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Decision Support

A differential oligopoly game with differentiated goods and sticky prices

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Abstract

We investigate a dynamic oligopoly game where goods are differentiated and prices are sticky. We study the open-loop and the closed-loop memoryless Nash equilibrium, and show that the latter equilibrium entails a larger level of steady state production as compared to the former; both equilibria entail a larger level of production in steady state than the static game. We also study the effects of price stickiness and product differentiation upon the steady state equilibrium allocation and profits. The per-firm equilibrium output is increasing in both product differentiation and price stickiness, while profits are increasing in both product differentiation and the speed of price adjustment. The steady state social welfare monotonically increases in the speed of price adjustment, and the overproduction entailed by dynamic competition has beneficial effect from a social standpoint.

Keywords: Game theory; Dynamic programming; Differential games; Price stickiness; Product differentiation

1. Introduction

The aim of this paper consists in studying the properties of the equilibria in a dynamic oligopoly model with price stickiness, along the lines first introduced by Simaan and Takayama (1978) and then extended by

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Fershtman and Kamien (1987) and Cellini and Lambertini (2004).[2] The novelty of this paper rests on the fact that we analyse an oligopoly where goods are differentiated, and we study how good differentiation and price stickiness interact in shaping the equilibrium allocation.

We take into consideration both the open-loop and the closed-loop memoryless Nash equilibrium. In both cases, an economically meaningful symmetric steady state exists; this equilibrium is a saddle. We focus on the steady state equilibrium allocation, and study its determinants. The already known properties of the differential game involving homogenous goods are confirmed; in particular, (i) the static game entails a lower level of production as compared to the steady-state equilibrium production levels of the differential game, and the steady-state Nash equilibrium production under the open-loop information structure is smaller than under the closed-loop rule; (ii) the stickier are the prices, the higher the steady state Nash equilibrium production levels. This consistency is not surprising, provided that the homogenous oligopoly case can be interpreted as a particular case of the present model.

In addition, we show that the degree of differentiation among goods is effective in determining both the production levels, and the responsiveness of quantities and profits to price stickiness. In particular, we show that the higher is the (symmetric) degree of differentiation among goods, the lower is the steady state level of production. Moreover, the higher is the degree of differentiation, the lower is the sensitivity of the steady state level of production to the price stickiness. This means that the degree of differentiation and the degree of price stickiness affect the steady state equilibrium level of production in much the same way. Concerning equilibrium profits, we show that they are affected in the same way by price flexibility and product differentiation.

Finally, we proceed to investigate the first best allocation, where a planner controls firms' output decisions so as to maximise social welfare. We show that the welfare distortion caused by the strategic interaction of profit-maximising Cournot firms is smaller at the steady state under the closed-loop memoryless Nash equilibrium than in the steady state under the open-loop solution. Both steady states are associated to a lower welfare loss, as compared to the equilibrium of the corresponding static game. Thus, the welfare assessment usually drawn from the static analysis is too pessimistic.

The outline of the paper is as follows. Section 2 introduces the basics of the model. Section 3 develops the differential game under the open-loop information structure, while Section 4 solves the closed-loop game. Both sections focus on the steady-state level of production, showing that the steady state is a saddle, and presenting comparative statics exercises on the production levels. The socially optimal production plan is characterised in Section 5. Section 6 concludes the paper.

2. The setup

A simple way to model price stickiness is to imagine that price adjusts, in response to the difference between its "notional" level and its current level. Under this perspective, price can be seen as the state variable of a dynamic system. Only a part of the difference between the "notional" and the "current" level of price can be corrected, in the presence of stickiness. This can be motivated, for instance, by costly adjustment. We formalise this idea, borrowing from Simaan and Takayama (1978), the following motion law concerning the price of any good i:[3]

[2] Several other authors have used the same model for more specific aims. Relevant such examples include Dockner and Haug (1990, 1991), where optimal trade policies are characterised; and Dockner and Gaunersdorfer (2001) and Benchekroun (2003), where the dynamic oligopoly model with sticky prices is used to investigate the incentives towards horizontal mergers.

[3] See also Fershtman and Kamien (1987). Mehlmann (1988, Chapter 5) provides an exhaustive exposition of both contributions. Fershtman and Kamien (1990) and Tsutsui and Mino (1990) present further results on the same model, in the case of a finite horizon.

$$\frac{\mathrm{d}p_i(t)}{\mathrm{d}t} \equiv \dot{p}_i(t) = s\{\widehat{p}_i(t) - p_i(t)\}, \tag{1}$$

where $\widehat{p}_i(t)$ denotes the notional level of price of good i at time t, while $p_i(t)$ denotes its current level. Notice that the speed of adjustment is captured by parameter s, with $0 < s < 1$. In particular, s measures which part of the difference between the notional price level and the current price level is immediately corrected. The lower is s, the higher is the degree of price stickiness.

