Optimal clearance-markdown pricing of retail goods using demand-response functions under variance-sensitive and risk-neutral criteria

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Abstract

This paper considers a dynamic pricing problem encountered during the clearance-markdown period of a retail product. Much of the literature assumes knowledge of reservation prices to solve such a problem. However, practicing retailers are more comfortable with the notion of consumers' demand-response functions than with reservation prices. Hence we introduce two models that incorporate such functions into the framework of dynamic pricing. In particular, we study in detail the Cobb-Douglas function, but our models are sufficiently general to capture the effects of any other demand-response function. Our first model is a risk-neutral model designed to maximize the *expected* value of the total revenues earned over a finite time horizon. Our second model is a risk-sensitive model designed to maximize the expected revenues subject to a constraint on the maximum allowable variance in the revenues. To the best of our knowledge, this is the first attempt at risk-sensitive pricing of retailer goods in the clearance-markdown period. Finally, we show with the help of data collected from a major grocery chain in the northeastern USA how our models may be used in practice.

1. Introduction

Seasonality of demand, product perishability, an increasing competition among retailers, and shortening product life cycles have made sales induced by markdowns for clearance a widely-prevalent and necessary practice in the US retail industry. In fact, according to the National Retail Federation, marked-down goods, which accounted for just 8% of retail sales three decades ago, have now increased to around 20% [32]. In spite of their increasing use, "retailers hate markdowns" [32] because of the difficulties in determining the discounts and the often-adverse consequences of sub-optimal markdowns. If the prices are not cut deep enough, then they may be stuck with truckloads of "worthless" inventory. On the other hand, if the price discounts are too deep, then they may lose profits to customers who would have otherwise paid higher prices. The challenges and pitfalls of markdown-pricing strategies are quite apparent every year with scores of retailers, including Gap and Neiman Marcus, often attributing their poor financial performances to sub-optimal markdown-pricing strategies [32]. Figure 1 shows the different elements of the time horizon over which sales occur.

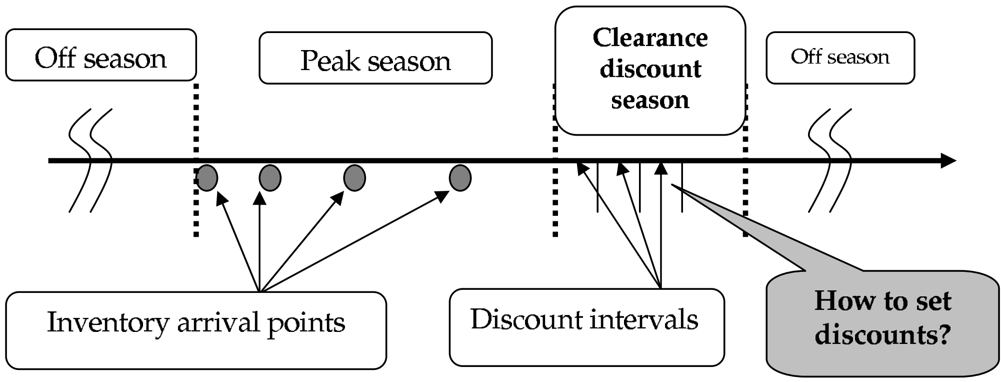

Figure 1: Time horizon of interest to us.

Despite the critical import of successful markdown pricing, in practice, retailers depend mostly on rules of thumb, gut-feelings, or what the competition is doing, rather than on any scientific decision processes [32, 45]. This is changing in recent years as a handful of retailers like J.C. Penny and L.L. Bean have adopted more scientific approaches. They have started "experimenting with sophisticated new software programs to test principles similar to yield management, which airlines mastered years ago to eke out the maximum profit from every seat" [32, 44]. Prior related research in the area of revenue management and dynamic pricing can be categorized into work done in economics, marketing, consumer psychology and operations research. Economists typically seek to understand how markets work [4, 7, 18]; a target industry is chosen and appropriate assumptions are made of the market structure (competition, oligopoly, etc) to generate defining mathematical expressions, based on proven demand-supply tenets. Empirical data collected on the number and types of market participants and elasticities of demand/supply are employed in the mathematical expressions. When the equilibrium market-conditions are identified, the behavior of one or more suppliers (firms) within the market is analyzed. Pashigan [35] was one of the first works to study consumer behavior in the context of price variations. Bagwell et al. [4] set up a game-theory-based competitive market equilibrium for retail goods, conducting a mathematical analysis on directions that prices and investment decisions take. An analysis of the gasoline markets by Borenstein and Shepard [7] reveals higher collusion between market players to be an important causative agent for increased price margins and dynamic pricing trends. The paper by Courty and Li [11] reveals the importance of understanding changing customer information and behavior to increase firm profit.

The literature on marketing and consumer psychology, notably the works by Kalyanaraman and Winer [23] and Briesch et al. [8], provides compelling evidence for the dependence of demand on past (reference) prices. The research is often empirical, although in many cases [24, 23, 26] theoretical generalizations are also derived. Other examples of similar work are [27, 19]. Exercises of this nature are aimed at understanding consumer behavior and dynamics, and have focussed on using such understanding to develop optimal markdown pricing strategy for firms. Adaptation-level theory [22] has been used to analyze the customers' response to the current price of a product, and prospect theory [12] has been employed to understand how individual consumers perceive current prices as *gains* or *losses* relative to a reference price, resulting in a "loss-aversion" asymmetry. These results have been validated empirically as being consistent with how consumers actually behave in Putler [38] and Kalyanaraman and Winer [23].

