = C_k x_{k+1} - A_k^T u_k \tag{H}$$

with symmetric matrices B, C and nonsingular $I-A$ can be written as SdS (S) with

$$\mathcal{S} = \begin{pmatrix} (I-A)^{-1} & (I-A)^{-1}B \\ C(I-A)^{-1} & C(I-A)^{-1}B + I - A^T \end{pmatrix}.$$

Since the higher order Sturm–Liouville equation

$$\sum_{\nu=0}^{n} \Delta^\nu (r_k^{(\nu)} \Delta^\nu y_{k+n-\nu}) = 0 \tag{SL}$$

with $r_k^{(n)} \neq 0$ can be written in the form of (H) with

$$A = (A_{ij}), \qquad A_{ij} = \begin{cases} 1 & \text{if } \; j = i+1, \; i = 1, ..., n-1 \\ 0 & \text{elsewhere,} \end{cases}$$

$$B_k = \operatorname{diag}\left\{0, ..., 0, \frac{1}{r_k^{(n)}}\right\}, \qquad C_k = \operatorname{diag}\{r_k^{(0)}, ..., r_k^{(n-1)}\},$$

using the substitution

$$x_k = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \qquad u_k = \begin{pmatrix} \sum_{\nu=1}^{n} (-1)^{\nu-1} \Delta^{\nu-1} (r_k^{(\nu)} \Delta^\nu y_{k+n-\nu}) \\ \vdots \\ -\Delta(r_k^{(n)} \Delta^n y_k) + r_k^{(n-1)} \Delta^{n-1} y_{k+1} \\ r_k^{(n)} \Delta^n y_k \end{pmatrix},$$

this equation is a special case of (S) as well.

Now, consider the trigonometric matrix SdS

$$S_{k+1} = \mathcal{P}_k S_k + \mathcal{Q}_k C_k, \qquad C_{k+1} = -\mathcal{Q}_k S_k + \mathcal{P}_k C_k \tag{T}$$

with $\mathcal{P}$, $\mathcal{Q}$ satisfying (see (1.6)) $\mathcal{P}^T\mathcal{P} + \mathcal{Q}^T\mathcal{Q} = I$ and $\mathcal{P}^T\mathcal{Q} = \mathcal{Q}^T\mathcal{P}$. The fact that trigonometric systems are self-reciprocal, i.e., transformation (2.4) with $\mathcal{R} = \mathcal{J}$ transforms this system into itself, implies that if (S, C) is a solution of (T), then $(C, -S)$ solves this system as well. Consequently, if the matrix $\left(\begin{smallmatrix} S & C \\ C & -S \end{smallmatrix}\right)$ is symplectic for some k, then it is symplectic everywhere so that both C^TS and SC^T are symmetric and $C^TC + S^TS = CC^T + SS^T = I$. Basic properties of solutions of trigonometric systems are established in the recent paper of Anderson [2].

Finally, for the sake of comparison, let us recall the concepts of trigonometric systems and trigonometric transformations for the differential Hamiltonian system

$$x' = A(t)\,x + B(t)\,u, \qquad u' = C(t)\,x - A^T(t)\,u \tag{2.5}$$

which represents the continuous counterpart of (S). In this system we suppose that the matrices B and C are symmetric. The trigonometric transformation of (2.5) which is the main result of [9] reads as follows.

THEOREM 2.1 [9, Theorem 1]. *There exist continuously differentiable matrices* $H, K: I \subset \mathbb{R} \to \mathbb{R}^{n\times n}$ *such that* H *is nonsingular,* $H^T K \equiv K^T H$ *and the transformation*

$$\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} H(t) & 0 \\ K(t) & (H^T(t))^{-1} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}$$

transforms (2.5) *into the trigonometric system* (1.7). *The matrix* Q *is given by the formula* $Q(t) = H^{-1}(t)\,B(t)(H^T(t))^{-1}$.

To show some applications of this transformation in oscillation theory of differential Hamiltonian systems, recall that the system (H) is said to be *oscillatory* if there exists a conjoined basis (X, U) of this system (defined in the same way as for (S)) and a real sequence $t_n \to \infty$ such that $\det X(t_n) = 0$. Eventual controllability of (2.5) is also defined in the same way as for (S).

It is known (see [15]) that an eventually controllable trigonometric differential system (1.7) is oscillatory iff $\int^\infty \operatorname{Tr} Q(s)\,ds = \infty$, where Tr denotes the trace, i.e., the sum of the diagonal elements of a matrix. Combining this result with Theorem 2.1, we have the following oscillation criterion for (2.5).

THEOREM 2.2 [9, Theorem 3]. *Suppose that* $B \geqslant 0$ *for large* t, $\int^\infty \operatorname{Tr} B(s)\,ds = \infty$, *system* (2.5) *is eventually controllable and there exists a constant* M *such that* $\|X(t)\| \leqslant M$ *eventually for any conjoined basis* (X, U) *of* (2.5). *Then the system* (2.5) *is oscillatory.*

3. THE TRIGONOMETRIC TRANSFORMATION

In this section we state and prove the main result of our paper.

