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Weyl–Titchmarsh theory for symplectic difference systems

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ABSTRACT

In this work, we establish Weyl–Titchmarsh theory for symplectic difference systems. This paper extends classical Weyl–Titchmarsh theory and provides a foundation for studying spectral theory of symplectic difference systems.

1. Introduction

Difference equations arise naturally as discretised analogues of differential equations, and they also appear in their own right, e.g., in the recurrence formulae for special functions and orthogonal polynomials. In spectral theory, difference operators are studied as models which avoid the inherent unboundedness of differential operators, the most prominent example being Jacobi matrices, with three-term recurrence formulae as eigenvalue equations (cf. the detailed account in [2]).

Consider the symplectic difference system

$$z_{k+1} = \mathcal{S}_k z_k, \quad k \in \mathbb{Z}, \tag{1.1}$$

where $\mathcal{S}_k \in C^{n\times n}$ are symplectic matrices for $k \in \mathbb{Z}$, i.e.,

$$\mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J} \quad \text{with} \quad \mathcal{J} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where I_n denotes the $n \times n$ identity matrix.

Symplectic difference systems (1.1) cover a large variety of difference equations and systems, such as linear Hamiltonian difference systems

$$\begin{cases} \Delta x_k = A_k x_{k+1} + B_k u_k, \\ \Delta u_k = C_k x_{k+1} - A_k^* u_k, \end{cases}$$

where $B_k, C_k \in \mathbb{C}^{n\times n}$ are symmetric and $I_n - A_k$ are nonsingular. In turn, systems (1.1) also cover higher-order Sturm–Liouville difference equations

$$\sum_{j=0}^{n} (-\Delta)^j \left\{ p_j(k) \Delta^j y_{k+1-j} \right\} = 0 \quad \text{with} \quad p_n(k) \neq 0,$$

in particular its special case, Sturm–Liouville second order difference equations

$$\Delta(p_k \Delta x_k) + q_k x_{k+1} = 0 \quad \text{with} \quad p_k \neq 0.$$

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This paper is devoted to Weyl–Titchmarsh theory for linear symplectic difference systems

$$z_{k+1} = \mathcal{S}_k(\lambda) z_k \quad \text{with} \quad \mathcal{S}_k(\lambda) = S_k - \lambda \widehat{\mathcal{S}}_k \quad \text{for } k \in \mathbb{Z}, \tag{1.2}$$

where λ is a parameter. Here

$$\mathcal{S}_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix} \quad \text{and} \quad \widehat{\mathcal{S}}_k = \begin{pmatrix} 0 & 0 \\ \mathcal{W}_k A_k & \mathcal{W}_k B_k \end{pmatrix},$$

where $A_k, B_k, C_k, D_k, \mathcal{W}_k \in \mathbb{C}^{n\times n}$.

The paper is organized as follows. Some fundamental theory for symplectic difference systems is given in Section 2. For symplectic difference systems we refer the reader to [1,3–6]. Weyl matrix disks are constructed and their properties are studied in Section 3. These matrix disks are nested and converge to a limiting set of the matrix circle. The results are some generalizations of Weyl–Titchmarsh theory for Hamiltonian difference systems [8,17]. The basic Weyl–Titchmarsh theory of regular Hamiltonian systems can be found in [2], Weyl–Titchmarsh theory of singular Hamiltonian systems and their basic spectral theory was developed by Hinton and Shaw and many others (cf. [7,9–15,19] and the references therein).

2. Preliminary results

Consider the symplectic system (1.2). Set

$$z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix} \quad \text{with} \quad x_k, u_k \in \mathbb{R}^n.$$

Then system (1.2) can be rewritten as

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k, \\ u_{k+1} = C_k x_k - \lambda \mathcal{W}_k x_{k+1} + D_k u_k, \end{cases} \quad \text{for } k \in \mathbb{Z}. \tag{2.1}$$

In this section, we shall study the fundamental theory and properties of solutions for the system (1.2), i.e., (2.1).

Assumption 2.1. Throughout we assume that S_k is symplectic, i.e.,

$$S_k^* \mathcal{J} S_k = \mathcal{J} \quad \text{for } k \in \mathbb{Z}.$$

Assumption 2.2. We always assume that $\mathcal{W}_k$ is symmetric and nonnegative definite, i.e., $\mathcal{W}_k \geqslant 0$, and for any nontrivial solution $z_k = \begin{pmatrix} x_k \\ u_k \end{pmatrix}$ of (2.1), we have

$$\sum_{k=p}^{q} x_{k+1}^* \mathcal{W}_k x_{k+1} > 0 \quad \text{for any } p, q \in \mathbb{Z} \text{ with } q > p.$$

Remark 2.1. $\mathcal{S}_k$ is symplectic if and only if

$$A_k^* C_k = C_k^* A_k, \quad B_k^* D_k = D_k^* B_k, \quad A_k^* D_k - C_k^* B_k = I_n. \tag{2.2}$$

From (2.2), it is easy to show that $S_k(\lambda)$ is symplectic for all $\lambda \in \mathbb{C}$, i.e.,

$$\mathcal{S}_k^*(\bar{\lambda}) \mathcal{J} \mathcal{S}_k(\lambda) = \mathcal{J}.$$

Now we consider the existence of solutions for (1.2) or (2.1). From [1, Theorem 3.1] and noting that $S_k(\lambda)$ is symplectic, we have the following.

Lemma 2.1 (Existence and Uniqueness Theorem). *For arbitrary initial data $m \in \mathbb{Z}$, $\xi \in \mathbb{C}^{2n}$, the initial value problem of (1.2) with $z_m = \xi$ has a unique solution for all $k \in \mathbb{Z}$.*

Now we consider the structure of solutions for the system (1.2), i.e., (2.1). For this purpose, we introduce the following definitions, which are extracted from [1].

