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Variable Change for Sturm–Liouville Differential Expressions on Time Scales

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Dedicated to Professor Allan Peterson on the occasion of his 60th birthday.

We study second order scalar delta derivative expressions of Sturm–Liouville type on our newly defined Sturmian time scales. Sturmian time scales include the discrete and continuous cases studied by Sturm. A form of second order differential expression on a Sturmian time scale considered here satisfies a Green's identity and hence is "formally self-adjoint". A unified variable change method is developed which allows simultaneous change of independent and dependent variable for expressions which include continuous and discrete theories as special cases. This unifies a continuous result of Coppel with a discrete result of Voepel. For the fourth order case, we explore unification of a continuous result of Ahlbrandt, Hinton and Lewis [4] with a discrete result of Voepel [32].

Keywords: Sturm–Liouville equations; Time scales; Delta derivatives; Variable change

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1. DIFFERENTIAL EXPRESSIONS ON TIME SCALES

We define a time scale as in [3, Section 1], i.e., let $\mathbb{X}$ be a nonempty subset of the real numbers with the property that every Cauchy sequence in $\mathbb{X}$ converges to a point of $\mathbb{X}$ with the possible exception of Cauchy sequences converging to a finite infimum or a finite supremum of $\mathbb{X}$. Use Hilger's notation [21] as employed in the recent monograph of Bohner and Peterson [10] or in the first book on time scales by Kaymakçalan *et al.* [27]. In particular, let σ be the right jump function on $\mathbb{X}$ and let ρ be the left jump function on $\mathbb{X}$. Hilger used σ in two ways: (i) as a mapping from $\mathbb{X}$ into $\mathbb{X}$, and (ii) as a shift operator on a function which was denoted by $f^{\sigma}(x) = f(\sigma(x))$. When working with composites of operators, we may choose to replace his f^{σ} by Sf. We sometimes use Df to denote f^{Δ}. We will consider even order differential equations defined on functions $y(x)$ with $x \in \mathbb{X}$. Background material can be found in [1,4,8,17,22–26,33].

Time scale differential equations of the form

$$\left(ry^{\Delta}\right)^{\Delta}(t) + (py^{\sigma})(t) = 0 \tag{1}$$

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were introduced by Erbe and Hilger [18] and have been studied by several authors in [2,9,10,19].

Atkinson [7, Section 0.7] considered "the discrete case presented by the recurrence relation"

$$c_{n+1}y_{n+1} + (a_n\lambda + b_n)y_n + c_{n-1}y_{n-1} = 0.$$

That form stimulated an enormous literature (see the references in [6]). In particular, the associated coefficient matrix is symmetric and this recurrence relation can be viewed as arising from discrete variational theory. Discrete nonsymmetric equations of the form $\Delta(r_n\Delta x_n) + p_n x_{n+1} = 0$ and matrix versions of this equation have been studied in [11–13], where they were called "self-adjoint". The form of (1), or the form of expression determined by the left side, even though called "self-adjoint" in [2,10,19] appears to be unsatisfactory from the classical viewpoint of formal self-adjointness and from the classical viewpoint of self-adjointness of standard boundary value problems. The concept of a self-adjoint extension of a symmetric operator which is densely defined on a Hilbert space is an important topic in functional analysis and spectral theory. Refer to [28, pp. 84, 255], [29, Ch. X], and [14, pp. 308–309]. See [16, p. 2 and Section 6] for a discussion about some difficulties caused by calling the differential expressions arising in (1) "self-adjoint".

In the discrete equations considered in [11–13], one can make the expression formally self-adjoint by replacing n by $m = n - 1$, i.e., set $m = \rho(n)$. In order to carry out a similar idea on (1), let us restart by considering a time scale differential expression of a form arising as a special case of a Jacobi equation in the calculus of variations on time scales (see Section 4 below or [3, Section 1])

$$L_2[y](t) = -\left(ry^{\Delta}\right)^{\Delta}(t) + (py^{\sigma})(t). \tag{2}$$

We will make the substitution $t = \rho(x)$ in (2). However, we find it useful to introduce shift operator notation so as to avoid ambiguous expressions like $y^{\sigma}(\rho(x))$.

For a time scale $\mathbb{X}$, let $\mathbb{X}^{\kappa}$ be Hilger's truncated above (kappen = lop off) set consisting of $\mathbb{X}$ except for a possible maximal isolated point. Use the notation $\mathbb{X}_{\kappa}$ for $\mathbb{X}$ truncated below by deleting a possible minimal isolated point. Also, let $\mathbb{X}_{\kappa}^{\kappa} = \mathbb{X}^{\kappa} \cap \mathbb{X}_{\kappa}$. Let $\mathscr{V}(\mathbb{X})$ be the vector space of real (or complex) valued functions defined on $\mathbb{X}$. Let S be the shift operator on $\mathscr{V}(\mathbb{X})$ defined by $v = Su$ if $v(x) = u(\sigma(x))$ for $x \in \mathbb{X}$. Introduce the notation Λ for the shift operator on $\mathscr{V}(\mathbb{X})$ defined by $g = \Lambda f$ with

$$g(x) = \begin{cases} f(\rho(x)), & \text{for } x \in \mathbb{X}_{\kappa}, \\ 0, & \text{for } x \in \mathbb{X} \setminus \mathbb{X}_{\kappa}. \end{cases} \tag{3}$$

In the example of $\mathbb{X} = \mathbb{N}$ (the naturals), the subspace of $\mathscr{V}(\mathbb{N})$ consisting of sequences $(\alpha_1, \alpha_2, \ldots)$ with convergent $\sum_1^{\infty}|\alpha_i|^2$, together with inner product $\langle\alpha, \beta\rangle = \sum_1^{\infty}\alpha_i\bar{\beta}_i$, is the Hilbert space ℓ^2. Our operator Λ agrees with the "unilateral shift" on ℓ^2 [14, Prop. 2.10, p. 32] (our Λ is Conway's S), where $\Lambda(\alpha_1, \alpha_2, \ldots) = (0, \alpha_1, \alpha_2, \ldots)$, and the Hilbert space adjoint $\Lambda^* = S$ is given by $\Lambda^*(\alpha_1, \alpha_2, \ldots) = S(\alpha_1, \alpha_2, \ldots) = (\alpha_2, \alpha_3, \ldots)$.