THEOREM 3.1. *There exist $n \times n$-matrices H and K such that H is nonsingular, $H^T K = K^T H$, and the transformation*

$$\begin{pmatrix} s \\ c \end{pmatrix} = \begin{pmatrix} H^{-1} & 0 \\ -K^T & H^T \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{3.1}$$

transforms the symplectic system (S) *into the trigonometric system* (T) *without changing the oscillatory behavior. Moreover, the matrices* $\mathscr{P}$ *and* $\mathscr{Q}$ *from* (T) *are given by* $\mathscr{P}_k = H_{k+1}^{-1} \mathscr{A}_k H_k + H_{k+1}^{-1} \mathscr{B}_k K_k$ *and* $\mathscr{Q}_k = H_{k+1}^{-1} \mathscr{B}_k H_k^{T-1}$.

Proof. If (X, U), $(\tilde{X}, \tilde{U})$ are normalized conjoined bases of (S), then $XX^T + \tilde{X}\tilde{X}^T > 0$ and there exists a nonsingular $n \times n$-matrix H with

$$HH^T = XX^T + \tilde{X}\tilde{X}^T. \tag{3.2}$$

We put

$$K = (UX^T + \tilde{U}\tilde{X}^T)\, H^{T-1} \tag{3.3}$$

(where $^{T-1}$ denotes the inverse of the transpose of the matrix indicated) so that

$$H^T K - K^T H = H^{-1}[HH^T K H^T - HK^T HH^T]\, H^{T-1} = 0$$

by (2.1) and (2.2). Finally, let $Z = H^{-1}(X + \mathrm{i}\tilde{X})$. First of all we have (use (2.2))

$$(X + \mathrm{i}\tilde{X})(X + \mathrm{i}\tilde{X})^* = HH^T \tag{3.4}$$

(where * denotes the conjugate transpose of the matrix indicated), and hence

$$ZZ^* = I = Z^* Z \tag{3.5}$$

shows that Z is unitary. Next, from (3.1) we have

$$Z^{-1} = (X + \mathrm{i}\tilde{X})^{-1}\, H = (X + \mathrm{i}\tilde{X})^*\, H^{T-1} = (X^T - \mathrm{i}\tilde{X}^T)\, H^{T-1},$$

and therefore (use again (2.2))

$$\begin{aligned} Z_{k+1} Z_k^{-1} &= H_{k+1}^{-1}(X_{k+1} + \mathrm{i}\tilde{X}_{k+1})(X_k^T - \mathrm{i}\tilde{X}_k^T)\, H_k^{T-1} \\ &= H_{k+1}^{-1}\{\mathscr{A}_k(X_k X_k^T + \tilde{X}_k \tilde{X}_k^T + \mathrm{i}[\tilde{X}_k X_k^T - X_k \tilde{X}_k^T]) \\ &\quad + \mathscr{B}_k(U_k X_k^T + \tilde{U}_k \tilde{X}_k^T + \mathrm{i}[\tilde{U}_k X_k^T - U_k \tilde{X}_k^T])\}\, H_k^{T-1} \\ &= H_{k+1}^{-1} \mathscr{A}_k H_k + H_{k+1}^{-1} \mathscr{B}_k K_k + \mathrm{i} H_{k+1}^{-1} \mathscr{B}_k H_k^{T-1} = \mathscr{P}_k + \mathrm{i}\mathscr{Q}_k. \end{aligned}$$

Since both Z_k and Z_{k+1} are unitary according to (3.5), also $\mathcal{P}+\mathrm{i}\mathcal{Q}$ is unitary, i.e.,

$$I=(\mathcal{P}+\mathrm{i}\mathcal{Q})(\mathcal{P}^T-\mathrm{i}\mathcal{Q}^T)=\mathcal{P}\mathcal{P}^T+\mathcal{Q}\mathcal{Q}^T+\mathrm{i}(\mathcal{Q}\mathcal{P}^T-\mathcal{P}\mathcal{Q}^T)$$

so that

$$\mathcal{P}\mathcal{P}^T+\mathcal{Q}\mathcal{Q}^T=I \qquad \text{and} \qquad \mathcal{P}\mathcal{Q}^T=\mathcal{Q}\mathcal{P}^T. \tag{3.6}$$

From (3.6) it follows that the matrix $\begin{pmatrix} \mathcal{P} & \mathcal{Q} \\ -\mathcal{Q} & \mathcal{P} \end{pmatrix}$ is symplectic. According to our transformation we put

$$S=H^{-1}X=\operatorname{Re} Z \qquad \text{and} \qquad C=-K^TX+H^TU=-H^{-1}\tilde{X}=-\operatorname{Im} Z$$

which holds because of (use (2.1))

$$\tilde{X}-HK^TX+HH^TU=\tilde{X}-(XU^T+\tilde{X}\tilde{U}^T)\,X+(XX^T+\tilde{X}\tilde{X}^T)\,U=0.$$

