A continuum $X$ whose hyperspace of subcontinua is not its continuous image

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Abstract

We construct a metric continuum $X$ such that the hyperspace of subcontinua, $C(X)$, of $X$ is not a continuous image of $X$. This answers a question by I. Krzemińska and J. R. Prajs.

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INTRODUCTION

A continuum is a nonempty, nondegenerate compact connected metric space. Given a continuum $X$, we consider the following hyperspaces of $X$.

$$2^X = \{ A \subset X : A \text{ is closed and nonempty} \},$$

$$C(X) = \{ A \in 2^X : A \text{ is connected} \}.$$ 

Both are endowed with the Hausdorff metric $H$.

The problem of determining conditions for the existence of onto mappings between $X$, $C(X)$ and $2^X$ has been considered in [1], [2] and [3]. In [2, 3.6] it was proved that there is always a mapping from $2^X$ onto $C(X)$. In [2, 3.5], was also proved that there is a mapping from $X$ onto $2^X$ if and only if (a) $X$ is locally connected or (b) $X$ contains an open set with countably many components. Thus, if $X$ satisfies (a) or (b), then $C(X)$ is a continuous image of $X$. For a more detailed
discussion on this topic see Chapter IV of [3]. In [1, Question 2], I. Krzemińska and J. R. Prajs asked if there exists a continuum X with no continuous surjection from X onto \( C(X) \) (this question also was posted on the Open Problems on Continuum Theory website of J. R. Prajs and W. J. Charatonik). Here, we answered this question in the negative by constructing a continuum X such that \( C(X) \) is not a continuous image of X. The related questions: Is \( C(C(X)) \) a continuous image of \( C(X) \) for each continuum X? ([1, Question 3]) and; if \( Y \) is a continuous image of X, is \( C(Y) \) a continuous image of \( C(X) \)? ([1, Question 1]) remain open.

THE EXAMPLE

Let \( Z \) be the \( \sin(\frac{1}{x}) \)-continuum defined as the closure in the plane of the graph of the function \( \sin(\frac{1}{x}) \) with the interval \((0, 1]\) as its domain. Then \( Z \) is the union of the limit arc \( J = \{(0) \times [-1, 1]\) and the ray \( T = Z - J \). Consider the continuum \( Y \) obtained by identifying the points \( (0; 0), (0; 1) \) in \( Z \) and let \( h : Z \rightarrow Y \) be the quotient map. Then \( Y \) is the union of the simple closed curve \( C = h(J) \) and the ray \( h(T) \). The continuum \( Y \) constructed in this way is going to be called a packman having its limit circle \( C \), its ray \( h(T) \) and its peak point \( h((0; 1]) \).

Consider the circle \( S = \{(x, y, 0) \in E^3 : x^2 + y^2 = 1\} \) in the Euclidean space \( E^3 \). Fix a sequence \( \{z_n\}_{n=1}^{\infty} \) in \( S \) and a point \( z_0 \in S \) such that \( \lim z_n = z_0 \) and the points \( z_0, z_1, z_2, \ldots \) are pairwise different.

Consider a sequence of packman continua \( \{Y_n\}_{n=1}^{\infty} \) in \( E^3 \) with the following properties:
(a) \( S \) is the limit circle of \( Y_n \) for each \( n \in \mathbb{N} \),
(b) \( z_n \) is the peak point of \( Y_n \) for each \( n \in \mathbb{N} \),
(c) the rays \( R_1 = Y_1 - S, R_2 = Y_2 - S, \ldots \) are pairwise disjoint,
(d) \( \lim Y_n = S \).

The continuum X is then defined as \( X = Y_1 \cup Y_2 \cup \ldots \).

We can assume also that X satisfies the following property:
(e) given a path \( \alpha : [0, 1] \rightarrow S \) and \( \lambda > 0 \), there exists \( \delta > 0 \) such that, if \( z_n \in \alpha([0, 1]), q \in R_n \) and \( \|\alpha(0) - q\| < \delta \), then there exists a path \( \beta : [0, 1] \rightarrow R_n \) such that \( \beta(0) = q \) and \( \|\alpha(t) - \beta(t)\| < \lambda \) for each \( t \in [0, 1] \),
Observe that $X$ satisfies the following properties:

(f) if $J$ is an arc in $S$, $z_n \in J$ and $z_n$ is not an end point of $J$, then $J \notin \text{cl}_{C(X)}(C(R_n))$, where $\text{cl}_{C(X)}(C(R_n)) = \{ A \in C(X) : A \subset R_n \}$;

(g) if $J$ is an arc in $S - \{ z_n \}$, then $J \in \text{cl}_{C(X)}(C(R_n))$;

(h) $S$ is a terminal subcontinuum of $X$, that is, if $A \in C(X)$ and $A \cap S \neq \emptyset$, then $A \subset S$ or $S \subset A$.