As far as the notional price concerns, it is dictated by the demand condition, deriving from the preference structure of consumers. We assume that the notional price in any instant t is defined as follows:

$$\widehat{p}_i(t) = A - Bq_i(t) - D\sum_{j\neq i} q_j(t). \tag{2}$$

This function is borrowed from Spence (1976) and employed by Singh and Vives (1984), Vives (1985), Cellini and Lambertini (1998, 2002), inter alia. The number of available varieties is assumed to be constant over time and equal to N, with $i \in [1, N]$. Parameter A measures the market size or the reservation price, which is assumed to be equal across varieties for the sake of simplicity. As for parameters B and D, assume $0 \leqslant D \leqslant B$. Notice that parameter D captures the degree of substitutability between any pair of different goods. In the limit case $D = 0$, goods are independent and each firm becomes a monopolist. In the opposite limit case $D = B$, the goods produced by different firms are perfect substitutes and the model collapses into the homogenous oligopoly model. Thus, the higher is parameter D, the lower is the (symmetric) degree of differentiation.[4]

Consider a population of N single-product firms. The instantaneous production cost function of firm i is assumed to be quadratic:

$$C_i(t) = cq_i(t) - \frac{1}{2}[q_i(t)]^2, \quad 0 < c < A. \tag{3}$$

As a consequence, the instantaneous profit function of firm i is

$$\pi_i(t) = q_i(t) \cdot \left[p_i(t) - c - \frac{1}{2}q_i(t)\right]. \tag{4}$$

For future reference, we report the solution of the static problem, where the notional price is equal to the current price, and each firm simultaneously chooses the quantity to be produced. In such a case, the maximisation of the function $\pi_i = q_i[A - Bq_i - D\sum_{j\neq i}q_j] - cq_i - \frac{1}{2}q_i^2$, with respect to q_i, provides the reaction function

$$q_i = \left[A - c - D\sum_{j\neq i} q_j(t)\right] \bigg/ (1 + 2B). \tag{5}$$

Solving the system of N best replies summarised in (5) w.r.t. the vector of output levels, one finds the unique Cournot–Nash equilibrium quantity:[5]

$$q_{\mathrm{CN}} = \frac{(A - c)}{[1 + 2B + D(N - 1)]}, \tag{6}$$

which is clearly symmetric since firms are ex ante symmetric and the system of reaction functions (5) is linear in the choice variables.

[4] A model where D is a variable, whose dynamics is driven by the investment efforts of firms devoted to product differentiation is in Cellini and Lambertini (2002).

[5] A shortcut would consists in imposing the symmetry conditions $q_i = q_j = q$ and then solving (5).

In the problem we are interested here, however, the current price of any good is generally different from its notional level. The production decisions of firms affect notional prices, but current prices evolve subject to the existence of price stickiness. We assume that firms choose the quantity to be produced, so that we are in a Cournot framework. More precisely, each player (i.e., each firm) chooses the path of his control variable $q_i(t)$ over time, from the present to infinity, i.e., $t \in [0, \infty)$, in order to maximize the present value of the profit flow, subject to (i) the motion laws regarding the state variables, and (ii) the initial conditions. Formally, the problem of player i may be written as follows:

$$\max_{q_i(t)} \quad J_i = \int_0^\infty \mathrm{e}^{-\rho t} q_i(t) \cdot \left[p_i(t) - c - \frac{1}{2} q_i(t) \right] \mathrm{d}t \tag{7}$$

$$\text{s.t.} \quad \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} = s \left[A - Bq_i(t) - D \sum_{j \neq i} q_j(t) - p_i(t) \right], \quad i \in [1, N], \tag{8}$$

$$\text{s.t.} \quad p_i(0) = p_{i,0} > 0, \quad i \in [1, N]. \tag{9}$$

Notice that the number of the state variables in the problem of each player i is equal to N, corresponding to the price of the N available varieties, while the control variable is one for each player, specifically, the quantity to be produced. The factor $\mathrm{e}^{-\rho t}$ discounts future gains, and the discount rate ρ is assumed to be constant and equal across firms.

We solve the problem by considering—in turn—the open-loop solution, and the closed-loop memoryless solution.

3. The open-loop solution

Here we look for the open-loop Nash equilibrium, i.e., we examine a situation where firms commit to a production plan at $t = 0$ and stick to that plan forever.

The Hamiltonian function is

$$\begin{aligned} \mathscr{H}_i(t) = \mathrm{e}^{-\rho t} \cdot \Bigg\{ & q_i(t) \cdot \left[p_i(t) - c - \frac{1}{2} q_i(t) \right] + \lambda_i^i(t) s \left[A - Bq_i(t) - D \sum_{j \neq i} q_j(t) - p_i(t) \right] \\ & + \sum_{j \neq i} \lambda_j^i(t) s \left[A - Bq_j(t) - D \sum_{h \neq j} q_h(t) - p_j(t) \right] \Bigg\}, \end{aligned} \tag{10}$$

where $\lambda_i^i(t) = \mu_i^i(t)\mathrm{e}^{\rho t}$, and $\mu_i^i(t)$ is the co-state variable associated by player i to the price of his product, $p_i(t)$; similarly, $\lambda_j^i(t) = \mu_j^i(t)\mathrm{e}^{\rho t}$, with $\mu_j^i(t)$ being the co-state variable associated by player i to the price of the good $j \neq i$. As usual, supplementary variables λ represent co-states in current value, and are introduce to ease calculation.

The outcome of the open-loop game is summarised by the following:

Proposition 1. *When the open-loop Nash equilibrium solution concept is adopted, a symmetric steady state exists, where the individual output and the market price are*

$$q_{\mathrm{OL}}^\infty = \frac{(s+\rho)(A-c)}{(s+\rho)[1+B+D(N-1)]+sB},$$

$$p_{\mathrm{OL}}^\infty = A - \frac{[B+D(N-1)](s+\rho)(A-c)}{(s+\rho)[1+B+D(N-1)]+sB}.$$

Such a steady state is a saddle.