Now, since

$$\begin{aligned} S_{k+1}-\mathrm{i}C_{k+1}&=Z_{k+1}=Z_{k+1}Z_k^{-1}Z_k=(\mathcal{P}_k+\mathrm{i}\mathcal{Q}_k)\,Z_k \\ &=(\mathcal{P}_k+\mathrm{i}\mathcal{Q}_k)(S_k-\mathrm{i}C_k)=\mathcal{P}_kS_k+\mathcal{Q}_kC_k+\mathrm{i}(\mathcal{Q}_kS_k-\mathcal{P}_kC_k), \end{aligned}$$

we have

$$S_{k+1}=\mathcal{P}_kS_k+\mathcal{Q}_kC_k \qquad \text{and} \qquad C_{k+1}=-\mathcal{Q}_kS_k+\mathcal{P}_kC_k,$$

so (S, C) is really a solution of (T). ∎

Remark 3.2. Observe that any trigonometric system (T) can be transformed into another trigonometric system with symmetric and positive semidefinite matrices $\tilde{\mathcal{Q}}_k$ at the position of the matrices $\mathcal{Q}_k$, using the transformation

$$\begin{pmatrix} \tilde{s} \\ \tilde{c} \end{pmatrix}=\begin{pmatrix} H_k^{-1} & 0 \\ 0 & H_k^T \end{pmatrix}\begin{pmatrix} s \\ c \end{pmatrix},$$

where the matrices H_k are recursively defined by $H_0=I$ and $H_{k+1}=G_k^{-1}H_k$ with orthogonal matrices G_k, i.e., $G_k^TG_k=I$, such that $G_k\mathcal{Q}_k$ are symmetric and positive semidefinite. Such matrices G_k exist according to the well-known principle of polar decomposition, see, e.g., [12, Theorem 3.1.9(c)]. This setting implies that all matrices H_k are orthogonal and hence that the transformation matrices $\begin{pmatrix} H_k^{-1} & 0 \\ 0 & H_k^T \end{pmatrix}$ are symplectic. The transformed system then reads

$$\tilde{S}_{k+1} = \tilde{\mathscr{P}}_k \tilde{S}_k + \tilde{\mathscr{Q}}_k \tilde{C}_k, \qquad \tilde{C}_{k+1} = -\tilde{\mathscr{Q}}_k \tilde{S}_k + \tilde{\mathscr{P}}_k \tilde{C}_k,$$

where $\tilde{\mathscr{P}}_k = H_{k+1}^{-1} \mathscr{P}_k H_k$ and $\tilde{\mathscr{Q}}_k = H_{k+1}^{-1} \mathscr{Q}_k H_k = H_k^{-1} G_k \mathscr{Q}_k H_k^{T-1}$ so that indeed all matrices $\tilde{\mathscr{Q}}_k$ are symmetric and positve semidefinite.

4. OSCILLATION PROPERTIES OF TRIGONOMETRIC SYSTEMS

In this section we establish a necessary and sufficient condition for oscillation of the trigonometric system (T) with positive definite matrices $\mathscr{Q}$. We need the following two auxiliary statements from matrix theory.

LEMMA 4.1. *Let $\mathscr{M}_j$, $j = 1, ..., m$, be unitary $n \times n$-matrices, denote by* $\exp\{i\varphi_j^{(l)}\}$, $l = 1, ..., n$, *their eigenvalues and suppose that*

$$\frac{\pi}{m} > \varphi_j^{(1)} \geqslant \cdots \geqslant \varphi_j^{(n)} \geqslant 0, \qquad j = 1, ..., m.$$

Then

$$\left\| \prod_{j=1}^{m} \mathscr{M}_j - I \right\|^2 \leqslant 2 \left[1 - \cos\left(\sum_{j=1}^{m} \varphi_j^{(1)} \right) \right]. \tag{4.1}$$

Proof. We proceed by induction. Suppose that $m = 2$ and denote by $\exp\{i\theta^{(l)}\}$, $l = 1, ..., n$, eigenvalues of the product $\mathscr{M}_1 \mathscr{M}_2$. Then

$$\begin{aligned}
\|\mathscr{M}_1 \mathscr{M}_2 - I\|^2 &= \max_{\|u\|=1} \langle (\mathscr{M}_1 \mathscr{M}_2 - I)\, u, (\mathscr{M}_1 \mathscr{M}_2 - I)\, u \rangle \\
&= \max_{\|u\|=1} \{ \|\mathscr{M}_1 \mathscr{M}_2 u\|^2 + \|u\|^2 - \langle \mathscr{M}_1 \mathscr{M}_2 u, u \rangle - \langle u, \mathscr{M}_1 \mathscr{M}_2 u \rangle \} \\
&= 2 - \min_{\|u\|=1} \langle (\mathscr{M}_1 \mathscr{M}_2 + (\mathscr{M}_1 \mathscr{M}_2)^*)\, u, u \rangle \\
&= 2 - 2 \min_{1 \leqslant j \leqslant n} \cos \theta^{(j)} = 2 - 2 \cos(\max_{1 \leqslant j \leqslant n} \theta^{(j)}).
\end{aligned}$$