Definition 2.1. Let U_k, V_k and $\widetilde{U}_k, \widetilde{V}_k$ be pairs of $n \times m$ and $n \times p$ matrix-valued functions for $k \in \mathbb{Z}$. Then

$$\begin{Bmatrix} U_k & \widetilde{U}_k \\ V_k & \widetilde{V}_k \end{Bmatrix} = U_k^* \widetilde{V}_k - V_k^* \widetilde{U}_k$$

is called the *Wronskian* of $\begin{pmatrix} U_k \\ V_k \end{pmatrix}$ and $\begin{pmatrix} \widetilde{U}_k \\ \widetilde{V}_k \end{pmatrix}$. If U_k, V_k is a solution of (2.1) such that

$$\begin{Bmatrix} U_k & U_k \\ V_k & V_k \end{Bmatrix} = 0 \quad \text{for } k \in \mathbb{Z},$$

then U_k, V_k is said to be a *prepared* solution of (2.1) and $\begin{pmatrix} U_k \\ V_k \end{pmatrix}$ is called a prepared matrix solution of (2.1).

If $\begin{pmatrix} U_k \\ V_k \end{pmatrix}$ and $\begin{pmatrix} \widetilde{U}_k \\ \widetilde{V}_k \end{pmatrix}$ $(a \leqslant k \leqslant b+1)$ are $2n \times n$ matrix-valued functions each of which is a prepared solution on $a \leqslant k \leqslant b$ and the Wronskian of $\begin{pmatrix} U_k \\ V_k \end{pmatrix}$ and $\begin{pmatrix} \widetilde{U}_k \\ \widetilde{V}_k \end{pmatrix}$ is nonsingular, then $\begin{pmatrix} U_k \\ V_k \end{pmatrix}$ and $\begin{pmatrix} \widetilde{U}_k \\ \widetilde{V}_k \end{pmatrix}$ is called a *fundamental solution set*. The matrix-valued function $\begin{pmatrix} U_k & \widetilde{U}_k \\ V_k & \widetilde{V}_k \end{pmatrix}$ is called a *fundamental matrix* for the system (2.1) defined for $k \in [a, b] \cap \mathbb{Z}$.

Lemma 2.2. *If U_k, V_k and $\widetilde{U}_k, \widetilde{V}_k$ $(a \leqslant k \leqslant b+1)$ are $n \times n$ matrix-valued functions each of which is a prepared solution on $a \leqslant k \leqslant b$, then the following conditions are equivalent:*

Case (I). The Wronskian of $\begin{pmatrix} U_k \\ V_k \end{pmatrix}$ and $\begin{pmatrix} \widetilde{U}_k \\ \widetilde{V}_k \end{pmatrix}$ is nonsingular.

Case (II). If $\begin{pmatrix} X_k \\ Y_k \end{pmatrix}$ is a $2n \times n$ solution, then there exist unique $n \times n$ matrices C_1 and C_2 such that

$$\begin{pmatrix} X_k \\ Y_k \end{pmatrix} = \begin{pmatrix} U_k \\ V_k \end{pmatrix} C_1 + \begin{pmatrix} \widetilde{U}_k \\ \widetilde{V}_k \end{pmatrix} C_2 \quad \text{for } a \leqslant k \leqslant b.$$

The following lemma is an immediate corollary of [1, Corollary 3.3].

Lemma 2.3. *Let $Z_k(\lambda)$ be $2n \times n$ matrix-valued solutions of (2.1) for $a \leqslant k \leqslant b$. Then*

$$Z_k^*(\bar{\lambda}) \mathcal{J} Z_k(\lambda) = Z_a^*(\bar{\lambda}) \mathcal{J} Z_a(\lambda) \quad \text{for all } a \leqslant k \leqslant b.$$

In the rest of the paper we use the following notation for the imaginary part of a complex number or matrix:

$$\Im\lambda = \frac{\lambda - \bar{\lambda}}{2i} \quad \text{and} \quad \Im M = \frac{M - M^*}{2i}.$$

Now we consider the system (2.1) for $a \leqslant k \leqslant b+1$ with the formally self-adjoint boundary conditions

$$\alpha z_a = 0, \quad \beta z_{b+1} = 0, \tag{2.3}$$

where α and β are $n \times 2n$ matrices satisfying the self-adjoint conditions

$$\begin{aligned} &\operatorname{rank} \alpha = n, \quad \alpha\alpha^* = I_n, \quad \alpha\mathcal{J}\alpha^* = 0, \\ &\operatorname{rank} \beta = n, \quad \beta\beta^* = I_n, \quad \beta\mathcal{J}\beta^* = 0. \end{aligned} \tag{2.4}$$

Let $\theta_k(\lambda)$ and $\phi_k(\lambda)$ be the $2n \times n$ matrix-valued solutions of (2.1) satisfying

$$\theta_a(\lambda) = \alpha^* \quad \text{and} \quad \phi_a(\lambda) = \mathcal{J}\alpha^*.$$

It is clear that $\alpha\theta_a(\lambda) = I_n$ and $\alpha\phi_a(\lambda) = 0$. Set $\Phi = (\theta \ \phi)$. Then it is easy to see that $\Phi_k(\lambda)$ is the fundamental matrix for (2.1) satisfying $\Phi_a(\lambda) = (\alpha^* \ \mathcal{J}\alpha^*)$.

Lemma 2.4. *Let $Z_k(\lambda)$ be the fundamental matrix for (2.1) satisfying $Z_a(\lambda) = (\alpha^* \ \mathcal{J}\alpha^*)$. Then*

$$Z_k^*(\bar{\lambda}) \mathcal{J} Z_k(\lambda) = Z_k(\lambda) \mathcal{J} Z_k^*(\bar{\lambda}) = \mathcal{J} \quad \text{for all } k \in \mathbb{Z}.$$

Proof. From Lemma 2.3,

$$Z_k^*(\bar{\lambda}) \mathcal{J} Z_k(\lambda) = Z_a^*(\bar{\lambda}) \mathcal{J} Z_a(\lambda) = \begin{pmatrix} \alpha \\ \alpha\mathcal{J}^* \end{pmatrix} \mathcal{J} (\alpha^* \quad \mathcal{J}\alpha^*) = \mathcal{J} \quad \text{for all } k \in \mathbb{Z}.$$

Furthermore, $-\mathcal{J} Z_k^*(\bar{\lambda}) \mathcal{J} Z_k(\lambda) = I_{2n}$ for $k \in \mathbb{Z}$ implies $\mathcal{J} Z_k(\lambda)(-\mathcal{J} Z_k^*(\bar{\lambda})) = I_{2n}$ for $k \in \mathbb{Z}$. It follows that $Z_k(\lambda) \mathcal{J} Z_k^*(\bar{\lambda}) = \mathcal{J}$ for $k \in \mathbb{Z}$. This completes the proof. □