Proof. In order to find the open-loop Nash equilibria, we have to solve the following first-order condition:

$$\frac{\partial \mathscr{H}_i(t)}{\partial q_i(t)} = 0 \tag{11}$$

along with the adjoint equations

$$-\frac{\partial \mathscr{H}_i(t)}{\partial p_i(t)} = \frac{\partial \lambda_i^i(t)}{\partial t} - \rho \lambda_i^i(t), \tag{12}$$

$$-\frac{\partial \mathscr{H}_i(t)}{\partial p_j(t)} = \frac{\partial \lambda_j^i(t)}{\partial t} - \rho \lambda_j^i(t). \tag{13}$$

The first order condition and the adjoint equations have to be considered along with the initial conditions $\{p_i(0) = p_{i,0}\}_{i=1}^N$ and the transversality conditions, which set the final value of the state and/or co-state variables:

$$\lim_{t\to\infty} \lambda_i^i(t) \cdot p_i(t) = 0, \qquad \lim_{t\to\infty} \lambda_j^i(t) \cdot p_j(t) = 0. \tag{14}$$

From (11)–(13) we obtain respectively:

$$q_i(t) = p_i(t) - c - \lambda_i^i(t)sB - sD\sum_{j\neq i} \lambda_j^i(t), \tag{15}$$

$$\frac{\partial \lambda_i^i(t)}{\partial t} = (s+\rho)\lambda_i^i(t) - q_i(t), \tag{16}$$

$$\frac{\partial \lambda_j^i(t)}{\partial t} = (s+\rho_i)\lambda_j^i(t), \tag{17}$$

Eq. (15) can be differentiated w.r.t. time to obtain the dynamics of firm *i*'s output:

$$\frac{\mathrm{d}q_i(t)}{\mathrm{d}t} = \frac{\mathrm{d}p_i(t)}{\mathrm{d}t} - s\left[B\frac{\mathrm{d}\lambda_i^i(t)}{\mathrm{d}t} + D\sum_{j\neq i}\frac{\mathrm{d}\lambda_j^i(t)}{\mathrm{d}t}\right], \tag{18}$$

where $\mathrm{d}p_i(t)/\mathrm{d}t$ is given by (1). Therefore the above differential equation may be rewritten as follows:

$$\frac{\mathrm{d}q_i(t)}{\mathrm{d}t} = s\left[A - Bq_i(t) - D\sum_{j\neq i} q_j(t) - p_i(t)\right] - s\left[B\frac{\mathrm{d}\lambda_i^i(t)}{\mathrm{d}t} + D\sum_{j\neq i}\frac{\mathrm{d}\lambda_j^i(t)}{\mathrm{d}t}\right]. \tag{19}$$

Now we show that a unique symmetric Nash equilibrium exists.[6] On the basis of the ex ante symmetry of firms, we impose $p_{j,0} = \overline{p}_0$, $q_j = \overline{q}$ and $p_j = \overline{p}$ for all $j \neq i$. Then, we also impose $\lambda_j^i = \lambda_{\text{other}}$ for any $j \neq i$; using expressions (16), (17) and (19) becomes

$$\begin{aligned}\frac{\mathrm{d}q_i(t)}{\mathrm{d}t} &= s[A - Bq_i(t) - D(N-1)\overline{q}(t) - p_i(t)] \\ &\quad - s\big[B\big((s+\rho)\lambda_i^i(t) - q_i(t)\big) + D(N-1)(s+\rho_i)\lambda_{\text{other}}(t)\big].\end{aligned} \tag{20}$$

This amounts to saying that we take all but one firm's policies as fixed in order to derive the Nash equilibrium. To this regard, it is worth stressing that we *cannot* postulate $\lambda_i^i(t) = \lambda_{\text{other}}(t)$, since the effect of the price of the variety produced by each firm on his own profit, is obviously different from the effect of the price of the varieties produced by the opponents. Note, however, that (20) applies to any firm $i = 1, 2, 3 \ldots, N$ alike. This entails that the system of N equations (20) gives rise to a symmetric Nash

[6] In general, other asymmetric equilibria may exist for $N > 2$, unlike what happens in the static game.

equilibrium where each firm charges the same price and produces the same output as any of its rivals. Accordingly, as a further step we also set $q_i = \overline{q}$ and $p_i = \overline{p}$ and $\lambda_i^i = \lambda_{\text{own}}$. This gives the following system of differential equations:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{\mathrm{d}p(t)}{\mathrm{d}t} - sB\frac{\mathrm{d}\lambda_{\text{own}}(t)}{\mathrm{d}t} - (N-1)sD\frac{\mathrm{d}\lambda_{\text{other}}(t)}{\mathrm{d}t}, \tag{21}$$

$$\frac{\mathrm{d}p(t)}{\mathrm{d}t} = s[A - Bq(t) - D(N-1)q(t) - p(t)], \tag{22}$$

$$\frac{\mathrm{d}\lambda_{\text{own}}(t)}{\mathrm{d}t} = (s+\rho)\lambda_{\text{own}}(t) - q(t), \tag{23}$$

$$\frac{\mathrm{d}\lambda_{\text{other}}(t)}{\mathrm{d}t} = (s+\rho)\lambda_{\text{other}}(t). \tag{24}$$

This means that, for any player, the dynamics of all relevant variables may be described by a system of four dynamic equations. The system can be written in matrix form as follows:

$$\begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \\ \dot{\lambda}_{\text{own}}(t) \\ \dot{\lambda}_{\text{other}}(t) \end{bmatrix} = \begin{bmatrix} -s & -s[B+D(N-1)] & 0 & 0 \\ 1 & 0 & -sB & -(N-1)sD \\ 0 & -1 & (s+\rho) & 0 \\ 0 & 0 & 0 & (s+\rho) \end{bmatrix} \begin{bmatrix} p(t) \\ q(t) \\ \lambda_{\text{own}}(t) \\ \lambda_{\text{other}}(t) \end{bmatrix}. \tag{25}$$