The body of literature on inventory-order optimization and joint ordering-and-production pricing is quite significant (see [13]) for a survey), but our focus on clearance markdown pricing renders this beyond the scope of this work. A nice review of the viewpoints of optimization theorists can be found in the works by Elmaghraby and Keskinocak [14] and by Bitran and Caldentey [5]. Gallego and van Ryzin [17] and Bitran and Mondschein [6] are seminal works that introduced the ideas of stochastic customer arrival patterns and reservation prices, jump-starting a whole new branch of research involving these concepts. They describe basic dynamic-programming models that yield optimal prices for *one* product. Probability functions are assumed for consumer arrival rates and reservation prices. Gallego and van Ryzin [17] obtain some insightful structural results relating price to stock levels and the length of the horizon, and also present an excellent analysis of some good heuristics.

Several improvements to these were introduced later, through models of varying complexity, that strove to closely replicate different real-life situations, such as multiple product scenarios (see [25, 9, 30]). A Markov decision process in a heuristic [3] and deterministic [20] sense has also been used to solve the pricing problem. Xu and Hopp [52] show that dynamic pricing coordinated with demand forecasting and inventory decisions achieves higher profits than static pricing. Lin [29] shows how real-time learning enables fine-tuning of customer arrival-rates and the precise forecasting of future demand to maximize the total revenue. Other works that detail dynamic pricing based on customer preferences and responses (with or without demand learning) include those by Chod and Rudi [10] and Van Miegham and Data [48]. A few have modeled consumer arrivals as diffusion processes using Brownian motion: Raman and Chatterjee [39], Sapra and Jackson [42] and Xu and Hopp [52]. Multi-supplier formulations using game-theory represent extensions but are complex to model.

Popescu and Wu [37] combine marketing and behavioral theory approaches to solve the dynamic pricing problem of a monopolistic supplier, whose consumers are sensitive to its pricing history. The work by Anjos, Chung and Currie [2] presents families of continuous functions that enable explicit characterization and easy implementation of optimal pricing strategies. An earlier work on *risk-sensitive* pricing was the one by Feng and Xiao [16] where they presented a risk-sensitive pricing model to maximize sales revenue for perishable commodities with fixed capacity.

In the light of this literature review, our contributions are mainly twofold: (i) we present a generic dynamic programming model for risk-neutral pricing that can incorporate complicated *demand-response functions* instead of reservation prices and (ii) we present a dynamic optimization model, based on Lagrangean relaxation, for *risk-sensitive* pricing. Both models take a modest amount of computational time. Practicing retailers are comfortable with the use of demand-response functions, especially in the clearance-markdown period, and hence our framework is amenable for direct use with real-world data to develop clearance-pricing strategies. In particular, it is not necessary to compute the distributions of the reservation prices or those of the demand-arrival patterns. Also, our second model is developed for a risk-averse retailer. Such a model is psychologically more reassuring to practitioners, who are usually risk-sensitive. We conclude this paper by showing how the methods developed here can be used with real-world data collected from a local grocery chain.

The rest of this article is organized as follows. Section 2 presents a dynamic programming model that is risk-neutral, while Section 3 presents a model that is risk-sensitive. Computational results with our models are described in Section 4. Section 5 concludes this paper.

2. A risk-neutral approach

The price elasticity of demand, as noted above, is a well-studied parameter in the literature on economics. Let S denote the total demand for a product and q the price for that product. The relationship between S and q that we have used in our computations is given by the generic function $g(.)$, such that $S = g(q)$, where $g(.)$ depends on various factors such as time of the season and the gender and the income level of the buyers [36, 46]. One example of $g(.)$ is the Cobb-Douglas function (standard literature in Microeconomics [36, 49, 50]), which is used to model a downward-sloping convex function with two substitutable entities. The function contains a constant elasticity parameter η. We will describe this function in more detail below; we first present a DP model that can incorporate such demand-response functions within itself.

We begin with some notation. $\mathcal{Q}$ will denote the discrete and finite ordered set of prices allowed, whose cardinality will be denoted by R, and $q(r)$ will denote the rth element of the set $\mathcal{Q}$. K will denote the number of periods. Typically, the duration of a period is one day, but it could also be a week or a fortnight. Let C represent the starting inventory for the product. The period will be indexed by k. The demand realized (or goods sold) in the kth period, when the price of the product equals the rth element of the ordered set $\mathcal{Q}$, will be denoted by S_k^r. This quantity is a random variable. Since the DP formulation will be backward, we will begin computing from $k = K$. Let $V_k(i)$ denote the value function of DP in the beginning of the kth period if the inventory to be sold is i. The algorithm will work as follows:

Step 1: Set $k = K$. Set $V_{K+1}(i) = 0$ for $i = 0, 1, 2 \ldots, C$.

Step 2: If $k = 1$, perform the computation in (1) for $i = C$. Otherwise, do the same for $i = 0, 1, 2, \ldots, C$.

$$V_k(i) \leftarrow \max_{r=1,2,\ldots,R} \left[\sum_{j=0}^{i} \mathsf{P}[S_k^r = i - j] \left\{ (i-j)q(r) + V_{k+1}(j) \right\} \right]. \tag{1}$$

Step 3: Decrement k by 1. If $k > 0$, return to Step 2; otherwise go to Step 4.