As usual, λ is called an eigenvalue of (2.1) defined on $[a, b+1] \cap \mathbb{Z}$ with (2.3) if a nontrivial solution $z = \begin{pmatrix} x \\ u \end{pmatrix}$ exists on $[a, b+1] \cap \mathbb{Z}$. Let $z = \begin{pmatrix} x \\ u \end{pmatrix}$ and $\tilde{z} = \begin{pmatrix} \tilde{x} \\ \tilde{u} \end{pmatrix}$ solve (2.1). Define $\langle \cdot, \cdot \rangle$ by

$$\langle z, \tilde{z} \rangle := \sum_{k=a}^{b} x_{k+1}^* \mathcal{W}_k \tilde{x}_{k+1}.$$

Lemma 2.5. *λ is an eigenvalue of the problem (2.1) and (2.3) if and only if $\det(\beta\phi_{b+1}(\lambda)) = 0$, and $z_k(\lambda)$ is a corresponding eigenfunction if and only if there exists a vector $\xi \in \mathbb{C}^n$ such that $z_k(\lambda) = \phi_k(\lambda)\xi$ for $k \in \mathbb{Z}$, where ξ is a nonzero solution of the equation $\beta\phi_{b+1}(\lambda)\xi = 0$.*

Proof. Let λ be an eigenvalue of the eigenvalue problem (2.1) and (2.3) with corresponding eigenfunction $z_k(\lambda)$. Then there exists a unique constant vector $\eta \in \mathbb{C}^{2n} \setminus \{0\}$ such that

$$z_k(\lambda) = \Phi_k(\lambda)\eta \quad \text{for all } k \in \mathbb{Z}.$$

Then, using (2.3) and (2.4),

$$0 = \alpha z_a(\lambda) = \alpha\Phi_a(\lambda)\eta = \alpha(\, \alpha^* \quad \mathcal{J}\alpha^* \,)\eta = (\, I_n \quad 0 \,)\eta = \zeta,$$

where $\eta = \begin{pmatrix} \zeta \\ \xi \end{pmatrix}$ with $\zeta, \xi \in \mathbb{C}^n$. Thus $z_k(\lambda) = \phi_k(\lambda)\xi$, and (2.3) implies $\beta\phi_{b+1}(\lambda)\xi = 0$. Clearly, $\xi \neq 0$, since $z_k(\lambda) \neq 0$. Hence ξ is a nonzero solution of $\beta\phi_{b+1}(\lambda)\xi = 0$. Thus $\det(\beta\phi_{b+1}(\lambda)) = 0$.

Conversely, if λ satisfies $\det(\beta\phi_{b+1}(\lambda)) = 0$, then $\beta\phi_{b+1}(\lambda)\xi = 0$ has a nonzero solution ξ. Let $z_k(\lambda) = \phi_k(\lambda)\xi$. Then $\beta z_{b+1}(\lambda) = 0$. Moreover, $\alpha z_a(\lambda) = \alpha\phi_a(\lambda)\xi = \alpha\mathcal{J}\alpha^* = 0$ by (2.4). Taking into account rank $\phi_a(\lambda) = \text{rank}\,(\mathcal{J}\alpha^*) = n$, we get that $z_k(\lambda)$ is a nontrivial solution of (2.1). This completes the proof. □

Lemma 2.6. *Let $z(\lambda) = \begin{pmatrix} x(\lambda) \\ u(\lambda) \end{pmatrix}$ and $z(\nu) = \begin{pmatrix} x(\nu) \\ u(\nu) \end{pmatrix}$ be any $n \times p$ ($p \geqslant 1$) solutions of (2.1) corresponding to the parameters $\lambda,\ \nu \in \mathbb{C}$. Then*

$$z_m^*(\nu)\mathcal{J}z_m(\lambda) - z_l^*(\nu)\mathcal{J}z_l(\lambda) = (\lambda - \bar{\nu})\sum_{k=l}^{m-1} x_{k+1}^*(\nu)\mathcal{W}_k x_{k+1}(\lambda). \tag{2.5}$$

In particular,

$$z_m^*(\lambda)\mathcal{J}z_m(\lambda) - z_l^*(\lambda)\mathcal{J}z_l(\lambda) = 2i\Im\lambda\sum_{k=l}^{m-1} x_{k+1}^*(\lambda)\mathcal{W}_k x_{k+1}(\lambda). \tag{2.6}$$

Proof. From [1, Theorem 3.2], we have

$$z_m^*(\nu)\mathcal{J}z_m(\lambda) - z_l^*(\nu)\mathcal{J}z_l(\lambda) = \sum_{k=l}^{m-1} \left[z_k^*(\nu)(\mathcal{J}\Delta z_k(\lambda)) - (\mathcal{J}\Delta z_k(\nu))^* z_{k+1}(\lambda)\right]. \tag{2.7}$$

Using (2.1) and (2.2) in (2.7), we conclude that (2.5) holds. □

Lemma 2.7. *Under Assumption 2.2, all eigenvalues of (2.1) and (2.3) are real, and eigenvectors corresponding to different eigenvalues are orthogonal.*

Proof. Let λ be an eigenvalue of (2.1) and (2.3) with corresponding eigenfunction $z_k(\lambda)$. Hence $z_k(\lambda)$ satisfies (2.1) and

$$y_a(\lambda) \in \text{Ker}\ \alpha = \text{Im}\ \mathcal{J}\alpha^* \quad \text{and} \quad y_{b+1}(\lambda) \in \text{Ker}\ \beta = \text{Im}\ \mathcal{J}\beta^*,$$

which follows from (2.4) and [16, Corollary 3.1.3]. Thus there exist $c_1, c_2 \in \mathbb{C}^n$ such that

$$z_a(\lambda) = \mathcal{J}\alpha^* c_1 \quad \text{and} \quad z_{b+1}(\lambda) = \mathcal{J}\beta^* c_2.$$