The substitution $t = \rho(x)$ in (2) gives the differential expression

$$\ell_2[y] = -(\Lambda(ry^{\Delta}))^{\Delta} + \Lambda(p)S(\Lambda y)$$

for y defined on $\mathbb{X}$ and $\ell_2[y](x)$ defined for $x \in \mathbb{X}_\kappa^\kappa$. See [10, Exercise 1.12, p. 5] for a reason why we omit a maximum isolated point. If y is defined on a real interval $\mathbb{X}$, then $^\Delta$ is the usual derivative, $\rho(x) = \sigma(\rho(x)) = x$, and ℓ_2 is the usual Sturm–Liouville differential expression

$$\ell_2[y](x) = -(r y')'(x) + p(x)y(x).$$

In case y is defined on a strictly increasing discrete set of distinct points x_n, $n \in \mathbb{N}$, then $^\Delta$ is the forward difference quotient

$$y^\Delta(x_n) = \frac{y(x_{n+1}) - y(x_n)}{x_{n+1} - x_n}.$$

If $n > 1$, then $\rho(x_n) = x_{n-1}$, $\sigma(\rho(x_n)) = x_n$, and

$$y^\Delta(\rho(x_n)) = \frac{y(\rho(x_{n+1})) - y(\rho(x_n))}{\rho(x_{n+1}) - \rho(x_n)} = \frac{y(x_n) - y(x_{n-1})}{x_n - x_{n-1}}.$$

Then ℓ_2 is the discrete expression

$$\ell_2[y](x_n) = -(r(x_{n-1})y^\Delta(x_{n-1}))^\Delta + p(x_{n-1})y(x_n), \quad n \in \mathbb{N}.$$

These continuous and discrete forms of ℓ_2 are special cases of Jacobi operators in, respectively, continuous variational theory [30, pp. 398–399] and discrete variational theory [6, Section 4.2]. However, the expression $L_2[y]$ in (2) appears to arise naturally from variational theory on time scales [5,9]. We will see in Section 6 that L_2, as well as another form, can be generated from Hamilton–Jacobi systems on time scales as presented in [3].

2. STURMIAN TIME SCALES

Because our goal is unification of discrete and continuous variable change methods for Sturm–Liouville expressions and because Sturm studied both continuous and discrete cases, we will subsequently consider only time scales $\mathbb{X}$ as defined in the previous section which have the properties

i) $\sigma(\rho(x)) = x$ for all $x \in \mathbb{X}_\kappa$;
ii) $\rho(\sigma(x)) = x$ for all $x \in \mathbb{X}^\kappa$.

We will call such time scales *Sturmian*.

If $\mathbb{X}$ is a real interval, then $\mathbb{X}$ is Sturmian. Note that in the discrete case, where $\mathbb{X}$ is a strictly increasing set of points $\{x_n |\ n = 1, 2, \ldots, N\}$ with $N \geq 2$, we have $\rho(\sigma(x_n)) = x_n$ for $n \neq N$ and $\sigma(\rho(x_n)) = x_n$ for $n \neq 1$. Thus the continuous and discrete problems studied by Sturm involved our so-called Sturmian time scales.

PROPOSITION 1 *Suppose $\mathbb{X}$ is Sturmian, u is any real or complex valued function defined on $\mathbb{X}_\kappa$, and v is any real or complex valued function defined on $\mathbb{X}^\kappa$. Then*

$$S(\Lambda(u(x))) = u(x) \text{ for } x \in \mathbb{X}_\kappa \quad \text{and} \quad \Lambda(S(v(x))) = v(x) \text{ for } x \in \mathbb{X}^\kappa.$$

Proof For $x \in \mathbb{X}_\kappa$, $\Lambda(u(x)) = u(\rho(x))$ and $S(\Lambda(u(x))) = u(\sigma(\rho(x))) = u(x)$. For $x \in \mathbb{X}^\kappa$, we know that $\sigma(x)$ is not a minimal isolated point of $\mathbb{X}$ and hence $\Lambda(S(v(x))) = \Lambda(v(\sigma(x))) = v(\rho(\sigma(x))) = v(x)$. □

3. FORMAL SELF-ADJOINTNESS

Under the assumptions that the time scale $\mathbb{X}$ is Sturmian and y is a function defined on $\mathbb{X}$, then $S(\Lambda(y(x))) = y(x)$ for $x \in \mathbb{X}_\kappa$, and we can write $\mathcal{L}_2$ in the *Sturm–Liouville* form

$$\mathcal{L}_2[y] = -\left(\Lambda\left(ry^\Delta\right)\right)^\Delta + (\Lambda p)y$$

on $\mathbb{X}_\kappa^\kappa$. For Sturmian $\mathbb{X}$ we now show that $\mathcal{L}_2$ is formally self-adjoint. If the expression $\mathcal{L}_2$ satisfies identity (4), Green's identity, we will say that $\mathcal{L}_2$ is *formally self-adjoint*. For the case of $\mathbb{X}$ a real interval $[a, b]$, (4) agrees with the second order case given in Hartman [20, pp. 398–399], also see [30, pp. 120–122]. For discrete case even order expressions, see [32] for definitions of formal adjoints and forms of formally self-adjoint expressions.