Step 4: Determine the optimal price, $r_k^*(i)$, for the kth period when the inventory is i as follows. For $k = 2, 3, \ldots, K$ and $i = 0, 1, 2, \ldots, C$,

$$p_k^*(i) = q \left(\arg\max_{r=1,2,\ldots,R} \left[\sum_{j=0}^{i} \mathsf{P}[S_k^r = i - j] \{ (i-j)q(r) + V_{k+1}(j) \} \right] \right).$$

And for the first period:

$$p_1^*(C) = q \left(\arg\max_{r=1,2,\ldots,R} \left[\sum_{j=0}^{C} \mathsf{P}[S_k^r = C - j] \{ (C-j)q(r) + V_2(j) \} \right] \right).$$

From the distribution of the demand and the nature of the function used, we can determine the transition probabilities defined above and thereby perform the computations required in the DP steps outlined above. We carried out a number of successful tests with the Cobb-Douglas function that we will describe in Section 4. A version of the Cobb-Douglas function is often defined as follows:

$$S_k^r = Z_{r,k}(q(r))^{-\eta}, \tag{2}$$

where Z is a random variable, whose distribution can be estimated from existing market data, that depends on the price $q(r)$ and k the time period. See Figure 2 for a graphical representation of the Cobb-Douglas function.

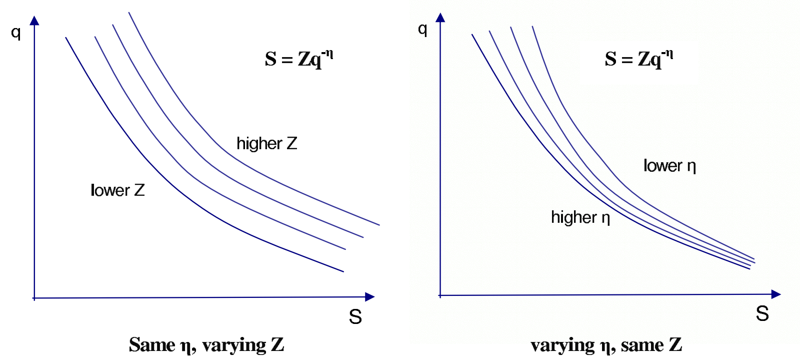

Figure 2: The influence of Z and η on the demand

3. A risk-sensitive approach

For a risk-sensitive formulation of pricing strategies, the DP model cannot be used easily, and one must resort to a more fundamental model based on mathematical programming.

Let $\mu_{r,k} \equiv \mathsf{E}[Z_{r,k}]$ and $\sigma^2_{r,k} \equiv \mathsf{V}[Z_{r,k}]$, where E and V denote the expectation and variance operators respectively. Further let $X_{r,k}$ denote a binary variable that assumes a value of 1 when price $q(r)$ is selected in period k and is 0 otherwise. If W^k_r denotes the revenue in the kth time period when the rth price is selected, then clearly $W^k_r = S^k_r q(r)$. Using the definition of the Cobb-Douglas function in (2), we have that $\mathsf{E}[W^k_r] = \mathsf{E}[(q(r))^{1-\eta} Z^k_r] = (q(r))^{1-\eta}\mathsf{E}[Z_{r,k}] = (q(r))^{1-\eta}\mu_{r,k}$.

The risk-neutral model can then be set up as the following mathematical program:

$$\text{Maximize } \sum_{r=1}^{R}\sum_{k=1}^{K} \mathsf{E}[W^k_r] X_{r,k} \equiv \sum_{r=1}^{R}\sum_{k=1}^{K} (q(r))^{1-\eta}\mu_{r,k} X_{r,k} \text{ such that} \tag{3}$$

$$\sum_{r=1}^{R}\sum_{k=1}^{K} (q(r))^{-\eta} \max(Z_{r,k}) X_{r,k} \leq C \text{ and} \tag{4}$$

$$X_{r,k} \in \{0,1\} \text{ for all } r = 1,2,\ldots,R, \text{ and } k = 1,2,\ldots,K.$$

The decision variables are $X_{r,k}$ for $r = 1,2,\ldots,R$ and $k = 1,2,\ldots,K$. The constraint in (4) ensures that the maximum demand generated by the pricing strategy does not exceed the total inventory in the system. If $\sigma^2_{r,k} \equiv \mathsf{V}[Z_{r,k}]$, then the variance of the revenues over the entire time horizon would be:

$$\sum_{r=1}^{R}\sum_{k=1}^{K} \mathsf{V}[W^k_r] X_{r,k} = \sum_{r=1}^{R}\sum_{k=1}^{K} \mathsf{V}[(q(r))^{1-\eta} Z_{r,k}] X_{r,k} = \sum_{r=1}^{R}\sum_{k=1}^{K} (q(r))^{2-2\eta}\sigma^2_{r,k} X_{r,k}.$$

Risk is an important element in the decision-making of managers in a variety of fields (see [40] for a survey). We use variance to measure risk in this paper. Variance was introduced as a measure of risk in the seminal work of Markowitz [31]. While variance exhibits a number of disadvantages with respect to measuring risk [31, 1], it is quite popular in the literature [16, 41]. The risk-sensitive pricing optimization problem (P) can then be set up as the above mathematical program subject to an additional risk constraint, which is as follows:

$$\sum_{r=1}^{R}\sum_{k=1}^{K} (q(r))^{2-2\eta}\sigma^2_{r,k} X_{r,k} \leq B, \tag{5}$$

where B is the ceiling on the risk, which is determined by managerial policy.