Using (2.4) and (2.6), we have

$$2i\Im\lambda\sum_{k=a}^{b} x_{k+1}^*(\lambda)\mathcal{W}_k x_{k+1}(\lambda) = z_{b+1}^*(\lambda)\mathcal{J}z_{b+1}(\lambda) - z_a^*(\lambda)\mathcal{J}z_a(\lambda) = c_2^*\beta\mathcal{J}^*\mathcal{J}\mathcal{J}\beta^* c_2 - c_1^*\alpha\mathcal{J}^*\mathcal{J}\mathcal{J}\alpha^* c_1 = 0,$$

so that $\Im\lambda = 0$ and $\lambda \in \mathbb{R}$ by Assumption 2.2. Now let $z(\lambda)$ and $z(\nu)$ be eigenfunctions corresponding to the eigenvalues $\lambda \neq \nu$. Then using Lemma 2.6 and proceeding as above, we have

$$(\lambda - \nu)\sum_{k=a}^{b} x_{k+1}^*(\nu)\mathcal{W}_k x_{k+1}(\lambda) = z_{b+1}^*(\nu)\mathcal{J}z_{b+1}(\lambda) - z_a^*(\nu)\mathcal{J}z_a(\lambda) = 0.$$

Hence $z(\lambda)$ and $z(\nu)$ are orthogonal. □

3. Weyl–Titchmarsh circles and disks

In this section, we consider the construction of Weyl–Titchmarsh disks and circles for symplectic systems (2.1). Assume (2.4) and let α, β be defined as in Section 2. Suppose $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and set

$$\chi_k(b,\lambda) = \begin{pmatrix} \theta_k(\lambda) & \phi_k(\lambda) \end{pmatrix} \begin{pmatrix} I_n \\ M_\beta(b,\lambda) \end{pmatrix}, \quad \text{where } M_\beta(b,\lambda) = -(\beta\phi_{b+1}(\lambda))^{-1}\beta\theta_{b+1}(\lambda)$$

(observe Lemmas 2.5 and 2.7). For any $n \times n$ matrix M, define

$$\mathcal{E}(M,b,\lambda) := -\mathrm{i}\ \mathrm{sgn}(\Im\lambda)(I_n\ M^*)\Phi^*_{b+1}(\lambda)\mathcal{J}\Phi_{b+1}(\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix}$$

and

$$\chi_k(\lambda) = \begin{pmatrix} \theta_k(\lambda) & \phi_k(\lambda) \end{pmatrix}\begin{pmatrix} I_n \\ M \end{pmatrix}, \quad \chi_k(b,\lambda) = \begin{pmatrix} \theta_k(\lambda) & \phi_k(\lambda) \end{pmatrix}\begin{pmatrix} I_n \\ M(b,\lambda) \end{pmatrix}.$$

It is clear that

$$\mathcal{E}(M,b,\lambda) = -\mathrm{i}\ \mathrm{sgn}(\Im\lambda)\chi^*_{b+1}(\lambda)\mathcal{J}\chi_{b+1}(\lambda).$$

Definition 3.1. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The sets

$$\mathfrak{D}(b,\lambda) = \{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}(M,b,\lambda) \leqslant 0\} \quad \text{and} \quad \mathcal{K}(b,\lambda) = \{M \in \mathbb{C}^{n\times n} \mid \mathcal{E}(M,b,\lambda) = 0\}$$

are called a *Weyl disk* and a *Weyl circle*, respectively.

Theorem 3.1. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathcal{K}(b,\lambda) = \{M_\beta(b,\lambda) \mid \beta \text{ satisfies } (2.4)\}.$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Assume that β satisfies (2.4). Let $\eta \in \mathbb{C}^{2n}$. Then $\beta\chi_{b+1}(b,\lambda)\eta = 0$ so that (use again (2.4) and [16, Corollary 3.1.3])

$$\chi_{b+1}(b,\lambda)\eta \in \mathrm{Ker}\ \beta = \mathrm{Im}\ \mathcal{J}^*\beta,$$

and thus there exists $c \in \mathbb{C}^n$ such that $\chi_{b+1}(b,\lambda)\eta = \mathcal{J}^*\beta c$. Hence

$$\eta^*\chi^*_{b+1}(b,\lambda)\mathcal{J}\chi_{b+1}(b,\lambda)\eta = c^*\beta^*\mathcal{J}\mathcal{J}\mathcal{J}^*\beta c = c^*\beta^*\mathcal{J}\beta c = 0$$

by (2.4). So $\chi^*_{b+1}(b,\lambda)\mathcal{J}\chi_{b+1}(b,\lambda) = 0$, that is, $\mathcal{E}(M_\beta(b,\lambda),b,\lambda) = 0$.

Conversely, if $\mathcal{E}(M,b,\lambda) = 0$, then

$$0 = (I_n\ M^*)\begin{pmatrix} \theta^*_{b+1}(\lambda) \\ \phi^*_{b+1}(\lambda) \end{pmatrix}\mathcal{J}\begin{pmatrix} \theta_{b+1}(\lambda) & \phi_{b+1}(\lambda) \end{pmatrix}\begin{pmatrix} I_n \\ M \end{pmatrix} = \gamma\mathcal{J}\gamma^*,$$

where

$$\gamma = (I_n\ M^*)\begin{pmatrix} \theta^*_{b+1}(\lambda) \\ \phi^*_{b+1}(\lambda) \end{pmatrix}\mathcal{J}.$$

Then rank $\gamma = n$ and $\gamma\chi_{b+1}(\lambda) = 0$. Since

$$\gamma\gamma^* = (I_n\ M^*)\begin{pmatrix} \theta^*_{b+1}(\lambda) \\ \phi^*_{b+1}(\lambda) \end{pmatrix}\begin{pmatrix} \theta_{b+1}(\lambda) & \phi_{b+1}(\lambda) \end{pmatrix}\begin{pmatrix} I_n \\ M \end{pmatrix} > 0,$$

we can define $\beta = (\gamma\gamma^*)^{-\frac{1}{2}}\gamma$. Then β satisfies the self-adjoint conditions (2.4) and $\beta(\theta_{b+1}(\lambda)\ \phi_{b+1}(\lambda))\begin{pmatrix} I_n \\ M \end{pmatrix} = 0$. It follows that $M = -(\beta\phi_{b+1}(\lambda))^{-1}\beta\theta_{b+1}(\lambda) = M_\beta(b,\lambda)$. □

Let

$$\mathcal{F}(b,\lambda) := -\mathrm{i}\ \mathrm{sgn}(\Im\lambda)\begin{pmatrix} \theta^*_{b+1}(\lambda) \\ \phi^*_{b+1}(\lambda) \end{pmatrix}\mathcal{J}\begin{pmatrix} \theta_{b+1}(\lambda) & \phi_{b+1}(\lambda) \end{pmatrix}. \tag{3.1}$$