Proposition 2 *Suppose that $\mathbb{X}$ is a Sturmian time scale and a, b are points of $\mathbb{X}_\kappa^\kappa$ with $a < b$. If y and z are such that $\mathcal{L}_2[y]$ and $\mathcal{L}_2[z]$ are uniquely defined and rd-continuous for $x \in [a, b]$ except at b if b is left scattered, then*

$$\int_a^b (\mathcal{L}_2[y](x)\bar{z}(x) - y(x)\mathcal{L}_2[\bar{z}](x))\Delta x = \left[-\left(\Lambda\left(ry^\Delta\right)\right)\bar{z} + y\Lambda\left(r\bar{z}^\Delta\right)\right]\Big|_a^b. \tag{4}$$

Proof Use the derivative of a product formula [10, Thm. 1.20 (iii)] on $(\Lambda f)g$ to write

$$[(\Lambda f)g]^\Delta = (\Lambda f)^\Delta g + (S(\Lambda f))g^\Delta$$

which gives the integration by parts formula

$$\int_a^b (\Lambda f)^\Delta g \Delta x = (\Lambda f)g\Big|_a^b - \int_a^b S(\Lambda f)g^\Delta \Delta x.$$

Then, in this notation,

$$\int_a^b -\left(\Lambda\left(ry^\Delta\right)\right)^\Delta \bar{z}\Delta x = -\left(\Lambda\left(ry^\Delta\right)\right)\bar{z}\Big|_a^b + \int_a^b \left(S\left(\Lambda\left(ry^\Delta\right)\right)\right)\bar{z}^\Delta \Delta x,$$

where

$$\left(S\left(\Lambda\left(ry^\Delta\right)\right)\right)\bar{z}^\Delta = \left(ry^\Delta\right)\bar{z}^\Delta = y^\Delta\left(r\bar{z}^\Delta\right) = y^\Delta S\left(\Lambda\left(r\bar{z}^\Delta\right)\right)$$

since $S(\Lambda u) = u$. The derivative formula

$$(yg)^\Delta = yg^\Delta + y^\Delta(Sg)$$

gives the integration by parts formula $\int_a^b y^\Delta(Sg)\Delta x = yg|_a^b - \int_a^b yg^\Delta \Delta x$. Set $g = \Lambda(r\bar{z}^\Delta)$ for the conclusion (4). □

4. CALCULUS OF VARIATIONS

Let c and d be given real numbers. For y defined at $t_0, t_1, \ldots, t_N$ with $y(t_0) = c$ and $y(t_N) = d$, the discrete variational problem of minimizing

$$J(y) = \sum_{n=1}^{N} f(t_{n-1}, y(t_n), y^\Delta(t_{n-1}))\,\Delta t_{n-1}$$

can be rewritten as

$$J(y) = \sum_{n=1}^{N} f(t_{n-1}, S(y(t_{n-1})), y^{\Delta}(t_{n-1}))\, \Delta t_{n-1}$$

or as

$$J(y) = \sum_{m=0}^{N-1} f(t_m, S(y(t_m)), y^{\Delta}(t_m))\, \Delta t_m,$$

which is the time scale integral

$$\int_{t_0}^{t_N} f\big(t, S(y(t)), y^{\Delta}(t)\big)\, \Delta t.$$

Thus we are led to consider time scale calculus of variations problems of the type considered in [9], see also [5]. Let $[a, b]$ be a time scale and consider the time scale integral

$$J(y) = \int_a^b f\big(t, S(y(t)), y^{\Delta}(t)\big) \Delta t \tag{5}$$

on the class of functions y defined on the time scale interval $[a, b]$ for which the indicated derivative and integral exist and y satisfies fixed end conditions $y(a) = c$, $y(b) = d$. Note that as in the discrete case, if b is left scattered, then the integrand of $J(y)$ does not depend upon the value of $t = b$.

Under suitable hypotheses on $f(t, u, v)$ and on a minimizing arc y the Euler–Lagrange equation

$$\big[f_v(t, S(y(t)), y^{\Delta}(t))\big]^{\Delta} = f_u\big(t, S(y(t)), y^{\Delta}(t)\big)$$

must hold for $t \in ([a, b]^{\kappa})^{\kappa}$. This is a consequence of setting $\phi(\varepsilon) = J(y + \varepsilon\eta)$ and setting $\phi'(0) = 0$. The condition $\phi''(0) \geq 0$ gives rise to the condition that the second variation

$$J_2(\eta) = \int_a^b 2\omega\,\big(t, S(\eta(t)), \eta^{\Delta}(t)\big) \Delta t,$$

where 2ω has the form $2\omega(t, u, v) = P(t)u^2 + 2Q(t)uv + R(t)v^2$, must be nonnegative on the class of admissible η. The Euler–Lagrange equation for J_2 is the Jacobi equation

$$-\big(Ry^{\Delta} + QS(y)\big)^{\Delta} + Qy^{\Delta} + PS(y) = 0.$$

Thus the form of the equation $L_2[y] = 0$ for L_2 of (2) is a special case of the Jacobi equation of the calculus of variations on time scales. However, we needed the assumption of a Sturmian time scale to obtain a Green's identity for the shifted form of the associated operator, namely $\mathcal{L}_2$.