Our pricing problem (P), with (3) as the objective function and (4) and (5) as constraints, is a constrained longest path problem. Figure 3 shows a simple network for $R = 3$ and $K = 3$; the weights along each path are the revenues. To be more specific, one of the constraints, (4), is a knapsack constraint, whereas the other constraint, (5), is a non-linear variance constraint. There are a few pieces of work that we draw upon to develop an exact solution method for (P). Handler and Zang [21] developed an exact procedure for the constrained shortest path problem when just a knapsack constraint is present. Sivakumar and Batta [43] built upon the work in [21] to develop an exact procedure for the constrained

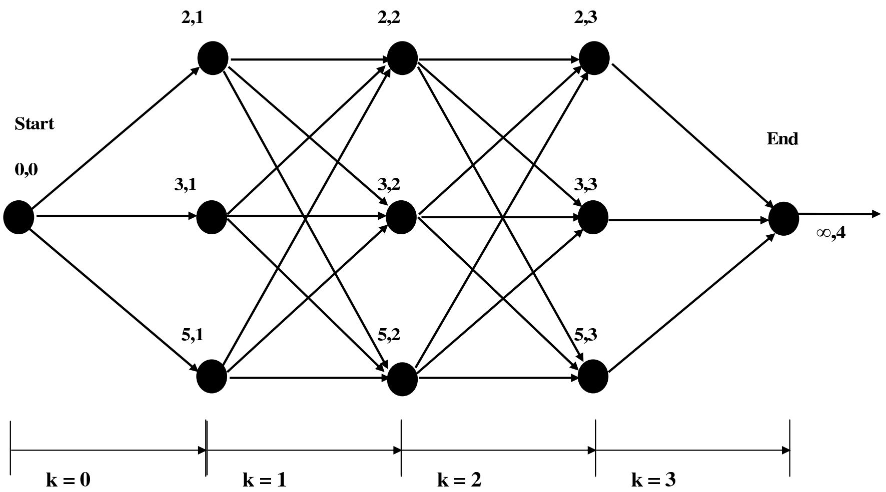

Figure 3: A directed graph for a small pricing problem in which $K = 3$ and $\mathcal{Q} = \{2, 3, 5\}$. Each node is defined by $(q(r), k)$ for $r = 1, 2, 3$ and $k = 1, 2, 3$.

shortest path problem when just a non-linear variance constraint is present. The works in both Handler and Zang [21] and Sivakumar and Batta [43] rely upon the ability to find the kth shortest path on a network, cf, [53]. Indeed, the shortest path problem is one of the very few optimization problems for which it is possible to find not just the best, but also the second best, third best, ..., solutions. This fact is exploited in [21, 43] to find an exact solution to their respective problems. A final piece of work that we draw upon is Naor and Brutlag [33], who present an algorithm to determine the kth *longest* path in a network.

Our procedure is essentially an amalgamation of the methods in [21, 43, 33]. The basic idea here is to solve the "routing problem" in two stages. The first stage involves Lagrangean dualization of the problem, yielding lower and upper bounds for the primal. In cases where the dual solution is not equal to the primal (i.e., the optimal Lagrangean multipliers are non-zero), the problem is taken to a second duality-gap-closure stage. A solution technique to the kth-longest-path problem (or shortest path in case of minimization [53, 28, 15]) is then employed for closing the duality gap to obtain an exact solution.

Following [43], we first solve for a Lagrangean dual with 2 penalty parameters. The Lagrangean relaxation of the original integer program becomes:

$$L(\lambda_1, \lambda_2) \equiv \sum_{r=1}^{R} \sum_{k=1}^{K} (q(r))^{1-\eta} \mu_{r,k} X_{r,k}$$

$$+\lambda_1 \left[C - \sum_{r=1}^{R} \sum_{k=1}^{K} (q(r))^{-\eta} \mu_{r,k} X_{r,k} \right] + \lambda_2 \left[B - \sum_{r=1}^{R} \sum_{k=1}^{K} (q(r))^{2-2\eta} \sigma_{r,k}^2 X_{r,k} \right]$$

where $\lambda_1, \lambda_2 \geq 0$. As is well-known [51], any solution of this Lagrangean relaxation will serve as an upper bound on the original problem, and the goal is to identify the best (lowest) upper bound. For this, one must solve the Lagrangean dual, i.e., minimize $L(\lambda_1, \lambda_2)$ such that

