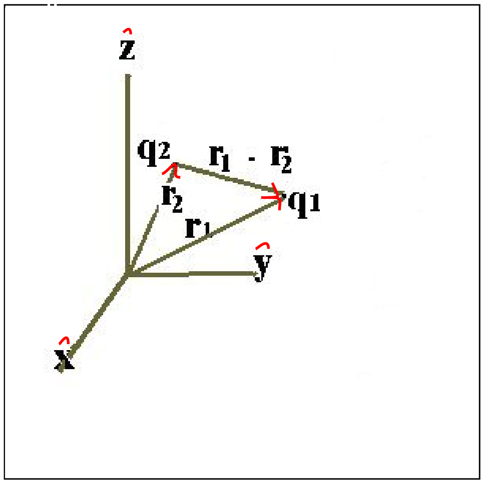
Chapter 1

Introduction and Survey

1.1 Maxwell's equations in a vacuum

1.1.1 Electrostatics The results of the numerous investigations of electromagnetic phenomena carried out during the 18th and 19th centuries led to the development of a set of equations which govern these phenomena. Coulomb's law (an action at a distance law) provided the description of the force, $\mathbf{F}_{12}$, on a stationary point particle at $\mathbf{r}_1$ with an electric charge q_1 due to a stationary point particle with charge q_2 located at $\mathbf{r}_2$ Note that the $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors. In the following equations of electrostatics the parameter k_1 is a constant determined by the system of units. In the modern system of units, SI, $k_1 = (4\pi\varepsilon_o)^{-1}$with ε_o the permittivity of free space ($=$ 8.854 x 10^{-12} farad per meter (F/m)). In Gaussian units , $k_1 = 1$ See Table 1 on page 779.



$$\mathbf{F}_{12} = k_1 \frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2).... \tag{1.01}$$

$$\text{where in SI units } k_1 = \frac{1}{4\pi\epsilon_o} = 10^{-7}c^2 \text{ and } q \text{ is in Coulombs (C)}$$

The concept of an electric field, $\mathbf{E}(\mathbf{r}_1 - \mathbf{r}_2)$, generated by electrically charged point particles

$$\mathbf{E}(\mathbf{r}_1 - \mathbf{r}_2) = k_1 \frac{q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.02}$$

allowed a local force law on point charges to be developed.

$$\mathbf{F}_{12} = q_1 \mathbf{E}(\mathbf{r}_1 - \mathbf{r}_2) \tag{1.03}$$

Superposition principle

The superposition principle holds for the electric field and if $\{q_i,\, i = 1, ..., N\}$ are the charges of N point particles located at the points $\{\mathbf{r}_i,\, i = 1, ..., N\}$ the electric field at a point $\mathbf{r} \neq \mathbf{r}_i$ is

$$\mathbf{E}(\mathbf{r}) = k_1 \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|^3} (\mathbf{r}-\mathbf{r}_i) = \frac{1}{4\pi\epsilon_o} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r}-\mathbf{r}_i|^3} (\mathbf{r}-\mathbf{r}_i) \text{. SI units} \tag{1.04}$$

The relationship between the charged particles and the fields which they generate is generalized by Gauss' law. This law relates electric fields to electric charge densities, $\rho(\mathbf{r})$, in the area of electrostatics.

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = 4\pi k_1 \rho(\mathbf{r}) = \rho(\mathbf{r})/\epsilon_o \text{ in SI units} \tag{1.05}$$

Point charges and the Dirac Delta function

In the case that $\rho(\mathbf{r}) = q\,\delta^{(3)}(\mathbf{r}-\mathbf{r}_2)$,where $\delta^{(3)}(\mathbf{r}-\mathbf{r}_2)$ is a three dimensional 'Dirac delta function' (see next two pages) [1], a solution to this partial differential equation is the electric field due to a charged point particle located at $\mathbf{r}_2$. The (one dimensional) Dirac delta function is defined by

$$\int_a^b f(x')\,\delta(x-x')\,dx' = f(x) \qquad \text{if } a<x<b \text{ and 0 otherwise.} \tag{1.06}$$

If $x=a$ or $x=b$ the integral is not well defined, but a common definition is

$$\int_a^b f(x')\,\delta(x-x')\,dx' = \frac{1}{2} f(x) \qquad x=a \text{ or } x=b$$

The three dimensional Dirac delta function (recognized henceforth with vectors as variables) is equal to the product of three one dimensional Dirac delta functions

$$\delta(\mathbf{r}-\mathbf{r}') \equiv \delta(x-x')\,\delta(y-y')\,\delta(z-z') \tag{1.07}$$

so that

$$\iiint_V f(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')dx'dy'dz' = f(\mathbf{r}) \quad \text{if } \mathbf{r} \text{ is interior to the volume, } V \text{ and } 0 \text{ otherwise} \tag{1.08}$$