5. VARIABLE CHANGE ON S–L EXPRESSIONS

We will now carry out a simultaneous change of independent and dependent variables on $L[y](x) \equiv L_2[y](x)$. We use $\mathbb{X}$ for our time scale so as to use the results of [3, Section 3].

Let $h(x) = \sigma(x) - x$ be the stepsize (or the graininess) of $\mathbb{X}$. Use the concept in [3, Section 3] (also see [10, Section 8.3]) of a "generalized time scale" $\mathbb{T}$ which is defined by $\mathbb{T} := f(\mathbb{X})$, with generalized jump function $\tau(t) \equiv f(\sigma(x))$, for $t = f(x)$, where we assume that f is a real valued class C^1 strictly monotone function defined on the smallest interval $[\mathbb{X}]$ which contains $\mathbb{X}$. Also, let $k(t) = \tau(t) - t$ and let $z^{\tau}(t)$ be the shift of $z(t)$ defined by $z^{\tau}(t) = z(\tau(t))$. Note that $t = f(x)$ implies $k(t) = h(x)f^{\Delta}(x)$. Let ^{o} be the alpha derivative with respect to t as defined in [3, Section 3] (also see [10, Section 8.3]). If $\mathbb{X}$ is discrete, x_n is strictly increasing, and $t_n = f(x_n)$, then

$$z^{o}(t_n) = \frac{z(t_{n+1}) - z(t_n)}{t_{n+1} - t_n} \quad \text{and} \quad z^{\tau}(t_n) = t_{n+1}.$$

Coppel obtained a general transformation result for second order Sturm–Liouville differential equations [15] under a simultaneous change of independent and dependent variables. We consider variable changes of the form

$$y(x) = m(x)z(t), \quad t = f(x), \quad x \in \mathbb{X}. \tag{6}$$

The corresponding discrete version of Coppel's result was given by Voepel [31]. Let us assume that m is in the domain of L_2 and the induced derivative $z^{o}(t)$ is the alpha derivative on the induced time scale as given in [3, Section 3]. A second expression $L_o[z]$ induced by this variable change is of the form

$$L_o[z](t) = -(Rz^{o})^{o}(t) + (Pz^{\tau})(t).$$

The following theorem generalizes the continuous and discrete transformation results to time scales.

THEOREM 3 (VARIABLE CHANGE FOR SECOND ORDER) *A variable change*

$$y(x) = m(x)z(t), \quad t = f(x), \quad x \in \mathbb{X},$$

defined for $x \in \mathbb{X}$ and $t \in \mathbb{T} = f(\mathbb{X})$ with f a strictly monotone class C^1 function on $\mathbb{X}$ is made on the expression $L[y] \equiv L_2[y]$. Define R and P by

$$R(t) = (rmm^{\sigma}f^{\Delta})(x) \quad \text{and} \quad P(t) = ((m^{\sigma}/f^{\Delta})L_2[m])(x), \tag{7}$$

where $t = f(x)$ for $x \in \mathbb{X}$. Then for y and z related by (6), the expressions L_o and L are related by

$$L_o[z](t) = \big((m^{\sigma}/f^{\Delta})L[y]\big)(x) \quad \text{for} \quad t = f(x). \tag{8}$$

(Note that if we intended to use these variable changes to induce unitarily equivalent self-adjoint operators, then our first step would be to interpret this result as starting with $\mathcal{L}_2$ rather then L_2.)

Proof Rewrite the systems transformation result of [3, Thm. 3.1] in neutral notation as follows. Let $\mathscr{L}$ be an expression defined on $\mathbb{X}$ by

$$\mathscr{L}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}^{\Delta} - M(x)\begin{bmatrix} u \\ v \end{bmatrix}$$

with M Hamiltonian on $\mathbb{X}^\kappa$, i.e., $M^*\mathscr{J} + \mathscr{J}M + hM^*\mathscr{J}M \equiv 0$, where

$$\mathscr{J} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Assume N is symplectic on $\mathbb{X}$, i.e., $N^*(t)\mathscr{J}N(t) = \mathscr{J}$. Then the matrix $\mathscr{Q}(t) := ((f^\Delta N^\sigma)^{-1} \times (MN - N^\Delta))(x)$ is Hamiltonian on $\mathbb{T}^i \equiv f(\mathbb{X}^\kappa)$, i.e., $\mathscr{Q}^*\mathscr{J} + \mathscr{J}\mathscr{Q} + k\mathscr{Q}^*\mathscr{J}\mathscr{Q} \equiv 0$ on $\mathbb{T}^i$, and the expression $\mathscr{L}_o$ defined by

$$\mathscr{L}_o\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}^o - \mathscr{Q}(t)\begin{bmatrix} U \\ V \end{bmatrix}$$

is related to $\mathscr{L}$ by the identity

$$\mathscr{L}_o\begin{bmatrix} U \\ V \end{bmatrix}(t) = \left((f^\Delta N^\sigma)^{-1}\mathscr{L}\begin{bmatrix} u \\ v \end{bmatrix}\right)(x) \tag{9}$$

when

$$\begin{bmatrix} u(x) \\ v(x) \end{bmatrix} = N(x)\begin{bmatrix} U(t) \\ V(t) \end{bmatrix} \quad \text{for} \quad t = f(x).$$