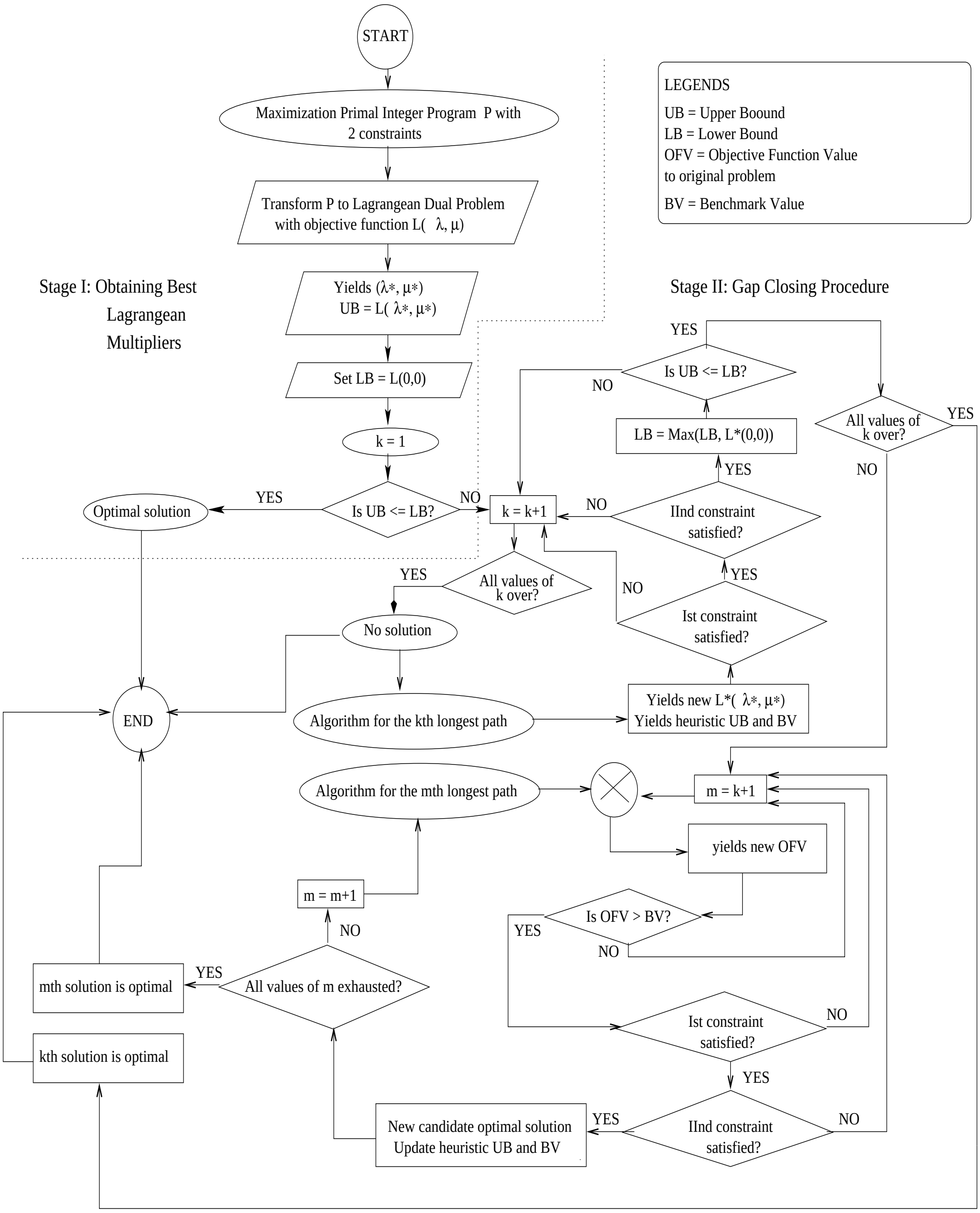

Figure 4: Flow chart for the Lagrangean Relaxation Technique

$\lambda_1, \lambda_2 \geq 0$. This is followed by the gap-closing procedure that involves iterative examination of whether the kth longest path, $(k+1)$st longest path, ... in a directed, acyclic network are optimal to the primal. The procedure is summarized in the flowchart shown in in Figure 4.

The validity of the exactness claim of [21], as formalized in [43], is briefly described here. The kth- best solution to the Lagrangean dual (with the optimal multipliers) is the best solution to the Lagrangean dual (using the same multipliers) of the *restricted primal* problem that excludes all *primal* solutions which occur among the 1st, 2nd ... kth- best solutions of the original Lagrangean dual. Furthermore, the upper bound generated in the first (Lagrangean dualization) stage is the value of the best *primal* feasible solution found so far. The progressive upper bounds, which are generated at every stage of the gap-closing procedure, will therefore be the best *primal* feasible solution among the 1st, 2nd, ..., kth best solutions of the Lagrangean. Thus when the upper bound is less than or equal to the lower bound, we conclude that an optimal solution has been obtained.

4. Computational Results

As a first step, we used some synthetic data to test the usefulness of our models. Since considerable experience has already been reported with DP in the literature on dynamic pricing, we present a more extensive account of the numerical results with the *integer program.* Finally, we conclude this section by showing how our models may be used with real-world data obtained from a national grocery chain.

4.1 Synthetic data

We first ran a basic numerical test with the risk-neutral DP model using artificial data. We were able to search for the optimal solution by exhaustive evaluation. The DP algorithm produced the optimal solution as expected. Table 1 shows the details of this experiment. More results with DP can be found in Neelakantan [34]. The transition probabilities can be found via

$$\mathsf{P}[S_k^r = l] = \mathsf{P}[Z_{r,k} = l \cdot (q(r))^{-n}],$$

where the probability can be easily determined from the distribution of $Z_{r,k}$. The distribution of $Z_{r,k}$ is assumed to be normal in our experiments.

For the risk-sensitive model, we used a problem structure with $K = 8$ and $R = 5$. The input data for μ and σ^2 of Z are detailed in Tables 2 and 3. The set of prices is $\mathcal{Q} = \{0.1, 0.2, 0.3, 0.4, 0.5\}$. The optimal price for the kth period will be denoted by $Q^*(k)$. Tables 4 and 5 present the solutions obtained from using the Lagrangean relaxation for Cases 1 though 10. Tables 6 and 7 are the corresponding tables for Cases 11 through 20. As is clear from Tables 5 and 7, the technique is able to generate an optimal solution after evaluating a fraction of the total number of solutions. The computer programs were written in C using a Pentium processor with 512 MB of RAM capacity. The computational time did not exceed 3 minutes in any of the cases.