If the argument of a Diract delta function is a function of x,

$$\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|\frac{dg}{dx}|_{x=x_i}} \text{ where } g(x_i)=0 \text{ denote simple zeros of } g(x) \tag{1.09}$$

[1] The Dirac delta function is defined by the condition that

$$\int_a^b f(x')\,\delta(x-x')\,dx' = f(x) \tag{1.1}$$

if $a<x<b$ and

$$\int_a^b f(x')\,\delta(x-x')\,dx' = 0 \tag{1.2}$$

if $x<a$ or $x>b$. If $x=a$ or $x=b$ the integral is not well defined.

For additional information on $\delta(\mathbf{r}-\mathbf{r}')$ see next two pages: See the Appendix.

Continuous distribution of charges (macroscopic picture)

If there are many densely distributed charges (so that macroscopically they can be described by a continuous charge density), $\frac{dq}{dv} = \rho(\mathbf{r})$, then

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\iiint_V \frac{\frac{dq(\mathbf{r})}{dV}}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,dV' = \frac{1}{4\pi\epsilon_o}\iiint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\,d^3x'. \tag{1.10}$$

Now we can write the integrand in terms of the gradient operator,∇ with respect to the $\bar{r}$ coordinate vector:

$$-\boldsymbol{\nabla}[\frac{1}{|\mathbf{r}-\mathbf{r}'|}] \quad = \quad \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\boldsymbol{\nabla}|\mathbf{r}-\mathbf{r}'| = \frac{1}{|\mathbf{r}-\mathbf{r}'|^2}\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{1.11}$$

$$\text{NOTE} \quad : \quad \boldsymbol{\nabla}|\mathbf{r}-\mathbf{r}'| \qquad \text{is a unit vector along the } \mathbf{r}-\mathbf{r}' \text{ direction}$$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\iiint_V -\boldsymbol{\nabla}[\frac{1}{|\mathbf{r}-\mathbf{r}'|}]\,\rho(\mathbf{r}')\,d^3x' \tag{1.12}$$

If one takes $\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r})$,

$$\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_o}\iiint_V -\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}[\frac{1}{|\mathbf{r}-\mathbf{r}'|}]\,\rho(\mathbf{r}')\,d^3x \tag{1.13}$$

and uses

$$\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}[\frac{1}{|\mathbf{r}-\mathbf{r}'|}] = \nabla^2[\frac{1}{|\mathbf{r}-\mathbf{r}'|}] = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{1.14}$$

the Eq 1.13 becomes,

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_o}\iiint_V 4\pi\delta(\mathbf{r}-\mathbf{r}')\,\rho(\mathbf{r}')\,d^3x \;\; \text{SI units} \\ &= \rho(\mathbf{r})/\epsilon_o \qquad \text{if } \mathbf{r} \text{ is inside the volume, V.} \\ &= 0 \quad \text{otherwise}\end{aligned} \tag{1.15}$$

One can also derive

$$\begin{aligned}\boldsymbol{\nabla}\mathbf{x}\mathbf{E}(\mathbf{r}) &= \boldsymbol{\nabla}\mathbf{x}\frac{1}{4\pi\epsilon_o}\iiint_V -\boldsymbol{\nabla}[\frac{1}{|\mathbf{r}-\mathbf{r}'|}]\,\rho(\mathbf{r}')\,d^3x' = 0 \\ \text{since} \qquad \boldsymbol{\nabla}\mathbf{x}\boldsymbol{\nabla}f(\mathbf{r}) &= 0\end{aligned} \tag{1.16}$$

The equations $\boldsymbol{\nabla}\cdot\mathbf{E}(\mathbf{r}) = \rho(\mathbf{r})/\epsilon_o$ and $\boldsymbol{\nabla}\mathbf{x}\mathbf{E}(\mathbf{r}) = 0$ define the behavior of $\mathbf{E}(\mathbf{r})$ in electrostatics. To determine $\mathbf{E}(\mathbf{r})$ uniquely, one must also have boundary conditions on $\mathbf{E}(\mathbf{r})$.