It is easy to show that a (non-trivial) steady state does exist in this dynamic system. We denote by p^∞, q^∞, $\lambda_{\text{own}}^\infty, \lambda_{\text{other}}^\infty$ the steady state levels of the relevant variables, namely, the price, the output level, and the co-state variables associated with the own price and with the price of different varieties, respectively. From Eq. (21) it is immediate to note that if $\mathrm{d}p/\mathrm{d}t = 0$, $\mathrm{d}\lambda_{\text{own}}/\mathrm{d}t = 0$, $\mathrm{d}\lambda_{\text{other}}/\mathrm{d}t = 0$ hold simultaneously, then $\mathrm{d}q/\mathrm{d}t = 0$. From Eq. (23) we note that $\mathrm{d}\lambda_{\text{own}}/\mathrm{d}t = 0$ entails $\lambda_{\text{own}}^\infty = q^\infty/(s+\rho)$, while from Eq. (24) we note that $\mathrm{d}\lambda_{\text{other}}/\mathrm{d}t = 0$ entails $\lambda_{\text{other}}^\infty = 0$. Eq. (22) shows that $\mathrm{d}p/\mathrm{d}t = 0$ entails $p^\infty = A - [B + D(N-1)]q^\infty$. Using Eq. (15) under the symmetry conditions, we obtain

$$q_{\text{OL}}^\infty = \frac{(s+\rho)(A-c)}{(s+\rho)[1+B+D(N-1)]+sB}. \tag{26}$$

Note that the solution is unique: in other words, only one symmetric Nash equilibrium does exist in this game.

Simple substitutions permit us to fully characterize the steady state market allocation under the open loop information structure:

$$p_{\text{OL}}^\infty = A - \frac{[B+D(N-1)](s+\rho)(A-c)}{(s+\rho)[1+B+D(N-1)]+sB}, \tag{27}$$

$$\lambda_{\text{own}}^\infty = \frac{(A-c)}{(s+\rho)[1+B+D(N-1)]+sB}. \tag{28}$$

As a final point, we investigate the stability property of the steady state. We already know that, in the case of homogenous oligopoly, the steady state is a saddle, completely described by a dynamic system of two differential equations in two variables, namely price and quantity (see Cellini and Lambertini, 2004). In the present, more general, setting with a differentiated oligopoly, four variables (and four differential equations) are necessary to fully characterise the dynamics of the system—even under the particular case of the symmetric equilibrium. In order to have that the steady state is stable in the saddle point sense, it is sufficient that exactly two out of the four characteristic roots of the Jacobian matrix associated with the dynamic system (that is, the Jacobian in system (25)), have a negative real part. This is the case indeed in the problem at hand. In particular, one characteristic root is $(s+\rho)$; the expressions of remaining three roots are rather heavy: one of them is real and positive; the two remaining roots are complex, with negative real parts. In sum, two out of four characteristic roots of the Jacobian matrix in (25) are real and positive; the other two roots have negative real parts, leading to the conclusion that the steady state is a saddle. This concludes the proof. □

Notice that the present result encompasses the result from the model with product homogeneity.[7] Notice also that the steady state level of production, under the open-loop information structure is larger than its counterpart in the static Cournot game. This is easily proved, by comparing (26) with (6). Consistently, the steady state level of price is smaller in the dynamic game as compared to the static game. This result is well known in the literature (see, e.g., Simaan and Takayama, 1978; Reynolds, 1987; Piga, 2000, inter alia). It means that the prolonged time of non-cooperative interaction leads to higher levels of production (and lower prices) as compared to static interaction.

Simple comparative statics exercises on the steady state level of production q^{∞}_{OL} lead to the following results, holding for all $N > 1$.

(i) $\partial q^{\infty}_{\mathrm{OL}}/\partial D < 0$: the higher is D, that is, the higher is the substitutability among goods (and the lower is the differentiation) the smaller is the steady state level of production in the symmetric open-loop Nash equilibrium. The intuition behind this result could be provided by the following argument: a higher substitutability among varieties means a smaller demand for any individual firm, and lower market-power. Consequently, the reaction of firms leads to an equilibrium where production is lower.
(ii) $\partial q^{\infty}_{\mathrm{OL}}/\partial s < 0$: the lower is the price stickiness (i.e., the larger is s), the smaller is the steady state production; put differently, the stickier are the prices, the larger the production in steady state is. This result is well known in the literature on differential oligopoly games with sticky prices (see also Piga, 2000; Cellini and Lambertini, 2004). A rough intuition for this result is provided by the following argument: when prices are sticky, the current production levels of firms are weakly effective in moving current prices; this fact leads firms to high levels of current and future production. On the contrary, when prices move largely in response to production decisions, firms choose to shrink the output levels.
(iii) $\partial^2 q^{\infty}_{\mathrm{OL}}/(\partial s \partial D) > 0$: this cross derivative measures how the sensitivity of steady state production to differentiation is affected by price stickiness. In this respect, note that the higher is s, i.e., the less sticky are prices, the higher is the sensitivity of production with respect to differentiation. Since differentiation and price stickiness have the same effect on production, their mutual interaction strengthens the effect on individual equilibrium output.