From a geometrical point of view, any unitary $n \times n$-matrix represents an isomorphism in $\mathbb{R}^{2n}$. The above computation shows that the square of the norm $\|\mathcal{M}_1\mathcal{M}_2 - I\|$ is determined by the maximal angle between u and $\mathcal{M}_1\mathcal{M}_2 u$. The matrix $\mathcal{M}_1\mathcal{M}_2$ corresponds to the composition of maps generated by $\mathcal{M}_1$ and $\mathcal{M}_2$, and obviously the maximal angle between u and $\mathcal{M}_1\mathcal{M}_2 u$ of this composition cannot exceed the sum of the "maximal angles" of maps $\mathcal{M}_1$ and $\mathcal{M}_2$ which are $\varphi_1^{(1)}$ and $\varphi_2^{(1)}$, respectively. Consequently, $\max_{1 \leqslant j \leqslant n} \theta^{(j)} \leqslant \varphi_1^{(1)} + \varphi_2^{(1)}$, and hence (4.1) holds for $m=2$. Now, suppose that the statement holds for $m-1$ and denote by $\exp\{i\psi^{(l)}\}$, $l=1, ..., n$, the eigenvalues of the product $\prod_{j=1}^{m-1} \mathcal{M}_j$ such that $\psi^{(1)} = \max_{1 \leqslant l \leqslant n} \psi^{(l)}$. Then by our induction assumption $\psi^{(1)} \leqslant \sum_{j=1}^{m-1} \varphi_j^{(1)}$ and using the same "geometrical" argument as in the first part of the proof we have

$$\left\| \left(\prod_{j=1}^{m-1} \mathcal{M}_j \right) \mathcal{M}_m - I \right\|^2 \leqslant 2[1 - \cos(\psi^{(1)} + \varphi_m^{(1)})]$$

$$\leqslant 2\left[1 - \cos\left(\sum_{j=1}^{m} \varphi_j^{(1)}\right)\right].$$

Our claim now follows by the principle of mathematical induction. ∎

LEMMA 4.2. *Let $\mathcal{P} + i\mathcal{Q}$ be a unitary $n \times n$-matrix with symmetric $\mathcal{Q}$ such that both $\mathcal{Q}$ and $\mathcal{Q}^{-1}\mathcal{P}$ are positive definite. If* $\exp\{i\varphi^{(j)}\}$, $j=1, ..., n$, *denote the eigenvalues of $\mathcal{P} + i\mathcal{Q}$ ordered in the same way as in the previous lemma, then*

$$\varphi^{(1)} = \operatorname{arccotg} \lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P}), \qquad \varphi^{(n)} = \operatorname{arccotg} \lambda^{(n)}(\mathcal{Q}^{-1}\mathcal{P}),$$

where $\lambda^{(1)}(\cdot)$ and $\lambda^{(n)}(\cdot)$ denote the minimal and maximal eigenvalue of the matrix indicated, respectively.

Proof. We will prove the first identity only, the proof of the second one is similar. We have (see (3.6)) $\mathcal{Q}^{-1}\mathcal{P}\mathcal{P}^T\mathcal{Q}^{-1} = \mathcal{Q}^{-1}(I - \mathcal{Q}^2)\,\mathcal{Q}^{-1} = \mathcal{Q}^{-2} - I$ so that

$$\operatorname{arccotg} \lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P}) = \arccos \frac{\lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P})}{\sqrt{1 + [\lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P})]^2}}$$

$$= \arccos \frac{\sqrt{\lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P}\mathcal{P}^T\mathcal{Q}^{-1})}}{\sqrt{1 + \lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P}\mathcal{P}^T\mathcal{Q}^{-1})}}$$

$$= \arccos \sqrt{1 - [\lambda^{(n)}(\mathcal{Q})]^2}.$$

On the other hand, since $\mathscr{P}$ is normal (observe that $\mathscr{P}^T\mathscr{P}+\mathscr{Q}^2=I=\mathscr{P}\mathscr{P}^T+\mathscr{Q}^2$),

$$\begin{aligned}\cos\varphi^{(1)}&=\tfrac{1}{2}\lambda^{(1)}(\mathscr{P}+\mathrm{i}\mathscr{Q}+(\mathscr{P}+\mathrm{i}\mathscr{Q})^*)=\tfrac{1}{2}\lambda^{(1)}(\mathscr{P}+\mathscr{P}^T)=\lambda^{(1)}(\mathscr{P})\\&=\sqrt{\lambda^{(1)}(\mathscr{P}\mathscr{P}^T)}=\sqrt{\lambda^{(1)}(I-\mathscr{Q}^2)}=\sqrt{1-[\lambda^{(n)}(\mathscr{Q})]^2}\end{aligned}$$

so that our claim follows. ∎

Now we can prove the main result of this section.