4.2 A case study

We obtained real-world data from the Center for Relationship Marketing in the Department of Marketing at the University of Buffalo. The data was related to a grocery item, and

k	1	2	3	4	5	6	7	8	9	10
$i = 20$	10	9	10	8	10	10	10	10	8	10
$i = 19$		9	10	8	10	10	10	10	8	10
$i = 18$		9	10	8	10	10	10	10	8	10
$i = 17$		9	10	8	10	7	10	10	8	10
$i = 16$		9	10	8	10	7	10	10	8	10
$i = 15$		9	10	8	10	7	10	10	8	10
$i = 14$		9	10	8	10	7	10	10	8	10
$i = 13$		9	10	8	10	7	10	10	8	10
$i = 12$		9	10	8	10	7	10	10	8	10
$i = 11$		9	10	8	10	7	10	10	1	1
$i = 10$		9	10	8	10	1	1	1	1	1
$i = 9$		9	10	1	1	1	1	1	1	1
$i = 8$		1	1	1	1	1	1	1	1	1
$i = 7$		1	1	1	1	1	1	1	1	1
$i = 6$		1	1	1	1	1	1	1	1	1
$i = 5$		1	1	1	1	1	1	1	1	1
$i = 4$		1	1	1	1	1	1	1	1	1
$i = 3$		1	1	1	1	1	1	1	1	1
$i = 2$		1	1	1	1	1	1	1	1	1
$i = 1$		1	1	1	1	1	1	1	1	1
$i = 0$		1	1	1	1	1	1	1	1	1

Table 1: The values of $p_k^*(i)$ using the DP algorithm with the following parameters: $\eta = 0.1$; $\mu_{r,k} = 100$; $C = 20$; $\sigma_{r,k} = 3$ for all r and k; $\mathcal{Q} = \{1, 2, \ldots, 10\}$.

k	1	2	3	4
$r = 1$	$\mu_{11} = 0$	$\mu_{12} = 10$	$\mu_{13} = 20$	$\mu_{14} = 30$
	$\sigma_{11}^2 = 1$	$\sigma_{12}^2 = 2$	$\sigma_{13}^2 = 3$	$\sigma_{14}^2 = 4$
$r = 2$	$\mu_{21} = 10$	$\mu_{22} = 20$	$\mu_{23} = 30$	$\mu_{24} = 40$
	$\sigma_{21}^2 = 3$	$\sigma_{22}^2 = 5$	$\sigma_{23}^2 = 7$	$\sigma_{24}^2 = 9$
$r = 3$	$\mu_{31} = 20$	$\mu_{32} = 30$	$\mu_{33} = 40$	$\mu_{34} = 50$
	$\sigma_{31}^2 = 5$	$\sigma_{32}^2 = 7$	$\sigma_{33}^2 = 9$	$\sigma_{34}^2 = 11$
$r = 4$	$\mu_{41} = 30$	$\mu_{42} = 40$	$\mu_{43} = 50$	$\mu_{44} = 60$
	$\sigma_{41}^2 = 7$	$\sigma_{42}^2 = 9$	$\sigma_{43}^2 = 11$	$\sigma_{44}^2 = 13$
$r = 5$	$\mu_{51} = 40$	$\mu_{52} = 50$	$\mu_{53} = 60$	$\mu_{54} = 70$
	$\sigma_{51}^2 = 9$	$\sigma_{52}^2 = 11$	$\sigma_{53}^2 = 13$	$\sigma_{54}^2 = 15$

Table 2: $\mu_{r,k}$ and $\sigma_{r,k}$ for $k = 1$ through $k = 4$

k	5	6	7	8
$r=1$	$\mu_{15}=40$	$\mu_{16}=50$	$\mu_{17}=60$	$\mu_{18}=70$
	$\sigma_{15}^2=5$	$\sigma_{16}^2=6$	$\sigma_{17}^2=7$	$\sigma_{18}^2=8$
$r=2$	$\mu_{25}=50$	$\mu_{26}=60$	$\mu_{27}=70$	$\mu_{28}=80$
	$\sigma_{25}^2=11$	$\sigma_{26}^2=13$	$\sigma_{27}^2=15$	$\sigma_{28}^2=17$
$r=3$	$\mu_{35}=60$	$\mu_{36}=70$	$\mu_{37}=80$	$\mu_{38}=85$
	$\sigma_{35}^2=13$	$\sigma_{36}^2=15$	$\sigma_{37}^2=17$	$\sigma_{38}^2=19$
$r=4$	$\mu_{45}=70$	$\mu_{46}=80$	$\mu_{47}=85$	$\mu_{48}=90$
	$\sigma_{45}^2=15$	$\sigma_{46}^2=17$	$\sigma_{47}^2=19$	$\sigma_{48}^2=21$
$r=5$	$\mu_{55}=80$	$\mu_{56}=85$	$\mu_{57}=90$	$\mu_{58}=100$
	$\sigma_{55}^2=17$	$\sigma_{56}^2=19$	$\sigma_{57}^2=21$	$\sigma_{58}^2=23$

Table 3: $\mu_{r,k}$ and $\sigma_{r,k}$ for $k=5$ through $k=8$

Case	B	C	η	$Q^*(k)$
				$(k=1,2,\dots,8)$
1	300	400	0.1	$0.1, 0.2, 0.2, 0.4, 0.5, 0.5, 0.5, 0.1$
2	300	500	0.1	$0.4, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.1$
3	300	750	0.5	$0.1, 0.4, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5$
4	300	1250	1.5	$0.1, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.1$
5	300	1500	1.5	$0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.4, 0.1$
6	300	2000	1.5	$0.5, 0.5, 0.5, 0.5, 0.5, 0.4, 0.3, 0.5$
7	300	5000	1.5	$0.5, 0.5, 0.5, 0.5, 0.1, 0.5, 0.1, 0.5$
8	400	400	0.1	$0.1, 0.2, 0.2, 0.4, 0.5, 0.5, 0.5, 0.1$
9	1000	400	0.1	$0.1, 0.2, 0.2, 0.4, 0.5, 0.5, 0.5, 0.1$
10	1000	750	0.5	$0.1, 0.4, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5$

Table 4: Optimal prices for cases 1 through 10.