Then $\mathcal{F}(b,\lambda)$ is a $2n \times 2n$ Hermitian matrix and

$$\mathcal{E}(M,b,\lambda) = (I_n\ M^*)\mathcal{F}(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix}. \tag{3.2}$$

Denote

$$\theta_k(\lambda) = \begin{pmatrix} \theta_k^{(1)}(\lambda) \\ \theta_k^{(2)}(\lambda) \end{pmatrix} \quad \text{with} \quad \theta_k^{(1)}(\lambda),\ \theta_k^{(2)}(\lambda) \in \mathbb{C}^{n\times n},$$
$$\phi_k(\lambda) = \begin{pmatrix} \phi_k^{(1)}(\lambda) \\ \phi_k^{(2)}(\lambda) \end{pmatrix} \quad \text{with} \quad \phi_k^{(1)}(\lambda),\ \phi_k^{(2)}(\lambda) \in \mathbb{C}^{n\times n}.$$

Define

$$\widehat{\mathcal{W}}_k := \begin{pmatrix} \mathcal{W}_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $\widehat{\mathcal{W}}_k$ is nonnegative definite and

$$\sum_{k=p}^{q} x_{k+1}^* \widehat{\mathcal{W}}_k x_{k+1} > 0 \quad \text{for all } p, q \in \mathbb{Z} \text{ with } q > p$$

for any nontrivial solution $z = \begin{pmatrix} x \\ u \end{pmatrix}$ of (2.1).

Lemma 3.1. *For $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $b \geqslant a$, we have*

$$\mathcal{F}(b,\lambda) = \operatorname{sgn}(\Im\lambda)\left[-\mathrm{i}\mathcal{J} + 2\Im\lambda \sum_{k=a}^{b} \begin{pmatrix} \theta_{k+1}^{(1)*}(\lambda) \\ \phi_{k+1}^{(1)*}(\lambda) \end{pmatrix} \mathcal{W}_k \begin{pmatrix} \theta_{k+1}^{(1)}(\lambda) & \phi_{k+1}^{(1)}(\lambda) \end{pmatrix} \right] \tag{3.3}$$

and

$$\sum_{k=a}^{b} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) = \frac{1}{2|\Im\lambda|} \mathcal{E}(M,b,\lambda) + \frac{\Im M}{\Im\lambda}. \tag{3.4}$$

Proof. From Lemma 2.6, we obtain

$$\begin{aligned} \Phi_{b+1}^*(\lambda)\mathcal{J}\Phi_{b+1}(\lambda) &= \Phi_a^*(\lambda)\mathcal{J}\Phi_a(\lambda) + 2\mathrm{i}\Im\lambda \sum_{k=a}^{b} \begin{pmatrix} \theta_{k+1}^{(1)*}(\lambda) \\ \phi_{k+1}^{(1)*}(\lambda) \end{pmatrix} \mathcal{W}_k \begin{pmatrix} \theta_{k+1}^{(1)}(\lambda) & \phi_{k+1}^{(1)}(\lambda) \end{pmatrix} \\ &= \mathcal{J} + 2\mathrm{i}\Im\lambda \sum_{k=a}^{b} \begin{pmatrix} \theta_{k+1}^{(1)*}(\lambda) \\ \phi_{k+1}^{(1)*}(\lambda) \end{pmatrix} \mathcal{W}_k \begin{pmatrix} \theta_{k+1}^{(1)}(\lambda) & \phi_{k+1}^{(1)}(\lambda) \end{pmatrix}, \end{aligned}$$

and so (3.3) follows from (3.1). From (3.2) and (3.3) we obtain

$$\begin{aligned} \sum_{k=a}^{b} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) &= (I_n\ M^*) \sum_{k=a}^{b} \begin{pmatrix} \theta_{k+1}^{(1)*}(\lambda) \\ \phi_{k+1}^{(1)*}(\lambda) \end{pmatrix} \mathcal{W}_k \begin{pmatrix} \theta_{k+1}^{(1)}(\lambda) & \phi_{k+1}^{(1)}(\lambda) \end{pmatrix} \begin{pmatrix} I_n \\ M \end{pmatrix} \\ &= \frac{1}{2|\Im\lambda|}(I_n\ M^*)[\mathcal{F}(b,\lambda) + \mathrm{i}\ \operatorname{sgn}(\Im\lambda)\mathcal{J}] \begin{pmatrix} I_n \\ M \end{pmatrix} \\ &= \frac{1}{2|\Im\lambda|}(I_n\ M^*)\mathcal{F}(b,\lambda)\begin{pmatrix} I_n \\ M \end{pmatrix} + \frac{\Im M}{\Im\lambda} = \frac{1}{2|\Im\lambda|}\mathcal{E}(M,b,\lambda) + \frac{\Im M}{\Im\lambda}. \end{aligned} \tag{3.5}$$

This completes the proof. □

Theorem 3.2. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathfrak{D}(b_2,\lambda) \subset \mathfrak{D}(b_1,\lambda) \quad \text{for any } b_1, b_2 \in \mathbb{Z} \text{ with } b_1 < b_2.$$

Proof. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $b_1 < b_2$. Assume $M \in \mathfrak{D}(b_2,\lambda)$. Then $\mathcal{E}(M,b_2,\lambda) \leqslant 0$. By Lemma 3.1 and Assumption 2.2

$$\mathcal{F}(b_2,\lambda) - \mathcal{F}(b_1,\lambda) = 2|\Im\lambda| \sum_{k=b_1+1}^{b_2} \begin{pmatrix} \theta_{k+1}^{(1)*}(\lambda) \\ \phi_{k+1}^{(1)*}(\lambda) \end{pmatrix} \mathcal{W}_k \begin{pmatrix} \theta_{k+1}^{(1)}(\lambda) & \phi_{k+1}^{(1)}(\lambda) \end{pmatrix} \geqslant 0,$$

which implies that $\mathcal{E}(M,b_2,\lambda) \geqslant \mathcal{E}(M,b_1,\lambda)$. From this we have $\mathcal{E}(M,b_1,\lambda) \leqslant 0$. Thus $M \in \mathfrak{D}(b_1,\lambda)$. □

Now we study convergence of the disks. For this purpose, we denote

$$\mathcal{F}(b,\lambda)=\begin{pmatrix}F_{11}(b,\lambda) & F_{12}(b,\lambda)\\ F_{12}^*(b,\lambda) & F_{22}(b,\lambda)\end{pmatrix}, \tag{3.6}$$

where $F_{11}(b,\lambda)$, $F_{12}(b,\lambda)$ and $F_{22}(b,\lambda)$ are $n\times n$ matrices.