We are going to prove that there is no a continuous map form $X$ onto $C(X)$.

By way of contradiction, suppose that there exists a continuous onto map $f : X \rightarrow C(X)$.

Given $n \in \mathbb{N}$ and a continuum $J \subset S - \{ z_n \}$, by Property (g), there exists a sequence of continua $\{ J_m \}_{m=1}^{\infty}$ such that $J_m \subset R_n$ for each $m \in \mathbb{N}$ and $\lim J_m = J$. Since $f$ is onto, for each $m \in \mathbb{N}$, there exists $p_m \in X$ such that $f(p_m) = J_m$. Since $X$ is compact, we can assume that $\lim p_m = p_0$, for some $p_0 \in X$. By continuity of $f$, $f(p_0) = J$. We need to prove the following.

**Claim 1.** If $\{ B_k \}_{k=1}^{\infty}$ is a sequence in $C(X)$ such that $p_0 \in B_k$ for each $k \in \mathbb{N}$ and $\lim B_k = \{ p_0 \}$, then there exists $k \in \mathbb{N}$ such that $B_k \cap \{ p_1, p_2, \ldots \} = \emptyset$.

In order to prove Claim 1, for each $k \in \mathbb{N}$, let $C_k = \bigcup \{ f(p) : p \in B_k \}$. It is easy to show that (see [3, Lemmas 1.43 and 1.48]) $J \subset C_k \subset C(X)$ for each $k \in \mathbb{N}$ and $\lim C_k = J$. Since $z_n \notin J$, there exists $k \in \mathbb{N}$ such that $z_n \notin C_k$. By Property (h), $C_k \subset S$. Thus $C_k \cap R_n = \emptyset$ and $C_k$ does not contain any set $J_m$. Hence, $B_k \cap \{ p_1, p_2, \ldots \} = \emptyset$.

As a consequence of Claim 1, we obtain that $X$ is not locally connected at $p_0$. Thus $p_0 \in S$. Another consequence of Claim 1 is that $S$ cannot contain infinitely many elements of the sequence $\{ p_m \}_{m=1}^{\infty}$.

In particular, we have that for each arc $J \subset S - \{ z_1 \}$, there exists a point $p_0 \in S$ such that $f(p_0) = S$. Taking a sequence of arcs in $S - \{ z_1 \}$ converging to $S$, we obtain that there exists a point $p \in S$ such that $f(p) = S$. Let $K = f^{-1}(\{ S \})$. Hence, $K \cap S \neq \emptyset$.

Since the set $F = \{ f(z_n) \in C(X) : n \in \{ 0, 1, 2, \ldots \} \}$ is countable, we can choose a point $q_0 \in S$ such that $\{ q_0 \} \notin F$.

For each $n \in \mathbb{N}$, fix a sequence of points $\{ q_m^{(n)} \}_{m=1}^{\infty}$ in $R_n$ such that $\lim q_m^{(n)} = q_0$. Since $f$ is onto, for each $m \in \mathbb{N}$, there exists a point $p_m^{(n)} \in X$ such that $f(p_m^{(n)}) =$
\{ q^{(n)}_m \}. Since \( X \) is compact, we may assume that the sequence \( \{ p^{(n)}_m \}_{m=1}^{\infty} \) converges to a point \( w_n \in X \). By the considerations we made before, \( w_n \in S \) and we may assume that \( p^{(n)}_m \in X - S \) for each \( m \in \mathbb{N} \). If there exists \( k \in \mathbb{N} \) such that \( R_k \) contains infinitely many elements of the sequence \( \{ p^{(n)}_m \}_{m=1}^{\infty} \), we choose one \( k_n \) with such a property and, taking a subsequence if necessary, we assume that the complete sequence \( \{ p^{(n)}_m \}_{m=1}^{\infty} \) is contained in \( R_{k_n} \). On the other hand, if each \( R_k \) contains finitely many elements of the sequence \( \{ p^{(n)}_m \}_{m=1}^{\infty} \), we make \( k_n = 0 \).