Case	$\mathsf{E}[Sales]$	$\mathsf{E}[Revenue]$	$\mathsf{V}[Revenue]$	X
1	397.92	174.71	19.67	129632
2	499.10	246.26	29.04	2699
3	749.46	368.28	57.7	194
4	1233.52	615.50	282	385701
5	1424.94	678.87	295.5	353776
6	1934.46	838.24	275.17	156732
7	4940.25	952.23	300	11931
8	397.92	174.71	19.67	129632
9	397.92	174.71	19.67	129632
10	749.46	368.28	57.7	194

Table 5: Results of the Lagrangean Relaxation technique for Cases 1 through 10: X denotes the number of solutions examined, while the total number of solutions in each problem is $5^8 = 390625$.

Instance	B	C	η	$Q^*(k)$
				$(k = 1, 2, \ldots, 8)$
11	1000	1250	1.5	$0.1, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.1$
12	1000	2000	1.5	$0.5, 0.5, 0.5, 0.5, 0.5, 0.4, 0.3, 0.5$
13	1500	400	0.1	$0.1, 0.2, 0.2, 0.4, 0.5, 0.5, 0.5, 0.1$
14	1500	750	0.5	$0.1, 0.4, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5$
15	1500	2000	1.5	$0.5, 0.5, 0.5, 0.5, 0.5, 0.4, 0.3, 0.5$
16	1500	2000	2	$0.5, 0.5, 0.5, 0.4, 0.5, 0.5, 0.5, 0.1$
17	2000	2000	2	$0.5, 0.5, 0.5, 0.4, 0.5, 0.5, 0.5, 0.1$
18	2000	3000	1.5	$0.5, 0.5, 0.5, 0.5, 0.5, 0.3, 0.2, 0.2$
19	2000	3000	2.0	$0.5, 0.5, 0.5, 0.5, 0.5, 0.4, 0.2, 0.5$
20	2000	5000	1.5	$0.5, 0.5, 0.5, 0.5, 0.5, 0.2, 0.1, 0.2$

Table 6: Optimal prices for cases 11 through 20.

Case	$\mathsf{E}[Sales]$	$\mathsf{E}[Revenue]$	$\mathsf{V}[Revenue]$	X
11	1233.52	615.50	282	385701
12	1934.46	838.24	275.17	156732
13	397.92	174.71	19.67	129632
14	749.46	368.28	57.7	194
15	1934.46	838.24	275.17	156732
16	1995.00	960	1241.25	390006
17	1995.00	960	1241.25	390006
18	2951.59	887.48	340	67386
19	2988.89	1266.67	647.14	353495
20	4627.37	958.67	350	9645

Table 7: Results of the Lagrangean Relaxation technique for Cases 11 through 20: X denotes the number of solutions examined, while the total number of solutions in each problem is $5^8 = 390625$.

Case	$\mu_{r,k}$	$\sigma_{r,k}$	Total Revenues
1	56.33	3.0	352.62
2	56.33	3.5	312.31
3	56.33	4.0	386.93
4	56.33	4.5	369.29
5	56.33	5.0	359.84
6	56.33	5.5	358.02
7	56.33	6.0	369.40
8	56.33	6.5	329.51
9	56.33	7.0	360.21

Table 8: The expected revenues from the risk-neutral model.

came from a major grocery chain from the Northeastern United States. Several pieces of information are required to compute optimal prices. More details on the weekly sales of the grocery item and how the data was processed to extract information can be found in [34]. It was clear that the retailer normally sets the price at any amongst the following 3 values: \$1.99, \$1.69, and \$1.5. \$1.99 is the price for the retailer's peak season when equilibrium conditions hold and \$1.69 and \$1.5 are the two discounted prices set during the retailer's clearance season. Each time-frame comprises of a 2-week peak season followed by a 1-week clearance season. Each clearance season can be divided into 7 time intervals of a day each, i.e., $K = 7$. The number of discount season prices to be chosen from was limited to the price set $\mathcal{Q} = \{1.29, 1.69, 1.99, 2.29, 2.69\}$. Table 8 lists the expected revenues generated during the discount-week for each mean-variance combination using the DP algorithm.

For the risk-sensitive model, we used various values for the means and the standard deviations ($\mu_{q,k}$ and $\sigma_{q,k}$) of the stochastic demand factor Z_q^k. Based on the actual data obtained, $\mu_{r,k}$ is varied from 10 to 75 and $\sigma_{r,k}$ from 2 to 7. In this way, 39 test instances (cases) were generated using different combinations of these values. The beginning inventory C was assumed to be 210 based on the data obtained, while the ceiling, B, on the total risk was allowed to take any one of the following 3 values: $100, 200$ and 500. The value of η used was 0.7. The optimal prices, the expected revenues, and the variance of the revenues generated during the discount-week are shown in Tables 9 and 10.