Now we examine the effects of a change in D and s on equilibrium profits $\pi^{\infty}_{\mathrm{OL}}$.

(a) $\partial \pi^{\infty}_{\mathrm{OL}}/\partial D < 0$: as intuition suggests, ceteris paribus, an increase in product substitutability hampers profits.
(b) $\partial \pi^{\infty}_{\mathrm{OL}}/\partial s > 0$: as the speed of adjustment becomes higher, profits increase.
(c) The cross derivative exhibits the following property:

$$\frac{\partial^2 \pi^{\infty}_{\mathrm{OL}}}{\partial s \partial D} \gtreqless 0 \quad \text{for all } D \lesseqgtr \frac{\rho + s + 2B(s-\rho)}{2(N-1)(\rho+s)}. \tag{29}$$

The above results can be interpreted as follows. First, it is worth noting that the speed of adjustment and product differentiation work in the same direction. Second, in general, firms' profits benefit from having

[7] In fact, when $D = B = 1$ the level of q^{∞}_{OL} given by Eq. (26) of the present paper coincides with the steady state level of production in the homogenous oligopoly model presented by Cellini and Lambertini (2004)—see in particular their Eq. (12), p. 307.

flexible prices. This benefit is enhanced by market power, which may obtain through either a high degree of concentration or a high degree of product differentiation. Therefore, the positive effect exerted by D on $\partial\pi_{\mathrm{OL}}^{\infty}/\partial s$ increases for sufficiently low values of D and N. As N becomes larger, the critical threshold of D in (29) decreases, so that there surely exists a critical number of firms beyond which the behaviour of $\partial\pi_{\mathrm{OL}}^{\infty}/\partial s$ w.r.t. D becomes non-monotone.

4. The memoryless closed-loop solution

Under the closed-loop information structure, firms do not precommit on any path and their strategies at any instant may depend on all the preceding history. In this situation, the information set used by firms in setting their strategies at any given time is often simplified to be only the current value of the state variables at that time. In this section we rely on the so-called *closed-loop memoryless Nash equilibrium* solution concept, under which players take into account the effect of the current level of state variables on the controls in every instant of time (see Basar and Olsder, 1982). This equilibrium is strongly time consistent.

The outcome of the memoryless closed-loop game is summarised by the following:

Proposition 2. *When the closed-loop memoryless Nash equilibrium solution concept is adopted, a symmetric steady state exists, where the individual output and the market price are*

$$q_{\mathrm{CL}}^{\infty}=\frac{\Psi(s+\rho)(A-c)}{\Psi[(s+\rho)(1+B+D(N-1))+sB]-(N-1)s^2D^2},$$
$$p_{\mathrm{CL}}^{\infty}=A-[B+D(N-1)]q_{\mathrm{CL}}^{\infty},$$

where $\Psi\equiv\rho+s(1+B+D(N-2))$. *Such a steady state is a saddle.*

Proof. The relevant Hamiltonian function is still (10), while the first order condition and the adjoint equations for the player i are as follows:

$$\frac{\partial\mathscr{H}_i(t)}{\partial q_i(t)}=0, \tag{30}$$
$$-\frac{\partial\mathscr{H}_i(t)}{\partial p_i(t)}-\sum_{h\neq i}\frac{\mathscr{H}_i(t)}{\partial q_h(t)}\frac{\partial q_h^*(t)}{\partial p_i(t)}=\frac{\partial\lambda_i^i(t)}{\partial t}-\rho\lambda_i^i(t), \tag{31}$$
$$-\frac{\partial\mathscr{H}_i(t)}{\partial p_j(t)}-\sum_{h\neq i}\frac{\mathscr{H}_i(t)}{\partial q_h(t)}\frac{\partial q_h^*(t)}{\partial p_j(t)}=\frac{\partial\lambda_j^i(t)}{\partial t}-\rho\lambda_j^i(t). \tag{32}$$

Also in this case the first order condition and the adjoint equations have to be considered along with the initial conditions $\{p_i(0)=p_{i,0}\}_{i=1}^{N}$ and the transversality conditions (14).

The terms

$$\frac{\partial\mathscr{H}_i(t)}{\partial q_h(t)}\frac{\partial q_h^*(t)}{\partial p_j(t)} \tag{33}$$

appearing in (31) and (32) capture strategic interaction, in any instant of time, through the feedback from states to controls, which is by definition absent under the open-loop solution concept. Whenever the expression in (33) is zero for all j, then the closed-loop memoryless equilibrium collapses into the open-loop Nash equilibrium (see, e.g., Driskill and McCafferty, 1989); this is not the case in the present setting.

In order to solve this problem, we take into account that $\frac{\partial q_h^*(t)}{\partial p_j(t)}=1$ iff $h=j$ while $\frac{\partial q_h^*(t)}{\partial p_j(t)}=0$ otherwise. From (30)–(32) we obtain respectively:

$$q_i(t) = p_i(t) - c - \lambda_i^i(t)sB - D\sum_{j\neq i}\lambda_j^i(t)s, \tag{34}$$

$$\frac{\partial\lambda_i^i(t)}{\partial t} = (s+\rho_i)\lambda_i^i(t) - q_i(t), \tag{35}$$

$$\frac{\partial\lambda_j^i(t)}{\partial t} = (\rho_i+s)\lambda_j^i(t) + sD\lambda_i^i(t) + D\sum_{h\neq i,j}\lambda_h^i(t)s + sB\lambda_h^i(t). \tag{36}$$