Let \( w_0 \in S \) be a limit point of the sequence \( \{ w_n \}_{n=1}^{\infty} \). By continuity of \( f \), for each \( n \in \mathbb{N} \), \( f(w_n) = \{ q_0 \} \). Thus \( f(w_0) = \{ q_0 \} \). In particular, \( f(w_0) \neq S \). Thus \( w_0 \in S - K \). Let \( G \) be the component of \( S - K \) which contains the point \( w_0 \).

Let \( \varphi : [0, 1] \to S \) be a continuous map such that \( \varphi|((0, 1) : (0, 1) \to G \) is a homeomorphism and \( \varphi(0), \varphi(1) \in K \). By the choice of \( w_0 \), we can choose three different numbers \( n, r \) and \( s \) in \( \mathbb{N} \) such that \( w_n, w_r \) and \( w_s \) belong to \( G \). One of the two components of \( G - \{ w_n \} \) (or both) does not contain the point \( z_{k_n} \). So one of the two intervals \( (0, \varphi^{-1}(w_n)) \) or \( (\varphi^{-1}(w_n), 1) \) (or both) does not intersect the set \( \varphi^{-1}(z_{k_n}) \). The same happens with each one of the numbers \( r \) and \( s \). Since we have three numbers and only two choices for the intervals, we may assume that the interval \( (0, \varphi^{-1}(w_n)) \) does not intersect the set \( \varphi^{-1}(z_{k_n}) \) and the interval \( (0, \varphi^{-1}(w_r)) \) does not intersect the set \( \varphi^{-1}(z_{k_r}) \). Let \( b = \varphi^{-1}(w_n) \) and \( c = \varphi^{-1}(w_r) \).

Let \( \eta = \| z_n - z_r \| > 0 \). Since \( \varphi(0) \in K \), \( f(\varphi(0)) = S \). Thus there exists \( a \in (0, \min\{b, c\}) \) such that \( H(f(\varphi(a)), S) < \frac{\eta}{3} \).

**Claim 2.** \( f(\varphi(a)) \subset S \) and \( f(\varphi(a)) \in \text{cl}_{C(X)}(C(R_n)) \cap \text{cl}_{C(X)}(C(R_r)) \).

Notice that there exists a retraction \( \rho : X \to S \). Since \( K = f^{-1}(\{S\}) \) and \( G \) is a component of \( S - K \), for each \( t \in [a, b] \), \( f(\varphi(t)) \neq S \). Since \( S \) is a terminal subcontinuum of \( X \), by the proof of Theorem 11.5 of [3], it follows that \( C(S) - \{S\} \) is an arcwise component of \( C(X) - \{S\} \). Since \( f(\varphi([a, b])) \subset C(X) - \{S\} \) and \( f(\varphi(b)) = f(w_n) = \{ q_0 \} \subset C(S) - \{S\} \), we conclude that \( f(\varphi([a, b])) \subset C(S) - \{S\} \). In particular, \( f(\varphi(a)) \in C(S) \) Thus, for each \( t \in [a, b] \), \( \rho(f(\varphi(t))) = f(\varphi(t)) \in C(S) - \{S\} \). Let \( \varepsilon_0 = \min\{H(\rho(f(\varphi(t))), S) : t \in [a, b]\} > 0 \).

We only prove that \( f(\varphi(a)) \in \text{cl}_{C(X)}(C(R_n)) \), the proof that \( f(\varphi(a)) \in \text{cl}_{C(X)}(C(R_r)) \) is similar. Let \( \varepsilon > 0 \).
By the continuity of $\rho$ and $f$, there exists $\lambda > 0$ such that, if $x, y \in X$ and $\|x - y\| < \lambda$, then $H(\rho(f(x)), \rho(f(y))) < \varepsilon_0$ and $H(f(x), f(y)) < \varepsilon$.

Apply Property (e) to the number $\lambda$ and the path $\varphi([a, b] : [a, b] \to S)$, to obtain a number $\delta > 0$ with the mentioned properties.