Lemma 3.2. *For $\lambda\in\mathbb{C}\setminus\mathbb{R},\ F_{11}(b,\lambda),\ F_{22}(b,\lambda)$ are positive definite and nondecreasing in b.*

Proof. From (3.3) and (3.6), we have

$$F_{11}(b,\lambda)=2|\Im\lambda|\sum_{k=a}^{b}\theta_{k+1}^{(1)*}(\lambda)\mathcal{W}(t)\theta_{k+1}^{(1)}(\lambda),$$
$$F_{22}(b,\lambda)=2|\Im\lambda|\sum_{k=a}^{b}\phi_{k+1}^{(1)*}(\lambda)\mathcal{W}(t)\phi_{k+1}^{(1)}(\lambda).$$

Using Assumption 2.2 completes the proof. □

Using the notation of (3.6), we find that (3.2) can be rewritten as

$$\begin{aligned}\mathcal{E}(M,b,\lambda)&=M^*F_{22}(b,\lambda)M+F_{12}(b,\lambda)M+M^*F_{12}^*(b,\lambda)+F_{11}(b,\lambda)\\&=\left[M+F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)\right]^*F_{22}(b,\lambda)\left[M+F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)\right]+F_{11}(b,\lambda)-F_{12}(b,\lambda)F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda).\end{aligned} \tag{3.7}$$

Lemma 3.3. *For $\lambda\in\mathbb{C}\setminus\mathbb{R},\ F_{12}(b,\lambda)F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)-F_{11}(b,\lambda)=F_{22}^{-1}(b,\bar\lambda)$.*

Proof. By applying Lemma 2.4 twice, we find

$$\begin{aligned}\mathcal{F}^*(b,\lambda)\mathcal{J}\mathcal{F}(b,\bar\lambda)&=(-\mathrm{i}^*\ \mathrm{sgn}(\Im\lambda))(-\mathrm{i}\ \mathrm{sgn}(\Im\bar\lambda))\Phi_{b+1}^*(\lambda)\mathcal{J}^*\Phi_{b+1}(\lambda)\mathcal{J}\Phi_{b+1}^*(\bar\lambda)\mathcal{J}\Phi_{b+1}(\bar\lambda)=-\Phi_{b+1}^*(\lambda)\mathcal{J}^*\mathcal{J}\mathcal{J}\Phi_{b+1}(\bar\lambda)\\&=-\Phi_{b+1}^*(\lambda)\mathcal{J}\Phi_{b+1}(\bar\lambda)=-\mathcal{J}.\end{aligned}$$

Hence

$$F_{12}(b,\lambda)F_{12}(b,\bar\lambda)-F_{11}(b,\lambda)F_{22}(b,\bar\lambda)=I_n\quad\text{and}\quad F_{22}(b,\lambda)F_{12}(b,\bar\lambda)-F_{12}^*(b,\lambda)F_{22}(b,\bar\lambda)=0. \tag{3.8}$$

From the second relation in (3.8), we have (observe Lemma 3.1)

$$F_{12}(b,\bar\lambda)F_{22}^{-1}(b,\bar\lambda)=F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda),$$

and hence, using also the first relation in (3.8) we obtain

$$\begin{aligned}F_{12}(b,\lambda)F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)-F_{11}(b,\lambda)&=F_{12}(b,\lambda)F_{12}(b,\bar\lambda)F_{22}^{-1}(b,\bar\lambda)-F_{11}(b,\lambda)=(I_n+F_{11}(b,\lambda)F_{22}(b,\bar\lambda))F_{22}^{-1}(b,\bar\lambda)-F_{11}(b,\lambda)\\&=F_{22}^{-1}(b,\bar\lambda),\end{aligned}$$

which completes the proof. □

From Lemma 3.3, (3.2), and hence (3.7) can be rewritten in the form

$$\mathcal{E}(M,b,\lambda)=(M-\mathfrak{C}(b,\lambda))^*\mathcal{R}^{-2}(b,\lambda)(M-\mathfrak{C}(b,\lambda))-\mathcal{R}^2(b,\bar\lambda), \tag{3.9}$$

where

$$\mathfrak{C}(b,\lambda)=-F_{22}^{-1}(b,\lambda)F_{12}^*(b,\lambda)\quad\text{and}\quad\mathcal{R}(b,\lambda)=F_{22}^{-\frac{1}{2}}(b,\lambda).$$

Definition 3.2. $\mathfrak{C}(b,\lambda)$ is called the *center* of the Weyl disk $\mathfrak{D}(b,\lambda)$ or the Weyl circle $\mathcal{K}(b,\lambda)$, while $\mathcal{R}(b,\lambda)$ and $\mathcal{R}(b,\bar\lambda)$ are called the matrix *radii* of $\mathfrak{D}(b,\lambda)$ or $\mathcal{K}(b,\lambda)$.

Remark 3.1. Theorems 3.3–3.6 and their proofs are the same as in [18, Theorems 4.8–4.10,4.12]. In order to maintain completeness, we still present them here.