THEOREM 4.3. *Suppose that the matrices $\mathscr{Q}_j$ in the trigonometric system* (T) *are symmetric and positive definite. Then this system is nonoscillatory if and only if*

$$\sum_{j=0}^{\infty}\operatorname{arccotg}\lambda^{(1)}(\mathscr{Q}_j^{-1}\mathscr{P}_j)<\infty. \tag{4.2}$$

Proof. First we prove that (4.2) is necessary for nonoscillation of (T). If (T) is nonoscillatory, then there exists a solution (S, C) of this system such that $S^TS+C^TC=I$, $S^TC=C^TS$, S_k are nonsingular and $S_{k+1}^{-1}\mathscr{Q}_kS_k^{T-1}$ is positive definite for large k. Since $\Delta(S_k^{-1}C_k)=-S_{k+1}^{-1}\mathscr{Q}_kS_k^{-1}$, the matrix sequence $M_k:=S_k^{-1}C_k$ is nonincreasing (in the sense that $M_{k+1}-M_k$ is nonpositive definite). Hence, if $\mu_k^{(1)}\leqslant\cdots\leqslant\mu_k^{(n)}$ denote the eigenvalues of M_k ordered by their size, each real sequence $\mu_k^{(j)}$, $j=1, ..., n$, is nonincreasing, i.e., $\mu_k^{(j)}\to\mu^{(j)}\in[-\infty, \infty)$ as $k\to\infty$. The fact that the matices $\mathscr{Q}_k$ and $S_{k+1}^{-1}\mathscr{Q}_kS_k^{T-1}$ are positive definite implies that $\det S_{k+1}\det S_k>0$, which means that $\det S_k$ is of one sign eventually and without loss of generality we may suppose that $\det S_k>0$. Now we have

$$\begin{aligned}\det(C_k+\mathrm{i}S_k)&=\det(C_0+\mathrm{i}S_0)\left\{\prod_{j=0}^{k-1}\det\mathscr{Q}_j\det(\mathscr{Q}_j^{-1}\mathscr{P}_j+\mathrm{i}I)\right\}\\&=\det(C_0+\mathrm{i}S_0)\left\{\prod_{j=0}^{k-1}\det\mathscr{Q}_j\left[\prod_{l=1}^{n}\lambda^{(l)}(\mathscr{Q}_j^{-1}\mathscr{P}_j+\mathrm{i}I)\right]\right\}\\&=\det(C_0+\mathrm{i}S_0)\left\{\prod_{j=0}^{k-1}\det\mathscr{Q}_j\left[\prod_{l=1}^{n}(\lambda^{(l)}(\mathscr{Q}_j^{-1}\mathscr{P}_j)+\mathrm{i})\right]\right\}.\end{aligned}$$

In the next computation, for the convenience of notation, we denote

$$K:=\det(C_0+\mathrm{i}S_0),\qquad \lambda_j^{(l)}:=\lambda^{(l)}(\mathscr{Q}_j^{-1}\mathscr{P}_j),\qquad \Lambda_j^{(l)}:=\frac{1}{\sqrt{1+(\lambda_j^{(l)})^2}},$$

$$\varphi_j^{(l)}:=\operatorname{arccotg}\lambda_j^{(l)},\quad q_j:=\det\mathscr{Q}_j,\quad \Lambda_j:=\prod_{l=1}^{n}\Lambda_j^{(l)},\quad L_k:=\prod_{j=0}^{k-1}\det\mathscr{Q}_j\Lambda_j.$$

Then, with this notation, using the Moivre theorem, we have

$$\begin{aligned}\det(C_k+\mathrm{i}S_k) &= K\left\{\prod_{j=0}^{k-1} q_j\left[\prod_{l=1}^{n}(\lambda_j^{(l)}+\mathrm{i})\right]\right\}\\ &= K\left\{\prod_{j=0}^{k-1} q_j\left[\prod_{l=1}^{n}\Lambda_j^{(l)}(\cos\varphi_j^{(l)}+\mathrm{i}\sin\varphi_j^{(l)})\right]\right\}\\ &= K\prod_{j=0}^{k-1} q_j\Lambda_j\left[\cos\left(\sum_{l=1}^{n}\varphi_j^{(l)}\right)+\mathrm{i}\sin\left(\sum_{l=1}^{n}\varphi_j^{(l)}\right)\right]\\ &= KL_k\left[\cos\left(\sum_{j=0}^{k-1}\sum_{l=1}^{n}\varphi_j^{(l)}\right)+\mathrm{i}\sin\left(\sum_{j=0}^{k-1}\sum_{l=1}^{n}\varphi_j^{(l)}\right)\right].\end{aligned}$$