Use the product rule and chain rule [3, Section 3] on $u(x) = y(x) = m(x)z(t)$ to obtain

$$ru^\Delta = rm^\Delta z + rm^\sigma f^\Delta z^o.$$

Now apply this to our present second order expression $L_2[y]$ by setting $L_2 = L$ and

$$\mathscr{L}\begin{bmatrix} y \\ ry^\Delta \end{bmatrix} := \begin{bmatrix} y \\ ry^\Delta \end{bmatrix}^\Delta - M(x)\begin{bmatrix} y \\ ry^\Delta \end{bmatrix} \quad \text{for} \quad M = \begin{bmatrix} 0 & 1/r \\ p & hp/r \end{bmatrix},$$

where

$$\begin{bmatrix} y \\ ry^\Delta \end{bmatrix}(x) = N(x)\begin{bmatrix} z \\ Rz^o \end{bmatrix}(t) \quad \text{for} \quad N = \begin{bmatrix} m & 0 \\ rm^\Delta & 1/m \end{bmatrix}.$$

Note that the (2,2) entry of N is determined by demanding that N is symplectic. Also,

$$\mathscr{L}_o\begin{bmatrix} z \\ Rz^o \end{bmatrix} := \begin{bmatrix} z \\ Rz^o \end{bmatrix}^o - \mathscr{Q}(t)\begin{bmatrix} z \\ Rz^o \end{bmatrix} \quad \text{for} \quad \mathscr{Q} = \begin{bmatrix} 0 & 1/R \\ P & Q_{22} \end{bmatrix}$$

with

$$Q_{22}(t) = k(t)\left(\frac{L_2[m]}{rm\left(f^\Delta\right)^2}\right)(x) = (kP/R)(t)$$

since $\mathscr{Q}$ is Hamiltonian on $\mathbb{T}^i$. Then

$$\mathscr{L}\begin{bmatrix} y \\ ry^\Delta \end{bmatrix} = \begin{bmatrix} 0 \\ -L[y] \end{bmatrix}, \quad \mathscr{L}_o\begin{bmatrix} z \\ Rz^o \end{bmatrix} = \begin{bmatrix} 0 \\ -L_o[z] \end{bmatrix},$$

and

$$\mathscr{L}_o\begin{bmatrix} z(t) \\ Rz^o(t) \end{bmatrix} = \left((f^\Delta N^\sigma)^{-1}\mathscr{L}\begin{bmatrix} y \\ ry^\Delta \end{bmatrix}\right)(x) \quad \text{for} \quad t = f(x)$$

gives the identity (8). □

6. GENESIS OF L_4

In this section, we again consider Sturmian time scales. A linear Hamiltonian system [3, Prop. 1.1] is given by

$$u^\Delta = Au^\sigma + Bv, \quad v^\Delta = Cu^\sigma - A^T v, \tag{10}$$

where A, B, and C are $n \times n$ matrices such that $I + \mu A$ is invertible. The matrices A, B, and C that correspond to Sturm–Liouville dynamic equations are given by

$$A = 0, \quad B = \frac{1}{r}, \quad C = p$$

in case $n = 1$, and by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix}, \quad C = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$$

in case $n = 2$. Before we present the case $n = 2$, we first consider $n = 1$, where the corresponding Hamiltonian system (10) is

$$u_1^\Delta = \frac{1}{r}v_1, \quad v_1^\Delta = pu_I^\sigma.$$

Letting $y = u_1$ gives

$$ry^\Delta = v_1, \quad v_1^\Delta = py^\sigma; \quad \text{hence} \quad \left(ry^\Delta\right)^\Delta = py^\sigma.$$

Again we arrive at

$$L_2[y] = -\left(ry^\Delta\right)^\Delta + py^\sigma.$$

Letting $y = u_1^\sigma$ gives (for Sturmian time scales)

$$ry^{\rho\Delta} = v_1, \quad v_1^\Delta = py; \quad \text{hence} \quad \left(ry^{\rho\Delta}\right)^\Delta = py$$

(where we abbreviated $(y^\rho)^\Delta$ by $y^{\rho\Delta}$). We therefore consider

$$m_2[y] = -\left(ry^{\rho\Delta}\right)^\Delta + py$$

and note that $m_2[y] = L_2[y^\rho]$ holds. Next apply the product rule to obtain

$$(ry^{\rho\Delta})^\Delta x = (ry^{\rho\Delta}x^\rho)^\Delta - ry^{\rho\Delta}x^{\rho\Delta},$$

and therefore a Green's identity for m_2 may be derived from

$$m_2[y]\bar{z} - ym_2[\bar{z}] = \left(r\bar{z}^{\rho\Delta}y^\rho - ry^{\rho\Delta}\bar{z}^\rho\right)^\Delta.$$

Next, we look at $n = 2$. The corresponding Hamiltonian system (10) is

$$u_1^\Delta = u_2^\sigma, \quad u_2^\Delta = \frac{1}{r}v_2, \quad v_1^\Delta = pu_1^\sigma, \quad v_2^\Delta = qu_2^\sigma - v_1.$$

We first let $y = u_1$, then

$$u_2 = y^{\Delta\rho}, \quad \frac{1}{r}v_2 = u_2^\Delta = y^{\Delta\rho\Delta}, \quad v_2 = ry^{\Delta\rho\Delta},$$

hence

$$qy^\Delta - v_1 = qu_2^\sigma - v_1 = v_2^\Delta = \left(ry^{\Delta\rho\Delta}\right)^\Delta,$$

and therefore

$$\left(qy^\Delta\right)^\Delta - py^\sigma = \left(ry^{\Delta\rho\Delta}\right)^{\Delta\Delta}.$$