Theorem 3.3. *Define the unit matrix circle and the unit matrix disc by*

$$\partial D=\{U\in\mathbb{C}^{n\times n}\mid U^*U=I_n\}\quad\text{and}\quad D=\{V\in\mathbb{C}^{n\times n}\mid V^*V\leqslant I_n\},$$

respectively. Then

$$\mathcal{K}(b,\lambda) = \{\mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)U\mathcal{R}(b,\bar{\lambda}) \mid U \in \partial D\}$$

and

$$\mathfrak{D}(b,\lambda) = \{\mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)V\mathcal{R}(b,\bar{\lambda}) \mid V \in D\}.$$

Proof. We only prove the first statement as the second one can be shown similarly. From (3.9),

$$\mathcal{E}(M,b,\lambda) = 0 \quad \text{if and only if } [\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\bar{\lambda})]^*[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\bar{\lambda})] = I_n. \tag{3.10}$$

First, let $M \in \mathcal{K}(b,\lambda)$ and put

$$U = \mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\bar{\lambda}).$$

Then

$$M = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)U\mathcal{R}(b,\bar{\lambda}),$$

and (3.10) yields $U^*U = I_n$. Conversely, let U be unitary and define

$$M = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda)U\mathcal{R}(b,\bar{\lambda}).$$

Then

$$U = \mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\bar{\lambda})$$

so that

$$[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\bar{\lambda})]^*[\mathcal{R}^{-1}(b,\lambda)(M - \mathfrak{C}(b,\lambda))\mathcal{R}^{-1}(b,\bar{\lambda})] = I_n,$$

and hence (3.10) yields $M \in \mathcal{K}(b,\lambda)$. □

Theorem 3.4. *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lim_{b\to\infty} \mathcal{R}(b,\lambda)$ *exists and* $\lim_{b\to\infty} \mathcal{R}(b,\lambda) \geqslant 0$.

Proof. From Lemma 3.2, $F_{22}(b,\lambda) > 0$ is Hermitian and nondecreasing in b. Thus $\mathcal{R}(b,\lambda) = F_{22}^{-\frac{1}{2}}(b,\lambda) > 0$ is Hermitian and nonincreasing in b. Hence $\lim_{b\to\infty} \mathcal{R}(b,\lambda)$ exists and is nonnegative definite. □

Theorem 3.5. *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\lim_{b\to\infty} \mathfrak{C}(b,\lambda)$ *exists.*

Proof. Let $b_1, b_2 \in \mathbb{Z}$ with $b_1 < b_2$. Let $V \in D$ and define

$$M = \mathfrak{C}(b_2,\lambda) + \mathcal{R}(b_2,\lambda)V\mathcal{R}(b_2,\bar{\lambda}).$$

By Theorem 3.3, $M \in \mathfrak{D}(b_2,\lambda)$. Hence by Theorem 3.2, $M \in \mathfrak{D}(b_1,\lambda)$. Again by Theorem 3.3, there exists $\Psi(V) \in D$ with

$$M = \mathfrak{C}(b_1,\lambda) + \mathcal{R}(b_1,\lambda)\Psi(V)\mathcal{R}(b_1,\bar{\lambda}).$$

Thus $\Psi\colon D \to D$ satisfies

$$\Psi(V) = \mathcal{R}^{-1}(b_1,\lambda)[\mathfrak{C}(b_2,\lambda) - \mathfrak{C}(b_1,\lambda) + \mathcal{R}(b_2,\lambda)V\mathcal{R}(b_2,\bar{\lambda})]\mathcal{R}^{-1}(b_1,\bar{\lambda}) \tag{3.11}$$

for all $V \in D$. This implies

$$\Psi(V_I) - \Psi(V_{II}) = \mathcal{R}^{-1}(b_1,\lambda)\mathcal{R}(b_1,\lambda)[V_I - V_{II}]\mathcal{R}(b_2,\bar{\lambda})\mathcal{R}^{-1}(b_2,\bar{\lambda})$$

for all $V_I, V_{II} \in D$. Thus $\Psi\colon D \to D$ is continuous and hence has a fixed point $\widetilde{V} \in D$ by Brouwer's fixed point theorem. Letting $\Psi(\widetilde{V}) = \widetilde{V}$ in (3.11), we have

$$\begin{aligned}
\|\mathfrak{C}(b_2,\lambda) - \mathfrak{C}(b_1,\lambda)\| &= \|\mathcal{R}(b_1,\lambda)\widetilde{V}\mathcal{R}(b_1,\bar{\lambda}) - \mathcal{R}(b_2,\lambda)\widetilde{V}\mathcal{R}(b_2,\bar{\lambda})\| \\
&\leqslant \|\mathcal{R}(b_1,\lambda)\widetilde{V}\mathcal{R}(b_1,\bar{\lambda}) - \mathcal{R}(b_1,\lambda)\widetilde{V}\mathcal{R}(b_2,\bar{\lambda})\| + \|\mathcal{R}(b_1,\lambda)\widetilde{V}\mathcal{R}(b_2,\bar{\lambda}) - \mathcal{R}(b_2,\lambda)\widetilde{V}\mathcal{R}(b_2,\bar{\lambda})\| \\
&\leqslant \|\mathcal{R}(b_1,\lambda)\|\|\mathcal{R}(b_2,\bar{\lambda}) - \mathcal{R}(b_1,\bar{\lambda})\| + \|\mathcal{R}(b_2,\lambda) - \mathcal{R}(b_1,\lambda)\|\|\mathcal{R}(b_2,\bar{\lambda})\|,
\end{aligned}$$

where $\|\cdot\|$ is a matrix norm. Using Theorem 3.4 completes the proof. □

Definition 3.3. Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and define

$$\mathfrak{C}_0(\lambda) := \lim_{b\to\infty} \mathfrak{C}(b,\lambda) \quad \text{and} \quad \mathcal{R}_0(\lambda) := \lim_{b\to\infty} \mathcal{R}(b,\lambda).$$

Then $\mathfrak{C}_0(\lambda)$ is called the *center* and $\mathcal{R}_0(\lambda)$ and $\mathcal{R}_0(\bar{\lambda})$ are called the matrix *radii* of the limiting set

$$\mathfrak{D}_0(\lambda) := \{\mathfrak{C}_0(\lambda) + \mathcal{R}_0(\lambda) V \mathcal{R}_0(\bar{\lambda}) \mid V \in D\}.$$

The following result gives another expression for $\mathfrak{D}_0(\lambda)$.

Theorem 3.6. *The set* $\mathfrak{D}_0(\lambda)$ *is given by* $\mathfrak{D}_0(\lambda) = \bigcap_{b\geqslant a} \mathfrak{D}(b,\lambda)$.

Proof. If $M \in \mathfrak{D}_0(\lambda)$, then there exists $V \in \mathfrak{D}$ such that $M = \mathfrak{C}_0(\lambda) + \mathcal{R}_0(\lambda) V \mathcal{R}_0(\bar{\lambda})$. Hence $M = \lim_{b\to\infty} M(b)$, where we have $M(b) = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda) V \mathcal{R}(b,\bar{\lambda})$. Let $\tilde{b} \geqslant a$. Then $M(b) \in \mathfrak{D}(b,\lambda) \subset \mathfrak{D}(\tilde{b},\lambda)$ for all $b \geqslant \tilde{b}$ by Theorem 3.2 and thus $M = \lim_{b\to\infty} M(b) \in \mathfrak{D}(\tilde{b},\lambda)$. Therefore $M \in \bigcap_{b\geqslant a} \mathfrak{D}(b,\lambda)$.