Now, as a shortcut, we introduce the symmetry assumptions $q_i = q_j = q$, and $p_i = p_j = p$, in order to focus on symmetric equilibrium.[8] Moreover, we pose $\lambda_i^i = \lambda_{\text{own}}$, and we postulate the symmetry assumption $\lambda_j^i = \lambda_h^i = \lambda_{\text{other}}$ for any $j \neq i$, $h \neq i$. These assumptions permit us to write Eq. (34) as follows:

$$q(t) = p(t) - c - sB\lambda_{\text{own}}(t) - sD(N-1)\lambda_{\text{other}}(t). \tag{37}$$

If we consider Eq. (37) and substitute for λ_{own} and λ_{other} the expressions deriving by integration over time of (35) and (36) respectively, we obtain the optimal rule followed by a player under the symmetric Nash equilibrium. It is worth mentioning that integrating (35) and (36) over time provides the expressions of $\lambda_{\text{own}}(t)$ and $\lambda_{\text{other}}(t)$ respectively; in both these functions $q(t)$ enters linearly and $p(t)$ does not appear. Consequently, expression (37) continues to be a function connecting $q(t)$ to $p(t)$ in a linear way, even after the substitution for $\lambda_{\text{own}}(t)$ and $\lambda_{\text{other}}(t)$. Put differently, the closed-loop rule followed by any player connects his control variable with the state variable linearly.[9] This is consistent with the linear-quadratic structure of the game, and with a quadratic value function for any player.

In the symmetric steady state the following relationships hold:

(i) from (31): $\lambda_{\text{own}}^{\infty} = q^{\infty}/(\rho+s)$;
(ii) from (32): $\lambda_{\text{other}}^{\infty} = -sD\lambda_{\text{own}}^{\infty}/[\rho+s(1+B+D(N-2))]$;
(iii) from the dynamic constraint: $p^{\infty} = A - [B + D(N-1)]q^{\infty}$.

Subsequent substitutions into Eq. (37) lead to find the following relation holding in steady state:

$$q^{\infty} = A - Bq^{\infty} - D(N-1)q^{\infty} - c - \frac{sBq^{\infty}}{(\rho+s)} + \frac{s^2D^2(N-1)q^{\infty}}{[\rho+s(1+B+D(N-2))](\rho+s)} \tag{38}$$

so that the steady state level of production, under the symmetric closed-loop memoryless Nash equilibrium turns out to be

$$q_{\text{CL}}^{\infty} = \frac{\Psi(s+\rho)(A-c)}{\Psi[(s+\rho)(1+B+D(N-1))+sB]-(N-1)s^2D^2}, \quad \Psi \equiv \rho + s(1+B+D(N-2)). \tag{39}$$

In the remainder of the section we discuss, in turn, the dynamic properties of the steady state, and the role of the parameters in determining the steady state production under the closed-loop information structure, as compared to the open-loop and the static game Nash equilibria.

As far the first point is concerned, the dynamic system (under symmetry) can be written in matrix form as follows:

[8] As in the previous section, one could alternatively take as given firm j's policy for all $j \neq i$, and then focus on the optimal behaviour of firm i in isolation. It can be easily shown that the two procedures lead to the same result.

[9] Remember that we are confining our attention to a symmetric equilibrium, so that the price is the same across goods. In general, other asymmetric equilibria may exist, in which the closed-loop rule connects the control variable of each player with the vector of state variables, i.e., with the vector of the price of all goods.

$$\begin{bmatrix} \dot{p}(t) \\ \dot{q}(t) \\ \dot{\lambda}_{\text{own}}(t) \\ \dot{\lambda}_{\text{other}}(t) \end{bmatrix} = \theta \begin{bmatrix} p(t) \\ q(t) \\ \lambda_{\text{own}}(t) \\ \lambda_{\text{other}}(t) \end{bmatrix}, \tag{40}$$

$$\theta = \begin{bmatrix} -s & -s[B+D(N-1)] & 0 & 0 \\ 1 & 0 & -sB & -(N-1)sD \\ 0 & -1 & (s+\rho) & 0 \\ 0 & 0 & 0 & [s+\rho+sB+D(N-2)] \end{bmatrix}.$$

Also in this case, it is possible to find the characteristic roots of the Jacobian matrix in (40), and it is possible to check that exactly two out of the four characteristic roots have negative real parts. Therefore, the steady state is stable in the saddle point sense—that is, given the initial conditions, the steady state is reached only for one appropriate combination of the control variable and co-state variables, at the initial time.[10] This concludes the proof. □

Simple comparative statics exercises on the steady state level of production q^{∞}_{CL} lead to the following points, confirming all the substantial conclusions about the steady state production under the open-loop information structure: for all $N > 1$, (i) $\partial q^{\infty}_{\text{CL}}/\partial D < 0$; (ii) $\partial q^{\infty}_{\text{CL}}/\partial s < 0$; (iii) $\partial^2 q^{\infty}_{\text{OL}}/(\partial s \partial D) > 0$. The effects of a change in D and/or s on equilibrium profits are qualitatively the same as in the open-loop case.

In this oligopoly with differentiated goods, like in the homogenous good case, the steady state level of production turns out to be larger under the closed-loop memoryless solution, than under the open-loop, as the comparison between Eqs. (26) and (39) makes clear. Both levels are larger than the production of the static Cournot game. This fact can be explained on the following grounds. The closed-loop output level is higher than the open-loop output level because, taking into account feedback effects, each firm tries to preempt the rivals. Since this holds for all firms alike, the outcome is that the closed-loop steady state production exceeds the open-loop steady state production.[11] In turn, the open-loop steady state output exceeds the static (or myopic) output because in the static game there is no time for adjustment and therefore firms have no way of trying to overproduce in order to preempt the rivals.