On the other hand (recall that $\mu_k^{(l)}$, $l=1, ..., n$, are the eigenvalues of M_k), if we denote $\psi_j^{(l)}=\operatorname{arccotg}\mu_j^{(l)}$ and $m_k=\prod_{l=1}^{n}(1/\sqrt{1+(\mu_k^{(l)})^2})$, then

$$\begin{aligned}\det(C_k+\mathrm{i}S_k) &= \det S_k\det(M_k+\mathrm{i}I)=\det S_k\prod_{l=1}^{n}(\mu_k^{(l)}+\mathrm{i})\\ &= m_k\det S_k\left[\cos\left(\sum_{l=1}^{n}\psi_k^{(l)}\right)+\mathrm{i}\sin\left(\sum_{l=1}^{n}\psi_k^{(l)}\right)\right].\end{aligned}$$

Consequently, comparing the arguments of these two expressions for $\det(C_k+\mathrm{i}S_k)$, we have

$$\sum_{l=1}^{n}\psi_k^{(l)}=\sum_{j=0}^{k-1}\sum_{l=1}^{n}\varphi_j^{(l)}+\arg K \qquad (\mathrm{mod}\ 2\pi).$$

Since $\lim_{k\to\infty}\mu_k^{(l)}$ exists (finite or $-\infty$) for every $l=1, ..., n$, we see that $\sum_{j=0}^{\infty}\varphi_j^{(l)}$ is finite for $l=1, ..., n$ so that (4.2) follows with $l=1$.

To prove sufficiency of (4.2), we show first that (4.2) implies convergence of the infinite matrix product $\prod_{j=0}^{\infty}(\mathscr{P}_j+\mathrm{i}\mathscr{Q}_j)$. To this end, it is sufficient to prove that the matrix sequence $\prod_{j=0}^{k}(\mathscr{P}_j+\mathrm{i}\mathscr{Q}_j)$ is a Cauchy sequence. Let $m, p\in\mathbb{N}$, then

$$\left\|\prod_{j=0}^{m+p}(\mathscr{P}_j+\mathrm{i}\mathscr{Q}_j)-\prod_{j=0}^{m}(\mathscr{P}_j+\mathrm{i}\mathscr{Q}_j)\right\|\leqslant\left\|\prod_{j=m+1}^{m+p}(\mathscr{P}_j+\mathrm{i}\mathscr{Q}_j)-I\right\|$$

since the spectral norm of any unitary matrix equals 1. Consequently, it suffices to show that the last term is less then any ε if $m\in\mathbb{N}$ is sufficiently large and $p\in\mathbb{N}$ is arbitrary. Let $\exp\{\mathrm{i}\varphi_j^{(l)}\}$, $l=1, ..., n$, be the eigenvalues of

$(\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j)$ ordered in the same way as in Lemma 4.1. Then by Lemma 4.1 and Lemma 4.2 we have

$$\begin{aligned}
\tfrac{1}{2}\left\| \prod_{j=m+1}^{m+p} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j) - I \right\|^2
&= 1 - \lambda^{(1)} \left[\prod_{j=m+1}^{m+p} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j) + \prod_{j=m+1}^{m+p} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j)^* \right] \\
&\leqslant 1 - \cos\left(\sum_{j=m+1}^{m+p} \varphi_j^{(1)} \right) \leqslant 1 - \cos\left(\sum_{j=m+1}^{\infty} \operatorname{arccotg} \lambda^{(1)} (\mathcal{Q}_j^{-1} \mathcal{P}_j) \right).
\end{aligned}$$

By (4.2), for sufficiently large m, $\| \prod_{j=m+1}^{\infty} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j) - I \|$ can be made arbitrarily small and hence $\prod_{j=0}^{k} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j)$ is really a Cauchy sequence. To finish the proof, we need to show that the convergence of the matrix product $\prod_{j=0}^{\infty} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j)$ implies nonoscillation of (T). Let $\prod_{j=0}^{\infty} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j) =: \mathcal{P} + \mathrm{i}\mathcal{Q}$ and (S, C) be the solution of (T) given by the initial condition $C_0 = \mathcal{Q}$, $S_0 = \mathcal{P}^T$. Then

$$(C_k + \mathrm{i}S_k) = \left\{ \prod_{j=0}^{k-1} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j) \right\} (C_0 + \mathrm{i}S_0) \to (\mathcal{P} + \mathrm{i}\mathcal{Q})(\mathcal{Q} + \mathrm{i}\mathcal{P}^T) = \mathrm{i}I$$

as $k \to \infty$, i.e., $C_k \to 0$ and $S_k \to I$, and this means that S_k is nonsingular if k is sufficiently large. Similarly as above we have

$$\arg\det \prod_{j=0}^{k} (\mathcal{P}_j + \mathrm{i}\mathcal{Q}_j) = \sum_{j=0}^{k} \sum_{l=1}^{n} \operatorname{arccotg} \lambda^{(l)} (\mathcal{Q}_j^{-1} \mathcal{P}_j) \to \arg\det(\mathcal{P} + \mathrm{i}\mathcal{Q})$$

as $k \to \infty$, hence

$$\sum^{\infty} \operatorname{arccotg} \lambda^{(l)} (\mathcal{Q}_j^{-1} \mathcal{P}_j) < \infty, \qquad l = 1, ..., n,$$