Thus L_4 arises as

$$L_4[y] = \left(ry^{\Delta\rho\Delta}\right)^{\Delta\Delta} - \left(qy^\Delta\right)^\Delta + py^\sigma.$$

Now let $y = u_1^\sigma$, then $y^\rho = u_1$ and

$$py = pu_1^\sigma = v_1^\Delta$$

so that

$$(ry^{\rho\Delta\rho\Delta})^{\Delta\Delta} = \left(ru_1^{\Delta\rho\Delta}\right)^{\Delta\Delta} = \left(ru_2^\Delta\right)^{\Delta\Delta} = v_2^{\Delta\Delta}$$

$$= \left(qu_2^\sigma\right)^\Delta - v_1^\Delta = \left(qu_1^\Delta\right)^\Delta - py = \left(qy^{\rho\Delta}\right)^\Delta - py.$$

Hence, define m_4 by

$$m_4[y] = \left(ry^{\rho\Delta\rho\Delta}\right)^{\Delta\Delta} - \left(qy^{\rho\Delta}\right)^\Delta + py$$

and note that $m_4[y] = L_4[y^\rho]$ holds. Next apply the product rule twice to obtain

$$\left(ry^{\rho\Delta\rho\Delta}\right)^{\Delta\Delta}x = \left\{\left(ry^{\rho\Delta\rho\Delta}\right)^\Delta x^\rho\right\}^\Delta - \left(ry^{\rho\Delta\rho\Delta}\right)^\Delta x^{\rho\Delta}$$

$$= \left\{\left(ry^{\rho\Delta\rho\Delta}\right)^\Delta x^\rho\right\}^\Delta - \left(ry^{\rho\Delta\rho\Delta}x^{\rho\Delta\rho}\right)^\Delta + ry^{\rho\Delta\rho\Delta}x^{\rho\Delta\rho\Delta},$$

and therefore a Green's identity for m_4 can be derived from

$$m_4[y]\bar{z} - ym_4[\bar{z}] = \Big\{ \big(ry^{\rho\Delta\rho\Delta}\big)^{\Delta}\bar{z}^{\rho} - \big(r\bar{z}^{\rho\Delta\rho\Delta}\big)^{\Delta}y^{\rho} + r\bar{z}^{\rho\Delta\rho\Delta}y^{\rho\Delta\rho}$$

$$- ry^{\rho\Delta\rho\Delta}\bar{z}^{\rho\Delta\rho} + q\bar{z}^{\rho\Delta}y^{\rho} - qy^{\rho\Delta}\bar{z}^{\rho} \Big\}^{\Delta}.$$

Thus both m_2 and m_4 are formally self-adjoint.

7. TRANSFORMATION OF FOURTH ORDER

THEOREM 4 (VARIABLE CHANGE FOR FOURTH ORDER) *A variable change*

$$y(x) = m(x)z(t), \quad t = f(x), \quad x \in \mathbb{X}, \tag{11}$$

defined for $x \in \mathbb{X}$, with $\mathbb{X}$ Sturmian, and $t \in \mathbb{T} = f(\mathbb{X})$ with $f^{\Delta}(x)$ never 0 on $\mathbb{X}$ (assume f is strictly monotone, f^{Δ}, $f^{\Delta p\Delta}$, and ρ^{Δ} exist) is made on the expression $L[y](x) \equiv L_4[y](x)$. Put

$$s = rm^{\Delta\rho\Delta}, \quad w = m^{\rho}f^{\Delta\rho}, \quad \beta = \frac{r(mw)^{\Delta}}{m}, \quad \alpha = \frac{sw - \beta m^{\Delta\rho}}{m}$$

and define R, P, and Q by

$$R(t) = (rm^{\rho}m^{\sigma}(f^{\Delta})^2 f^{\Delta\rho})(x), \quad P(t) = ((m^{\sigma}/f^{\Delta})L_4[m])(x),$$

and

$$Q(t) = (qmm^{\sigma}f^{\Delta})(x) - (\alpha m)^{\sigma}(x) - (\beta m)^{\Delta}(x),$$

where $t = f(x)$ for $x \in \mathbb{X}$. Then for y and z related by (11) the expressions L_o defined by

$$L_o[z](t) = (Rz^{o\varrho o})^{oo}(t) - (Qz^{o})^{o}(t) + (Pz^{\tau})(t)$$

(τ and ϱ are the forward and backward shift on $\mathbb{T}$) and L are related by

$$L_o[z](t) = ((m^{\sigma}/f^{\Delta})L[y])(x) \quad \text{for} \quad t = f(x). \tag{12}$$

Proof We use vectors u and v as in Section 6, i.e.,

$$u = \begin{bmatrix} y \\ y^{\Delta\rho} \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} -(ry^{\Delta\rho\Delta})^{\Delta} + qy^{\Delta} \\ ry^{\Delta\rho\Delta} \end{bmatrix}$$

and introduce U and V as the corresponding transformed variables

$$U = \begin{bmatrix} z \\ z^{o\varrho} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} -(Rz^{o\varrho o})^{o} + Qz^{o} \\ Rz^{o\varrho o} \end{bmatrix},$$

where ϱ is the generalized back shift on $\mathbb{T}$ defined by $\varrho(t) = f(\rho(x))$ for $t = f(x)$. Then one wishes to use relationships

$$u = EU \quad \text{and} \quad v = FU + E^{T-1}V,$$

where E occurs naturally by representing the entries of u in terms of the corresponding entries of U. In order to motivate the form of R and to generate F, we need to compute the corresponding derivatives of v in terms of derivatives of z. Since $y = mz$, we find

$$y^{\Delta} = m^{\Delta} z^{\tau} + mf^{\Delta} z^{o} = m^{\Delta} z + m^{\sigma} f^{\Delta} z^{o},$$

and

$$y^{\Delta\rho} = m^{\Delta\rho} z + m^{\rho} f^{\Delta\rho} z^{o\varrho} = m^{\Delta\rho} z^{\varrho} + mf^{\Delta\rho} z^{o\varrho}.$$