As a consequence, from the firms' viewpoint, the static situation (or, a situation where firms are myopic) is the most profitable one. On the contrary, the steady state allocation in the closed-loop memoryless equilibrium is socially preferred both to the open-loop steady state and to the static equilibria.

Finally, as the number of firms becomes infinitely large, optimal individual output tends to zero independently of the solution concept. As the market becomes perfectly competitive, open-loop and closed-loop steady state solutions coincide with the static Cournot–Nash solution, which is itself reproducing the perfectly competitive outcome.

5. The social optimum

Here we examine the socially optimal allocation, where a benevolent planner sets production plans so as to maximise the discounted social welfare, defined as the sum of industry profits and consumer surplus:

[10] This dynamic feature of the steady state is consistent with the analogous result in the particular case of homogenous oligopoly, where the 2×2 dynamic system presents a steady state, which is a proper saddle (see Cellini and Lambertini, 2004, Section 5).

[11] This is usually observed when firms control variables are output levels, investment levels, etc. (see, e.g., Reynolds, 1987).

$$\mathrm{SW}(t) \equiv \Pi(t) + \mathrm{CS}(t) = \sum_{i=1}^{N} \pi_i(t) + \frac{1}{2}\sum_{i=1}^{N}[A - p_i(t)]q_i(t)$$

under the set of constraints given by state Eq. (1) and the same initial and transversality conditions as in the previous sections. The outcome is summarised by:

Proposition 3. *Under social planning, the steady state levels of individual output and price are*

$$q_{\mathrm{SP}}^{\infty} = \frac{2(s+\rho)(A-c)}{[2+B+D(N-1)]\rho + 2[1+B+D(N-1)]s},$$
$$p_{\mathrm{SP}}^{\infty} = \frac{A[2s + 2c(B+D(N-1))(s+\rho) + (2-B-D(N-1))\rho]}{[2+B+D(N-1)]\rho + 2[1+B+D(N-1)]s}.$$

Such a steady state is a saddle.

Proof. The Hamiltonian function of the social planner is

$$\mathscr{H}_{\mathrm{SP}}(t) = \mathrm{e}^{-\rho t}\cdot\left\{\sum_{i=1}^{N}\left[q_i(t)\cdot\left[p_i(t) - c - \frac{1}{2}q_i(t)\right] + \frac{[A-p_i(t)]q_i(t)}{2}\right] + \lambda_i(t)s\left[A - Bq_i(t) - D\sum_{j\neq i}q_j(t) - p_i(t)\right] + \sum_{j\neq i}\lambda_j(t)s\left[A - Bq_j(t) - D\sum_{h\neq j}q_h(t) - p_j(t)\right]\right\}, \tag{41}$$

where $\lambda_i(t) = \mu_i(t)\mathrm{e}^{\rho t}$, and $\mu_i(t)$ is the co-state variable associated by the planner to the price of firm i's product. The first order conditions are[12]

$$\frac{\partial\mathscr{H}_{\mathrm{SP}}(t)}{\partial q_i(t)} = \frac{A - 2q_i(t) - 2c + p_i(t) - 2\left[Bs\lambda_i(t) + Ds\sum_{j\neq i}\lambda_j(t)\right]}{2} = 0, \tag{42}$$
$$-\frac{\partial\mathscr{H}_{\mathrm{SP}}(t)}{\partial p_i(t)} = \frac{\partial\lambda_i(t)}{\partial t} - \rho\lambda_i(t) \iff \frac{\partial\lambda_i(t)}{\partial t} = \lambda_i(t)(s+\rho) - \frac{q_i(t)}{2}. \tag{43}$$

Now, adopting the symmetry conditions $q_i(t) = q(t)$, $p_i(t) = p(t)$ and $\lambda_i(t) = \lambda(t)$, we obtain from (42)

$$\lambda(t) = \frac{A - 2q(t) - 2c + p(t)}{2s[B + D(N-1)]} \tag{44}$$

and

$$q(t) = \frac{A - 2c + p(t) - 2s\lambda(t)[B + D(N-1)]}{2}. \tag{45}$$

The latter expression can be differentiated w.r.t. time to yield

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t} = \frac{\mathrm{d}p(t)/\mathrm{d}t - 2s[B+D(N-1)]\cdot\mathrm{d}\lambda(t)/\mathrm{d}t}{2}, \tag{46}$$

[12] For brevity, we omit the indication of exponential discounting.

which, after straightforward substitutions using (1), (43) and (44), simplifies as follows:

$$\frac{\mathrm{d}q(t)}{\mathrm{d}t}=\frac{1}{2}\{A-p(t)+[B+D(N-1)]\Gamma\}, \tag{47}$$

where

$$\Gamma\equiv(s-1)q(t)-\frac{(A-2c+p(t)-2q(t))(s+\rho)}{B+D(N-1)}. \tag{48}$$

Then, solving the system $\{\mathrm{d}p(t)/\mathrm{d}t=0;\ \mathrm{d}q(t)/\mathrm{d}t=0\}$ w.r.t. individual price and output, we obtain their steady state values at the social optimum:

$$\begin{aligned} q_{\mathrm{SP}}^{\infty}&=\frac{2(s+\rho)(A-c)}{[2+B+D(N-1)]\rho+2[1+B+D(N-1)]s},\\ p_{\mathrm{SP}}^{\infty}&=\frac{A[2s+2c(B+D(N-1))(s+\rho)+(2-B-D(N-1))\rho]}{[2+B+D(N-1)]\rho+2[1+B+D(N-1)]s}. \end{aligned} \tag{49}$$