By the choice of $n$, $\zeta_{k_n} \notin \varphi([a, b])$. So $\varphi([a, b]) \subseteq S - \{z_{k_n}\}$. In the case that $k_n = 0$, there exists $N \in \mathbb{N}$ such that, for each $k \geq N$, $\zeta_k \notin \varphi([a, b])$. In this case $R_1 \cup \ldots \cup R_N$ contains finitely many elements of the sequence $\{p_m^{(n)}\}_{m=1}^\infty$. Thus, we can choose $m \in \mathbb{N}$ such that $p_m^{(n)} \notin R_1 \cup \ldots \cup R_N$ and $\|w_n - p_m^{(n)}\| < \delta$. Hence, $p_m^{(n)} \in R_{N_0}$ for some $N_0 > N$, so $\zeta_{N_0} \notin \varphi([a, b])$. In the case that $k_n \neq 0$, let $N_0 = k_n$ and choose $m \in \mathbb{N}$ such that $\|w_n - p_m^{(n)}\| < \delta$ and $p_m^{(n)} \in R_{k_n} = R_{N_0}$. In any case, numbers $N_0$ and $m$ can be chosen in such a way that $\|w_n - p_m^{(n)}\| < \delta$, $p_m^{(n)} \in R_{N_0}$ and $\zeta_{N_0} \notin \varphi([a, b])$.

By the choice of $\delta$, there exists a path $\beta : [a, b] \to R_{N_0}$ such that $\beta(b) = p_m^{(n)}$ and $\|\varphi(t) - \beta(t)\| < \lambda$ for each $t \in [a, b]$. By the choice of $\lambda$, given $t \in [a, b]$, $H(\rho(f(\varphi(t))), \rho(f(\beta(t)))) < \varepsilon_0$ and $H(f(\varphi(t)), f(\beta(t))) < \varepsilon$. Thus, $H(f(\varphi(a)), f(\beta(a))) < \varepsilon$ and, by the choice of $\varepsilon_0$, $\rho(f(\beta(t))) \neq S$ for each $t \in [a, b]$.

By definition, $f(\beta(b)) = f(p_m^{(n)}) = \{q_m^{(n)}\} \subset R_n$. We need to prove that $f(\beta(a)) \subset R_n$. In order to prove this inclusion, we prove that $f(\beta(t)) \subset R_n$ for each $t \in [a, b]$. Suppose to the contrary that there exists $t \in [a, b]$ such that $f(\beta(t)) \cap (X - R_n) \neq \emptyset$. Since $R_n$ is open in $X$, there exists $t_0 \in [a, b]$ such that $f(\beta(t_0)) \cap (X - R_n) \neq \emptyset$ and $f(\beta(t)) \subset R_n$ for each $t \in [t_0, b]$. Notice that $t_0 < b$. Since $S \cup R_n$ is compact, by continuity, $f(\beta(t_0)) \subset S \cup R_n$. Thus $f(\beta(t_0)) \cap S \neq \emptyset$. Given $t \in [t_0, b]$, consider the set $E_t = \bigcup\{f(\beta(u)) : u \in [t_0, t]\}$. Then $E_t \in C(X)$ (see [3, Lemmas 1.43 and 1.48]) and $\lim_{t \to t_0} E_t = E_{t_0} = f(\beta(t_0))$. Notice that, if $t > t_0$, then $f(\beta(t)) \subset E_t$, so $E_t \cap R_n \neq \emptyset$. Moreover, $f(\beta(t_0)) \subset E_t$, so $E_t \cap S \neq \emptyset$. By Property (h), $S \subset E_t$. Taking the limit as $t \to t_0$, we obtain that $S \subset E_{t_0} = f(\beta(t_0))$. Thus $S = \rho(f(\beta(t_0)))$, which is a contradiction with the conclusion in the previous paragraph. Therefore, $f(\beta(a)) \subset R_n$.

Hence, $f(\beta(a)) \subset R_n$ and $H(f(\varphi(a)), f(\beta(a))) < \varepsilon$. We have shown that $f(\varphi(a)) \in \text{cl}_{C(X)}(C(R_n))$. This ends the proof of Claim 2.

We are ready to obtain the final contradiction. By the first part of Claim 2, $f(\varphi(a))$ is a subarc of $S$. By Claim 2 and Property (f) $\zeta_n \notin f(\varphi(a))$ or $\zeta_n$ is an
end point of \( f(\varphi(a)) \) and the same happens for \( z_r \). Thus \( f(\varphi(a)) \) is contained in one of the two subarcs in which \( S \) is divided by the two points \( z_n \) and \( z_r \). But this is impossible since \( H(f(\varphi(a)), S) < \frac{2}{3} \). This completes the proof that there is no a continuous map from \( X \) onto \( C(X) \).

**BIBLIOGRAPHY**


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