i.e., $\lambda^{(1)}(\mathcal{Q}_k^{-1}\mathcal{P}_k) \to \infty$ as $k \to \infty$. In particular, for any symmetric matrix M the matrix $\mathcal{Q}_k^{-1}\mathcal{P}_k - M$ is positive definite for large k. Now,

$$\begin{aligned}
S_{k+1}^{-1} \mathcal{Q}_k S_k^{T-1} &= (\mathcal{P}_k S_k + \mathcal{Q}_k C_k)^{-1} \mathcal{Q}_k S_k^{T-1} \\
&= S_k^{-1} (\mathcal{P}_k + \mathcal{Q}_k C_k S_k^{-1})^{-1} \mathcal{Q}_k S_k^{T-1} \\
&= S_k^{-1} (\mathcal{Q}_k^{-1} \mathcal{P}_k - C_k S_k^{-1})\, S_k^{T-1}.
\end{aligned}$$

Since $C_k S_k^{-1} \to 0$ and $\mathcal{Q}_k^{-1}\mathcal{P}_k \to \infty$, the matrix $\mathcal{Q}_k \mathcal{P}_k - C_k S_k^{-1}$ is positive definite for large k and hence $S_{k+1}^{-1} \mathcal{Q}_k S_k^{T-1}$ has the same property. ∎

Remark 4.4. The previous theorem requires the matrices $\mathcal{Q}$ in (T) to be positive definite. By the trigonometric transformation of (S) given in

Theorem 3.1, the matrices $\mathcal{Q}$ are given by $\mathcal{Q}_k = H_{k+1}^{-1} \mathcal{B}_k H_k^{T-1}$, hence a necessary condition for positive definiteness of $\mathcal{Q}$ is nonsingularity of $\mathcal{B}$. However, symplectic systems (S) with $\mathcal{B}$ nonsingular do not cover a relatively large class of equations and systems, e.g., the higher order Sturm–Liouville difference equation (SL) which can be written in the form (S) with

$$\mathcal{B}_k = \frac{1}{r_k^{(n)}} \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

The extension of Theorem 3.1 to trigonometric systems with $\mathcal{Q}$ only nonnegative definite is a subject of present investigation.

5. APPLICATIONS

In this section we apply the trigonometric transformation to oscillation theory of symplectic difference and Hamiltonian systems. We start with an extension of the result of Anderson [2, Theorem 17] which is proved in [2] under the restriction that $\mathcal{Q}$ is nonsingular.

THEOREM 5.1. *Suppose that* (T) *is eventually controllable and*

$$\sum^{\infty} \operatorname{Tr} \mathcal{Q}_k = \infty. \tag{5.1}$$

Then (T) *is oscillatory.*

Proof. Suppose that (5.1) holds and that (T) is nonoscillatory. Then there exists a dominant solution of (T), i.e., a conjoined basis $(\tilde{S}, \tilde{C})$ such that $\tilde{S}_k$ are nonsingular for large k and $\lim_{m\to\infty}[\sum^m \tilde{S}_{k+1}^{-1} \mathcal{Q}_k \tilde{S}_k^{T-1}]^{-1}$ is a nonsingular matrix (c.f. [6]) which means that $\sum^\infty \operatorname{Tr} \tilde{S}_{k+1}^{-1} Q_k \tilde{S}_k^{T-1} < \infty$. In the remaining part of the proof we proceed in the same way as in [2]. Since for any conjoined basis (S, C) of (T), $S^T S + C^T C$ is a constant matrix, which implies that $\|S_k\|$ are bounded, we have with some constant $c > 0$

$$\infty > \sum^{\infty} \operatorname{Tr} \tilde{S}_{k+1}^{-1} \mathcal{Q}_k \tilde{S}_k^{T-1} \geqslant c \cdot \sum^{\infty} \operatorname{Tr} \mathcal{Q}_k,$$

a contradiction. ∎

COROLLARY 5.2. *Suppose that* (S) *is eventually controllable,* $\sum_k^\infty \operatorname{Tr} \mathcal{B}_k = \infty$, *and the norm of the first component* $\|X\|$ *is eventually bounded for any conjoined basis* (X, U) *of* (S). *Then* (S) *is oscillatory.*