Conversely, if $M \in \bigcap_{b\geqslant a} \mathfrak{D}(b,\lambda)$, then for all $b \geqslant a$, there exists $V_b \in \mathfrak{D}$ such that $M = \mathfrak{C}(b,\lambda) + \mathcal{R}(b,\lambda) V_b R(b,\bar{\lambda})$. Since $\mathfrak{D}$ is compact, there exist a sequence $\{b_k\}$ and $V \in \mathfrak{D}$ such that $V_{b_k} \to V$ as $k \to \infty$. Thus $M = \mathfrak{C}_0(\lambda) + \mathcal{R}_0(\lambda) V \mathcal{R}_0(\bar{\lambda}) \in \mathfrak{D}_0(\lambda)$. □

Theorem 3.7. *For all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$ *and for* $M \in \mathfrak{D}_0(\lambda)$, *we have* $\Im\lambda \cdot \Im M > 0$.

Proof. Assume that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and let $M \in \mathfrak{D}_0(\lambda)$. Fix an arbitrary $b > a$. From Theorem 3.6, $\mathfrak{D}_0(\lambda) \subset \mathfrak{D}(b,\lambda)$. Hence $M \in \mathfrak{D}(b,\lambda)$, and thus $\mathcal{E}(M,b,\lambda) \leqslant 0$. Therefore, (3.4) and Assumption 2.2 yield

$$\frac{\Im M}{\Im \lambda} \geqslant \sum_{k=a}^{b} \chi_{k+1}^*(\lambda) \mathcal{W}_k \chi_{k+1}(\lambda) > 0.$$

The proof is complete. □

Definition 3.4. Let M be an $n \times n$ matrix. We say that

(1) M lies in the limit circle, if $M \in \mathfrak{D}_0(\lambda)$;
(2) M lies on the boundary of the limit circle, if $M \in \mathfrak{D}_0(\lambda)$ and there exists a sequence $b_k \to \infty$ as $k \to \infty$ such that $\lim_{k\to\infty} \mathcal{E}(M, b_k, \lambda) = 0$.

Theorem 3.8. *Let* $M \in \mathbb{C}^{n\times n}$ *and* $\lambda \in \mathbb{C} \setminus \mathbb{R}$. *Then*

(1) M lies in the limit circle if and only if

$$\sum_{k=a}^{\infty} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) \leqslant \frac{\Im M}{\Im \lambda}. \tag{3.12}$$

(2) M lies on the boundary of the limit circle if and only if

$$\sum_{k=a}^{\infty} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) = \frac{\Im M}{\Im \lambda}. \tag{3.13}$$

Proof. First assume $M \in \mathfrak{D}_0(\lambda)$. Let $b > a$. Then $M \in \mathfrak{D}(b,\lambda)$ by Theorem 3.6. Hence $\mathcal{E}(M,b,\lambda) \leqslant 0$. From (3.4), we have that

$$\sum_{k=a}^{b} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) = \frac{1}{2|\Im\lambda|} \mathcal{E}(M,b,\lambda) + \frac{\Im M}{\Im \lambda} \leqslant \frac{\Im M}{\Im \lambda}.$$

Letting $b \to \infty$, we arrive at (3.12). Conversely, assume (3.12) holds. Let $b \geqslant a$. By Assumption 2.2,

$$\sum_{k=a}^{b} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) \leqslant \sum_{k=a}^{\infty} \chi_{k+1}^*(\lambda) \widehat{\mathcal{W}}_k \chi_{k+1}(\lambda) \leqslant \frac{\Im M}{\Im \lambda}.$$

So $\mathcal{E}(M,b,\lambda) \leqslant 0$ by (3.4). This shows that $M \in \mathfrak{D}(b,\lambda)$. Using Theorem 3.6 yields $M \in \mathfrak{D}_0(\lambda)$. This proves (1), and (2) can be concluded immediately by (1) and (3.4). □

Theorem 3.9. *Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then M lies on the boundary of the limit circle if and only if* $\lim_{c\to\infty} \chi_c^*(\lambda)\mathcal{J}\chi_c(\lambda) = 0$.

Proof. From Lemma 2.6, for any $t > a$, we have that

$$\chi_{c+1}^*(\lambda)\mathcal{J}\chi_{c+1}(\lambda) - \chi_a^*(\lambda)\mathcal{J}\chi_a(\lambda) = 2i\Im\lambda(I_n \quad M^*)\sum_{k=a}^{c}\begin{pmatrix}\theta_{k+1}^{(1)*} & \phi_{k+1}^{(1)*}\end{pmatrix}\mathcal{W}_k\begin{pmatrix}\theta_{k+1}^{(1)}\\ \phi_{k+1}^{(1)}\end{pmatrix}(I_n \quad M) = 2i\Im\lambda\sum_{k=a}^{c}\chi_{k+1}^*(\lambda)\widehat{\mathcal{W}}_k\chi_{k+1}(\lambda). \tag{3.14}$$

Since

$$\chi_a^*(\lambda)\mathcal{J}\chi_a(\lambda) = M^* - M = -2i\Im M,$$

we get

$$\chi_{c+1}^*(\lambda)\mathcal{J}\chi_{c+1}(\lambda) = 2i\Im\lambda\sum_{k=a}^{c}\chi_{k+1}^*(\lambda)\widehat{\mathcal{W}}_k\chi_{k+1}(\lambda). \tag{3.15}$$

From Theorem 3.8, $M(\lambda)$ is on the boundary of the limit circle if and only if

$$\Im\lambda\cdot\sum_{k=a}^{\infty}\chi_{k+1}^*(\lambda)\widehat{\mathcal{W}}_k\chi_{k+1}(\lambda) - \Im M = 0.$$

So by (3.15), we have that M is on the boundary of the limit circle if and only if

$$\lim_{c\to\infty}\chi_{c+1}^*(\lambda)\mathcal{J}\chi_{c+1}(\lambda) = 0.$$

This completes the proof. □

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