Electrostatic Potential, $\Phi(\mathbf{r})$

Sometimes it is easier to deal with scalar functions, and from $\boldsymbol{\nabla}\mathbf{x}\mathbf{E}(\mathbf{r}) = 0$ one can assume that

$$\mathbf{E}(\mathbf{r}) = -\boldsymbol{\nabla}\Phi(\mathbf{r}). \tag{1.17}$$

where the integral form for $\mathbf{E}(\mathbf{r})$ in Eq. 1.12, gives

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_o} \iiint_V \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} d^3x + \Phi_o; \qquad \nabla^2\Phi_o = 0 \tag{1.18}$$

The electrostatic potential, $\Phi(\mathbf{r})$ is the electrostatic potential energy (of a unit test charge) in the electric field, $\mathbf{E}(\mathbf{r})$ and is related to the work done by an external force in moving a charge, q, from $\mathbf{r}_o$ to $\mathbf{r}$:

$$\begin{aligned} \int_{\mathbf{r}_o}^{\mathbf{r}} -\mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' &= -\int_{\mathbf{r}_o}^{\mathbf{r}} q\mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \\ &= \int_{\mathbf{r}_o}^{\mathbf{r}} q\boldsymbol{\nabla}\Phi(\mathbf{r}) \cdot d\mathbf{r} \\ &= \int_{\mathbf{r}_o}^{\mathbf{r}} q[d\mathbf{r} \cdot \boldsymbol{\nabla}\Phi(\mathbf{r})] \\ &= \int_{\mathbf{r}_o}^{\mathbf{r}} q[d\,\Phi(\mathbf{r})] \\ &= q[\Phi(\mathbf{r}) - \Phi(\mathbf{r}_o)] \end{aligned} \tag{1.19}$$

Note that in $-\mathbf{F}(r') \cdot d\mathbf{r}'$ the negative sign denotes work done against the electric field, (not work done by the field, $\mathbf{E}(\mathbf{r})$). The following is a useful relation which you should know and be able to use:

$$d\mathbf{r} \cdot \boldsymbol{\nabla} f(\mathbf{r}) = df \quad \text{a perfect differential} \tag{1.20}$$

Magnetostatics

Lodestones have been used for millenniums as compass 'needles' for navigation. We now understand that the lodestones are magnets. These magnets are aligned by the magnetic field generated by the earth thereby setting up a north-south reference line. The compass was the principle application of magnetism until the development of electrical batteries which permitted researchers to generate electric currents.

Oersted discovered that magnetic fields were generated by electrical currents. Ampere investigated magnetic forces between charge currents thereby obtaining a quantitative relationship between charge currents and the magnetic fields they generate. He subsequently proposed a law, Ampere's law, which related magnetic (induction) fields, $\mathbf{B}(\mathbf{r})$, to electric current densities, $\mathbf{J}(\mathbf{r})$, in the area of magnetostatics.[2]

$$
\begin{aligned}
\nabla \times \mathbf{B}(\mathbf{r}) &= \alpha 4\pi k_2 \mathbf{J}(\mathbf{r}) = \mu_o \mathbf{J}(\mathbf{r}) \qquad \text{(SI units J in amp/m}^2\text{.)} \qquad (1.21)\\
\text{where } \mu_o &= \text{magnetic permeability of free space (henrys/m)}\\
\nabla \times \mathbf{B}(\mathbf{r}) &= \frac{4\pi}{c}\mathbf{J}(\mathbf{r}) \qquad \text{(Gaussian units; J in statamps/cm}^2\text{)}\\
\text{SI units} &: \quad k_2 = \frac{\mu_o}{4\pi} = 10^{-7}; \quad \alpha = 1 \qquad ; \text{ Gaussian : } k_2 = c^{-2}; \quad \alpha = c\\
\text{speed of light} &: \quad c = 2.998 \times 10^8 m/s. = \frac{1}{\sqrt{\mu_o \epsilon_o}}
\end{aligned}
$$

From Ampere's study of magnetic forces between charge currents it was deduced that the magnetic field exerts a force on moving charged objects. The magnetic force on a charge q moving with the velocity $\mathbf{v}$ at a point where the magnetic field is $\mathbf{B}$ is

$$
\begin{aligned}
\mathbf{F} &= k_3 q\, \mathbf{v} \times \mathbf{B} = q\, \mathbf{v} \times \mathbf{B} \qquad \text{(SI units.} B \text{ in tesla).} \qquad (1.22)\\
\text{SI units} &: \quad k_3 = 1; \qquad q \text{ in Coulombs}\\
\text{Gaussian cgs units} &: \quad k_3 = \frac{1}{c}
\end{aligned}
$$

1.1.2 Induced electric fields, time dependent magnetic fields

It was discovered by Faraday that time-varying magnetic fields would generate an electric field and he developed the relationship, Faraday's law,

$$
\nabla \times \mathbf{E}(\mathbf{r},t) = -k_3 \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \quad \text{(SI units.} \mathbf{B} \text{ in tesla)} \qquad (1.23)
$$