5. Conclusions

Dynamic pricing has attracted considerable attention in the last few years. A recent text by Talluri and van Ryzin [47] treats this topic in depth. In this paper, we considered a dynamic pricing problem encountered in the clearance period. We introduced dynamic-programming and integer-programming models for dynamic pricing using the demand-response functions that practicing retailers are comfortable with. Although our models focussed on the Cobb-Douglas functions, our models are general enough to accommodate any other demand-response function. To the best of our knowledge, this is the first work on risk-sensitive pricing in the clearance-markdown period. Finally, we demonstrated that our models can be used on real-world data, via a case study of a grocery item from a large grocery chain in Northeastern USA.

A number of extensions to this problem are being considered by us. First, an interesting extension is to consider the *joint* ordering-and-pricing problem in the context of clearance-markdowns. Secondly, an extension to *multiple* products, where joint discounts can be offered, seems to be quite appealing from the viewpoint of the real-world practitioner. Thirdly, we are considering the addition of constraints, such as the requirement of minimum inventory stocks into the demand-response functions, for future work.

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Case	B	$Q^*(k)$
1	100	1.99, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
2	200	1.29, 1.29, 1.69, 1.69, 2.29, 2.69, 2.29
3	500	1.69, 1.69, 1.69, 1.69, 1.69, 2.69, 1.99
4	100	2.69, 2.69, 2.69, 2.69, 2.69, 2.69, 1.99
5	200	2.69, 2.69, 1.69, 1.69, 1.69, 1.69, 1.69
6	500	1.69, 1.69, 1.69, 1.69, 1.69, 2.69, 1.99
7	100	2.69, 2.69, 2.69, 2.69, 2.69, 2.69, 1.99
8	200	2.69, 2.29, 2.29, 1.69, 1.69, 1.29, 1.29
9	500	1.99, 1.99, 1.99, 1.99, 1.69, 1.69, 1.69
10	100	1.99, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
11	200	1.69, 1.69, 1.69, 1.69, 1.69, 2.69, 2.69
12	500	1.99, 1.99, 1.99, 1.99, 1.69, 1.69, 1.69
13	100	1.99, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
14	200	1.29, 1.29, 1.29, 1.69, 2.29, 2.29, 2.69
15	500	1.69, 1.69, 1.69, 1.99, 1.69, 1.69, 1.69
16	100	1.99, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
17	200	1.29, 1.29, 1.29, 1.69, 2.29, 2.29, 2.69
18	500	1.69, 1.69, 1.69, 1.99, 1.69, 1.69, 1.69
19	100	2.69, 2.69, 2.69, 2.69, 2.69, 2.69, 1.99
20	200	2.69, 2.69, 1.69, 1.29, 1.29, 1.69, 1.99
21	500	1.99, 1.99, 1.69, 1.69, 1.69, 1.69, 1.99
22	100	1.99, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
23	200	1.69, 1.99, 1.69, 1.29, 1.29, 2.69, 2.69
24	500	1.99, 1.99, 1.69, 1.69, 1.69, 1.69, 1.99
25	100	2.69, 2.69, 2.69, 2.29, 2.69, 2.69, 2.69
26	200	2.69, 2.69, 1.69, 1.29, 1.29, 1.29, 2.69
27	500	1.99, 1.99, 1.69, 1.69, 1.69, 1.69, 1.99
28	100	2.29, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
29	200	1.29, 1.29, 2.69, 2.69, 2.69, 1.69, 1.29
30	500	1.69, 1.69, 1.69, 1.99, 1.69, 1.69, 1.69
31	100	2.29, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
32	200	1.29, 1.29, 1.29, 1.69, 2.69, 2.69, 2.69
33	500	1.69, 1.69, 1.69, 1.99, 1.99, 1.99, 1.99
34	100	2.69, 2.69, 2.69, 2.69, 2.69, 2.69, 2.29
35	200	2.69, 2.69, 2.69, 1.69, 1.29, 1.29, 1.29
36	500	1.99, 1.99, 1.99, 1.99, 1.69, 1.69, 1.69
37	100	2.29, 2.69, 2.69, 2.69, 2.69, 2.69, 2.69
38	200	2.69, 1.69, 1.29, 2.69, 2.69, 1.69, 1.29
39	500	1.99, 1.69, 1.69, 1.69, 1.99, 1.99, 1.99

Table 9: Optimal prices with the risk-sensitive model

Case	$\mathsf{E}[Revenue]$	$\mathsf{V}[Revenue]$	X
1	198.94	99.62	78077
2	363.41	198.15	17942
3	378.07	226.04	9375
4	249.27	99.62	75063
5	355.52	198.81	23646
6	378.07	246.39	9375
7	212.39	99.62	78093
8	357.00	197.67	20372
9	374.02	236.34	10217
10	208.90	99.62	78116
11	342.07	198.81	31926
12	374.02	247.74	10217
13	158.57	99.62	78099
14	308.37	199.34	20782
15	354.09	261.90	2149
16	158.57	99.62	78099
17	308.37	199.34	20782
18	354.09	261.90	2149
19	189.30	99.62	78115
20	348.34	199.51	16306
21	376.94	246.75	4210
22	189.30	99.62	78116
23	347.75	199.25	16663
24	376.94	250.54	4210
25	180.71	99.07	78117
26	333.48	198.72	26484
27	376.94	248.65	4210
28	147.07	99.07	78117
29	292.13	198.72	33059
30	354.09	261.90	2149
31	200.89	99.07	78117
32	341.54	198.72	32366
33	374.02	242.02	10217
34	200.89	99.07	78117
35	341.54	198.72	32366
36	374.02	242.02	10217
37	199.94	99.07	78117
38	327.31	196.30	44160
39	372.26	242.02	11020

Table 10: Expected value and variance of revenues obtained from the risk-sensitive model. X denotes the number of solutions examined.

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