Next,

$$ry^{\Delta\rho\Delta} = r\big(m^{\Delta\rho\Delta} z + m^{\Delta\rho} f^{\rho\Delta} z^{o\varrho} + (mf^{\Delta\rho})^{\Delta} z^{o\varrho} + m^{\sigma} f^{\Delta} f^{\Delta} z^{o\varrho o}\big) = rm^{\Delta\rho\Delta} z + \beta z^{o\varrho} + \frac{1}{w} R z^{o\varrho o}$$

(note that $m^{\Delta\rho} f^{\rho\Delta} = m^{\rho\Delta} f^{\Delta\rho}$ by e.g., [3, Theorem 2.7]). For these calculations it is good to remember that e.g., $(z^{\varrho})^{\Delta}$ really means, with the usual abuse of notation $(z^{\varrho} \circ f)^{\Delta}$, and then the chain rule [3, Theorem 2.7] helps to calculate this derivative as

$$(z^{\varrho})^{\Delta} = (z^{\varrho} \circ f)^{\Delta} = (z \circ f^{\rho})^{\Delta} = (f^{\rho})^{\Delta} (z^{o} \circ f^{\rho}) = f^{\rho\Delta} (z^{o\varrho} \circ f) = f^{\rho\Delta} z^{o\varrho}.$$

Finally

$$\begin{aligned}
-\big(ry^{\Delta\rho\Delta}\big)^{\Delta} + qy^{\Delta} &= -\big(rm^{\Delta\rho\Delta}\big)^{\Delta} z - \big(rm^{\Delta\rho\Delta}\big)^{\sigma} f^{\Delta} z^{o} - \beta^{\Delta} z^{o} - \beta f^{\Delta} z^{o\varrho o} - (1/w)^{\Delta} R z^{o\varrho o} \\
&\quad - \big(1/w\big)^{\sigma} \big(R z^{o\varrho o}\big)^{\Delta} + qm^{\Delta} z + qm^{\sigma} f^{\Delta} z^{o} \\
&= \Big[\big(-rm^{\Delta\rho\Delta}\big)^{\Delta} + qm^{\Delta}\Big] z - \left[\frac{\beta f^{\Delta}}{R} + \left(\frac{1}{w}\right)^{\Delta}\right] R z^{o\varrho o} \\
&\quad - \Big[\big(rm^{\Delta\rho\Delta}\big)^{\sigma} f^{\Delta} + \beta^{\Delta} - qm^{\sigma} f^{\Delta}\Big] z^{o} - \frac{f^{\Delta}}{w^{\sigma}} \big(R z^{o\varrho o}\big)^{o} \\
&= \Big[-\big(rm^{\Delta\rho\Delta}\big)^{\Delta} + qm^{\Delta}\Big] z + \frac{1}{m} \big[-\big(R z^{o\varrho o}\big)^{o} + Q z^{o}\big] \\
&\quad - \left[\big(rm^{\Delta\rho\Delta}\big)^{\sigma} f^{\Delta} + \beta^{\Delta} - qm^{\sigma} f^{\Delta} + \frac{Q}{m}\right] z^{o} - \left[\frac{\beta f^{\Delta}}{R} + \left(\frac{1}{w}\right)^{\Delta}\right] R z^{o\varrho o}.
\end{aligned}$$

Now we calculate

$$\begin{aligned}
\frac{\beta f^{\Delta}}{R} + \left(\frac{1}{w}\right)^{\Delta} = \frac{(mw)^{\Delta}}{m^{\sigma} w w^{\sigma}} - \frac{w^{\Delta}}{w w^{\sigma}} = \frac{m^{\Delta}}{m m^{\sigma} f^{\Delta}} = \left(\frac{m^{\Delta}}{m m^{\sigma} f^{\Delta}}\right)^{\rho} + h\left(\frac{m^{\Delta}}{m m^{\sigma} f^{\Delta}}\right)^{\rho\Delta}, \\
h\left(\frac{m^{\Delta\rho}}{m^{\rho} m f^{\Delta\rho}}\right)^{\Delta} R = \frac{kR}{f^{\Delta}} \left(\frac{m^{\Delta\rho}}{mw}\right)^{\Delta} = \frac{k(sw - \beta m^{\Delta\rho})}{m} = k\alpha, \\
h\left(\frac{m^{\Delta\rho}}{m^{\rho} m f^{\Delta\rho}}\right)^{\Delta} R z^{o\varrho o} = \alpha k z^{o\varrho o} = \alpha(z^{o} - z^{o\varrho}) = \alpha z^{o} - \alpha z^{o\varrho},
\end{aligned}$$

and

$$\alpha m + \beta^{\Delta} m + (rm^{\Delta\rho\Delta})^{\sigma} f^{\Delta} m - qm m^{\sigma} f^{\Delta} + Q = 0$$

to find

$$-\left(ry^{\Delta\rho\Delta}\right)^{\Delta}+qy^{\Delta}=[-\left(rm^{\Delta\rho\Delta}\right)^{\Delta}+qm^{\Delta}]z+\alpha z^{o\varrho}+\frac{1}{m}[-(Rz^{o\varrho o})^{o}+Qz^{o}]-\frac{m^{\Delta\rho}}{m^{\rho}f^{\Delta\rho}}Rz^{o\varrho o}.$$