1.1.3 Conservation of electric charge

One of the fundamental conservation laws has been found to be the conservation of electric charge. This is stated in the relationship between electric charge densities, $\rho(\mathbf{r})$, and electric current densities, $\mathbf{J}(\mathbf{r})$,

[2] The parameter k_2 is a constant which is determined by the system of units. In the MKSA system $k_2 = \mu_o$ with μ_o the magnetic permeability of free space. In the Gaussian cgs system $k_2 = 4\pi/c$ with c the speed of light (a defined constant).

$$\nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t} = 0. \tag{1.24}$$

1.1.4

Induced magnetic fields, time varying electric fields

Ampere's law implies that charge is not conserved. This is demonstrated by the following analysis. It is a mathematical identity that

$$\nabla \cdot \nabla \times \mathbf{V}(\mathbf{r}) = 0$$

where $\mathbf{V}(\mathbf{r})$ is a vector field. If we apply this identity to Eq. 1.21, Ampere's law, we find that

$$\begin{aligned} &\nabla \cdot \nabla \times \mathbf{B}(\mathbf{r}) = \\ &\alpha 4\pi k_2 \, \nabla \cdot \mathbf{J}(\mathbf{r}) = 0 \end{aligned} \tag{1.25}$$

Clearly this violates the conservation of electric charge, $\nabla \cdot \mathbf{J}(\mathbf{r},t) = -\frac{\partial \rho(\mathbf{r},t)}{\partial t}$ if the current density results in time varying charge densities. This commonly occurs in capacitive circuits when the charge current builds up a charge on the plates of an electrical capacitor. Since Gauss' law (Eq. 1.15) provides a relationship between charge densities and electric fields the problem can be solved, as it was by Maxwell, with the addition to $J(\mathbf{r})$ of a term as follows

$$\begin{aligned} & \nabla \cdot \nabla \times \mathbf{B}(\mathbf{r}) \\ = \; & \alpha 4\pi k_2 [\nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{\partial \rho(\mathbf{r},t)}{\partial t}] \\ = \; & \alpha 4\pi k_2 [\nabla \cdot \mathbf{J}(\mathbf{r},t) + \frac{1}{4\pi k_1} \frac{\partial}{\partial t} \nabla \cdot \mathbf{E}(\mathbf{r},t)] \\ = \; & \nabla \cdot [\alpha 4\pi k_2 \mathbf{J}(\mathbf{r},t) + \frac{k_2 \alpha}{k_1} \frac{\partial}{\partial t} \mathbf{E}(\mathbf{r},t)] \end{aligned}$$

This guarantees that $\nabla \cdot \nabla \times \mathbf{B}(\mathbf{r}) = 0$ but requires the addition: of $k_2 \mathbf{J}_d(\mathbf{r},t)$ to $J(\mathbf{r},t)$ in Eq. 1.21.

$$\begin{aligned} k_2 \mathbf{J}_d(\mathbf{r},t) &= \frac{k_2 \alpha}{k_1} \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} = \mu_o \epsilon_o . \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \\ &= \frac{1}{c^2} \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \quad \textbf{(SI units } \alpha = 1\textbf{)} \end{aligned} \tag{1.26}$$

In this term $\mathbf{J}_d$ is called the displacement current density. The resulting equation

$$\nabla \times \mathbf{B}(\mathbf{r},t) = \mu_o \mathbf{J}(\mathbf{r},t) + \frac{1}{c^2} \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t} \quad \textbf{(SI units)} \tag{1.27}$$

is often called the Ampere-Maxwell equation.

1.1.5 A scarcity of magnetic monopoles

In the following we show that, although Gauss' law and Faraday's law are sufficient to provide a unique boundary value problem for the electric field, $E(\mathbf{r},t)$, (when boundary conditions are supplied), the Ampere-Maxwell law by itself does not provide a unique boundary value problem for the magnetic field.

Any vector field, $\mathbf{V}(\mathbf{r})$, can be written as the sum of two vectors

$$\mathbf{V}(\mathbf{r}) = -\boldsymbol{\nabla}\varphi(\mathbf{r}) + \boldsymbol{\nabla}\times\mathbf{A}(\mathbf{r}) \tag{1.28}$$

where $\varphi(\mathbf{r})$ is any scalar field and $\mathbf{A}(\mathbf{r})$ is any vector field. In this case

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{V}(\mathbf{r}) &= -\nabla^2\varphi(\mathbf{r}) \\ \boldsymbol{\nabla}\times\mathbf{V}(\mathbf{r}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}(\mathbf{r})) - \nabla^2\mathbf{A}(\mathbf{r}). \end{aligned} \tag{1.29}$$