Proof. We use the trigonometric transformation given in Theorem 3.1. Let (X, U) be any conjoined basis of (S). Then there exists another conjoined basis $(\tilde{X}, \tilde{U})$ such that $X^T\tilde{U} - U^T\tilde{X} \equiv I$, and the transformation (3.1) with H and K given by (3.2) and (3.3) transforms (S) into a trigonometric system (T). Since $\|X_k\|$, $\|\tilde{X}_k\|$ are bounded for large k, the norm $\|H_k\|$ is bounded as well and hence there exists a constant $c>0$ such that $\lambda^{(1)}(H_k^{-1})>c$ for large k. Now, in the same way as in the proof of [2, Theorem 17], we have

$$\sum^{\infty} \operatorname{Tr} \mathcal{Q}_k = \sum^{\infty} \operatorname{Tr}(H_{k+1}^{-1} \mathcal{B}_k H_k^{T-1}) \geqslant c^2 \sum^{\infty} \operatorname{Tr} \mathcal{B}_k = \infty,$$

a contradiction. ∎

Remark 5.3. Condition (5.1) is only sufficient but not necessary for oscillation of (S). To show this, consider the second order "Fibonacci" difference equation $x_{k+2} - x_{k+1} - x_k = 0$. This equation can be written in self-adjoint form as

$$\Delta((-1)^k \Delta x_k) + (-1)^k x_{k+1} = 0. \tag{5.2}$$

Let $\mu_1 = (1+\sqrt{5})/2$ and $\mu_2 = (1-\sqrt{5})/2$. Then $x_k = (1/\sqrt{5})\, \mu_1^k$, $\tilde{x}_k = \mu_2^k$ are linearly independent solutions of (5.2) such that $(-1)^k [x_k \Delta\tilde{x}_k - \tilde{x}_k \Delta x_k] = 1$. By Theorem 3.1, the transformation $\binom{x}{\Delta x} = \begin{pmatrix} H & 0 \\ K & H^{-1}\end{pmatrix}\binom{s}{c}$ with $H = \sqrt{x^2+\tilde{x}^2}$ and $K = (-1)^k (x \Delta x + \tilde{x} \Delta \tilde{x})/(\sqrt{x^2+\tilde{x}^2})$ transforms (5.2) into the trigonometric 2×2 system (T) with

$$\mathcal{Q}_k = (-1)^k / \sqrt{(x_k^2 + \tilde{x}_k^2)(x_{k+1}^2 + \tilde{x}_{k+1}^2)}.$$

It is not difficult to verify that the series $\sum^{\infty} \mathcal{Q}_k$ is convergent (use the Leibniz criterion) but (5.2) is oscillatory since, e.g., the solution x has arbitrarily large generalized zeros, i.e., $(-1)^k x_{k+1} x_k < 0$ for arbitrarily large k.

Now we turn our attention to an application of Theorem 4.3 in oscillation theory of the second order difference system

$$\Delta^2 x_k + P_k x_{k+1} = 0, \tag{5.3}$$

where $x \in \mathbb{R}^n$, P is a real, symmetric $n\times n$-matrix. Note that (5.3) can be written in the form (S) with $\mathcal{A} = I$, $\mathcal{B} = I$, $\mathcal{C} = -P$ and $\mathcal{D} = -P + I$ and that (5.3) covers a large class of symplectic systems since any system (S) with $\mathcal{A}$, $\mathcal{B}$ nonsingular can be transformed into (5.3) without changing oscillation behaviour of transformed systems, cf. [10, Theorem 3.2; 6, Lemma 1]. Our final result is a discrete analogue of [8, Theorem 3] where it is proved that the second order *differential* system $x'' + P(t)\, x = 0$ is oscillatory if and only if there exist matrix solutions $X, \tilde{X}$ of this system such that $X^T X'$ and $\tilde{X}^T \tilde{X}'$ are symmetric (in "Hamiltonian systems terminology" these are just conjoined solutions) and

$$\int^{\infty} \operatorname{Tr}([X(t)\, X^T(t) + \tilde{X}(t)\, \tilde{X}^T(t)]^{-1})\, dt = \infty.$$

THEOREM 5.4. *The difference system* (5.3) *is oscillatory if and only if there exist matrix solutions* $X, \tilde{X}$ *such that* $X^T \Delta \tilde{X}$ *and* $\tilde{X}^T \Delta X$ *are symmetric and*

$$\sum^{\infty} \operatorname{arccotg} \lambda^{(1)}(X_{k+1} X_k^T + \tilde{X}_{k+1} \tilde{X}_k^T) = \infty.$$

Proof. We use again the trigonometric transformation given in Theorem 3.1. Let $U = \Delta X$, $\tilde{U} = \Delta \tilde{X}$ and H, K be the same as in Theorem 3.1 and such that the matrix $\mathcal{Q}$ is symmetric and positive definite (see Remark 3.2). Then (3.1) transforms the symplectic system corresponding to (5.3) into a trigonometric system (T) with $\mathcal{P} = H_{k+1}^{-1}(X_{k+1} X_k^T + \tilde{X}_{k+1} \tilde{X}_k^T)$ H_k^{T-1} and $\mathcal{Q}_k = H_{k+1}^{-1} H_k^T$, hence

$$\begin{aligned} \lambda^{(1)}(\mathcal{Q}^{-1}\mathcal{P}) &= \lambda^{(1)}(H^T(X_{k+1} X^T + \tilde{X}_{k+1} \tilde{X}^T)\, H^{T-1}) \\ &= \lambda^{(1)}(X_{k+1} X^T + \tilde{X}_{k+1} \tilde{X}^T). \end{aligned}$$

Now the statement immediately follows from Theorem 4.3. ∎

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