Consequently, we find that the matrix N with

$$\begin{bmatrix} u \\ v \end{bmatrix} = N \begin{bmatrix} U \\ V \end{bmatrix} \quad \text{is} \quad N = \begin{bmatrix} E & 0 \\ F & E^{T-1} \end{bmatrix},$$

where

$$E = \begin{bmatrix} m & 0 \\ m^{\Delta\rho} & m^{\rho}f^{\Delta\rho} \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} -(rm^{\Delta\rho\Delta})^{\Delta}+qm^{\Delta} & \alpha \\ rm^{\Delta\rho\Delta} & \beta \end{bmatrix}.$$

Note that F^TE is symmetric and hence N is symplectic. Next, let

$$M = \begin{bmatrix} \mathscr{A} & \mathscr{B} \\ \mathscr{C} & \mathscr{D} \end{bmatrix}$$

with

$$\mathscr{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathscr{B} = \begin{bmatrix} 0 & h/r \\ 0 & 1/r \end{bmatrix}, \mathscr{C} = \begin{bmatrix} p & hp \\ 0 & q \end{bmatrix}, \mathscr{D} = \begin{bmatrix} 0 & h^2p/r \\ -1 & hq/r \end{bmatrix}$$

and define the expression $\mathscr{L}$ by

$$\mathscr{L}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}^{\Delta} - M(x)\begin{bmatrix} u \\ v \end{bmatrix}$$

so that by some easy computations (also compare with Section 6) we find

$$\mathscr{L}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_4[y] \\ 0 \end{bmatrix}.$$

Then by [3, Theorem 3.1] the matrix

$$\mathscr{Q} = \left(f^{\Delta}N^{\sigma}\right)^{-1}(MN - N^{\Delta})$$

is Hamiltonian on $\mathbb{T}^{i}$, and the expression $\mathscr{L}_o$ defined by

$$\mathscr{L}_o\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} U \\ V \end{bmatrix}^o - \mathscr{Q}(t)\begin{bmatrix} U \\ V \end{bmatrix}$$

is related to $\mathscr{L}$ by the identify

$$\mathscr{L}_o\begin{bmatrix} U \\ V \end{bmatrix} = (f^{\Delta}N^{\sigma})^{-1}\mathscr{L}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{13}$$

We find

$$\mathscr{Q} = \frac{1}{f^{\Delta}}\begin{bmatrix} E^{-1} & 0 \\ -F^T & E^T \end{bmatrix}^{\sigma}\begin{bmatrix} \mathscr{A}E+\mathscr{B}F-E^{\Delta} & \mathscr{B}E^{T-1} \\ \mathscr{C}E+\mathscr{D}F-F^{\Delta} & \mathscr{D}E^{T-1}-(E^{T-1})^{\Delta} \end{bmatrix} = \begin{bmatrix} \tilde{\mathscr{A}} & \tilde{\mathscr{B}} \\ \tilde{\mathscr{C}} & \tilde{\mathscr{D}} \end{bmatrix},$$

where

$$\tilde{\mathscr{A}} = \mathscr{A}, \quad \tilde{\mathscr{B}} = \begin{bmatrix} 0 & k/R \\ 0 & 1/R \end{bmatrix}, \quad \tilde{\mathscr{C}} = \begin{bmatrix} P & kP \\ 0 & Q \end{bmatrix}, \quad \tilde{\mathscr{D}} = \begin{bmatrix} 0 & k^2P/R \\ -1 & kQ/R \end{bmatrix}$$

can be found after some lengthy, but easy calculations. For the convenience of the reader willing to work through these calculations we give the "intermediate" matrices

$$\mathscr{A}E+\mathscr{B}F-E^{\Delta} = f^{\Delta}\begin{bmatrix} 0 & m^{\sigma} \\ 0 & m^{\Delta} \end{bmatrix},$$

$$\mathscr{C}E+\mathscr{D}F-F^{\Delta} = \begin{bmatrix} L_4[m] & hpm^{\sigma}f^{\Delta}-\alpha^{\Delta} \\ 0 & qm^{\sigma}f^{\Delta}-\alpha-\beta^{\Delta} \end{bmatrix},$$

and (note that $\tilde{\mathscr{D}}^T = E^{-1}(\mathscr{D}^T E^{\sigma} - \mathscr{B}^T F^{\sigma} + E^{\Delta})$)

$$\mathscr{D}^T E^{\sigma} - \mathscr{B}^T F^{\sigma} + E^{\Delta} = \begin{bmatrix} 0 & -mf^{\Delta} \\ (h^2/r)L_4[m] & (hQ - h\alpha m - rm^{\Delta}m^{\rho}f^{\Delta\rho})/(rm^{\sigma}) \end{bmatrix}.$$

Thus, as before,

$$\mathscr{L}_o\begin{bmatrix} U \\ V \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -L_o[z] \\ 0 \end{bmatrix},$$

and an application of (13) yields relation (12). □

In the discrete case this transformation theorem can be compared with Example C of [32], where the present variable $r(x_n)$ would agree with that $r_2(x_{n+1})$, the present $R(t_n)$ would agree with $r'_2(t_{n+1})$, $f^{\Delta}(x_n)$ is γ_{n+1}, and m was labeled as μ. However, our middle term has a different representation than that given in the previous discrete or continuous cases.

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