Finally, examining the Jacobian matrix of this problem, it is easily checked that the pair $(p_{\mathrm{SP}}^{\infty},q_{\mathrm{SP}}^{\infty})$ identifies a saddle point. This is omitted for brevity, as the pertaining procedure is analogous to that illustrated in the previous sections. □

Additionally, it can be established that (i) $\partial q_{\mathrm{SP}}^{\infty}/\partial D<0$; (ii) $\partial q_{\mathrm{SP}}^{\infty}/\partial s<0$ for all $N>1$, while

$$\frac{\partial^2 q_{\mathrm{SP}}^{\infty}}{\partial D\partial s}\propto 2s[B+D(N-1)-1]+\rho[B+D(N-1)-2] \tag{50}$$

may take either sign. By plugging the pair $(p_{\mathrm{SP}}^{\infty},q_{\mathrm{SP}}^{\infty})$ into the social welfare function, we obtain the steady state welfare level at the first best

$$\mathrm{SW}_{\mathrm{SP}}^{\infty}=\frac{2N(s+\rho)(A-c)^2[\rho+(1+B+D(N-1))s]}{[(2+B+D(N-1))\rho+2(1+B+D(N-1))s]^2}, \tag{51}$$

which exhibits the following property:

$$\frac{\partial SW_{\mathrm{SP}}^{\infty}}{\partial s}=\frac{2N\rho^2(A-c)^2(B+D(N-1))^2}{[(2+B+D(N-1))\rho+2(1+B+D(N-1))s]^3}>0. \tag{52}$$

That is:

Remark 1. At the first best, the steady state social welfare level monotonically increases in the speed of price adjustment.

It can also be shown that $\partial SW_{\mathrm{SP}}^{\infty}/\partial D<0$ (as we know from the static oligopoly models with product differentiation), while $\partial^2 SW_{\mathrm{SP}}^{\infty}/\partial D\partial s$ may take either sign.

More importantly, at this point we can build upon the ranking of $q_{\mathrm{SP}}^{\infty}, q_{\mathrm{OL}}^{\infty}$ and q_{CL}^{∞} in order to assess the distortionary effect caused by oligopolistic interaction on the welfare performance of the market. In the dynamic setup, the relevant per-firm output level at the subgame perfect equilibrium is $q_{\mathrm{CL}}^{\infty}>q_{\mathrm{OL}}^{\infty}$ for all admissible N and s. Therefore, the dynamic game highlights a reassuring property of the (subgame perfect) Cournot–Nash equilibrium, namely that of reducing indeed the welfare loss brought about by the profit-seeking behaviour of firms, as compared to what we are accustomed to believe according to the static approach. This produces our final result:

Remark 2. Given that $q_{\mathrm{CL}}^{\infty}>q_{\mathrm{OL}}^{\infty}>q_{\mathrm{CN}}$, the welfare loss observed at the subgame perfect equilibrium of the differential game is always smaller than the one associated with the static equilibrium.

The above claim, in a sense, casts a shadow upon the reliability of assessments based upon a well known static oligopoly model. Yet, reading it from a more positive angle, it says that the overproduction resulting from dynamic oligopolistic interaction has a desirable effect on welfare, that cannot be grasped through a merely static analysis.

6. Concluding remarks

In this paper we have investigated the properties of a dynamic oligopoly game with sticky prices and differentiated products. It is important to stress that the rigidities we have dealt here with, are real rather than nominal, provided that we have taken a partial equilibrium approach, with sticky relative prices.

We have shown that the dynamic rule governing the price motion (and in particular the degree of price stickiness) affects the final allocation, i.e., the steady state under the Nash equilibrium of the dynamic game. In particular two properties are worth mentioning: (i) in the (subgame perfect) closed-loop memoryless Nash equilibrium a steady state exists, which is stable in the saddle point sense, where the production is larger and the price is lower as compared to the open-loop steady state solution; (ii) irrespective of the equilibrium concept one adopts, in the dynamic framework, the steady state output levels and price levels are, respectively, higher and lower then their counterparts in the static game. Property (i) can be reformulated by saying that, if firms are unable to initially commit to a given output plan for the whole time horizon, then subgame perfection entails overproduction (for analogous results see Spence, 1979; Reynolds, 1987). Property (ii) suggests that the dynamic nature of interaction leads forms to over-produce, as compared to the Nash equilibrium of a static interaction.

The above mentioned results are analogous with the findings from the homogenous oligopoly model. In the present paper, additional results have been found, concerning the effects of the differentiation among the goods produced by firms upon steady state allocations. Under both the open-loop and the closed-loop solution concepts, the higher is the substitutability among goods, the lower is the steady state level of production; the tougher is the price stickiness, the higher the steady state level of production. Under this respect, the degree of price stickiness and the degree of product differentiation exert the same qualitative effects on the steady state output level. Moreover, the degree of product differentiation interacts with the sensitivity of steady state profits to price stickiness. In particular, if the number of firms is low enough and products are sufficiently differentiated, an increase in substitutability enhances the positive effects of a higher speed of price adjustment on profits.

The welfare analysis has shown that the distortion associated with the dynamic oligopoly game is always lower than it appears when judging from the static version of the same model.

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