In order to uniquely specify a vector field by a set of differential equations we must give both the curl and divergence of the vector field.

$$\boldsymbol{\nabla}\cdot\mathbf{V}(\mathbf{r}) = f(\mathbf{r}) \tag{1.30}$$

$$\boldsymbol{\nabla}\times\mathbf{V}(\mathbf{r}) = \bar{j}(\mathbf{r}) \tag{1.31}$$

where $f(\mathbf{r})$ is a scalar field and $\mathbf{j}(\mathbf{r})$ is a vector field. Thus Gauss' law, Eq. 1.15, and Faraday's law, Eq. 1.23, will uniquely determine the electric field (once boundary conditions are supplied). However the Ampere-Maxwell law, Eq. 1.27, only gives the curl of the magnetic field.. It follows that for a given value of the magnetic field on the boundary of some region we can generate many solutions to Eq. 1.27 just by adding a vector $\nabla\varphi(\mathbf{r})$ to $\mathbf{B}(\mathbf{r},t)$ which vanishes on the boundary. Note that this added vector will not change Eq. 1.27 since $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\varphi(\mathbf{r}) = 0$. The additional condition is obtained by requiring that $\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{r},t) = f(\mathbf{r})$. In that case the divergence of the magnetic field ($f(\mathbf{r})$)should be related to the magnetic charge density. Although the search for a magnetic charge continues to this day, none has ever been found. Until magnetic charges have been shown to exist we assume, as did Maxwell, that

$$\boldsymbol{\nabla}\cdot\mathbf{B}(\mathbf{r},t) = 0 \tag{1.32}$$

1.1.6 The Lorentz force

The four equations , 1.15, 1.25, 1.27, and 1.32 were developed and investigated by Maxwell in the late 19th century. In modern times they are known as *Maxwell's equations*. These equations, together with the force equations, Eq. 1.03 and Eq. 1.22 form the basis of classical electrodynamics. The latter equations are added to form the equation known as the *Lorentz force* law

$$\mathbf{F} = q[\mathbf{E}+k_3\mathbf{v}\times\mathbf{B}] = q[\mathbf{E}+(\mathbf{v}\times\mathbf{B})]. \qquad \text{(SI units)} \tag{1.33}$$

The system of units

The International System (SI) of units is the MKSA. This is the system of units used in the journals. However, since in some later chapters the text also uses the Gaussian cgs system of units, we will also refer to this system so as to facilitate a comparison See pages 778f in 3rd Edition of Jackson

Maxwell's equations in a vacuum (SI units)

$$
\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}) &= \rho(\mathbf{r})/\epsilon_o \\
\nabla \times \mathbf{E}(\mathbf{r},t) &= -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0 \\
\nabla \times \mathbf{B}(\mathbf{r},t) &= \mu_o \mathbf{J}(\mathbf{r},t) + \frac{1}{c^2}\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}
\end{aligned}
\qquad (1.34)
$$

Maxwell's equations in a vacuum (in terms of k_1,k_2, and $k_3 = 1/\alpha$)

$$
\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}) &= 4\pi k_1 \rho(\mathbf{r})_o \\
\nabla \times \mathbf{E}(\mathbf{r},t) &= -k_3 \frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} \\
\nabla \cdot \mathbf{B}(\mathbf{r},t) &= 0 \\
\nabla \times \mathbf{B}(\mathbf{r},t) &= 4\pi k_2 \alpha \mathbf{J}(\mathbf{r},t) + \frac{k_2 \alpha}{k_1}\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}
\end{aligned}
\qquad (1.35)
$$

:

system	k_1	k_2	α	k_3	q	B	cgs/mks	comments
esu (Electrostatic)	1	$\frac{1}{c^2}$	1	1	statcoul.		cgs	$\frac{k_1}{k_2} = c^2$
emu (Electromag.)	c^2	1	1	1	statcoul.		cgs	$c = c_o 10^{10} cm/s$
Gaussian	1	$\frac{1}{c^2}$	c	$\frac{1}{c}$	statcoul.	gauss (G)	cgs	$10^4 G = 1$ tesla
Heaviside-Lorentz	$\frac{1}{4\pi}$	$\frac{1}{4\pi c^2}$	c	$\frac{1}{c}$	statcoul		cgs	$c_o = 2.997\ 924\ 58$
SI (MKSA)	$\frac{1}{4\pi\epsilon_o}$	$\frac{\mu_o}{4\pi}$	1	1	Coulomb (C)	tesla	mks	$\epsilon_o \mu_o = \frac{1}{c^2}$
SI (MKSA)	$10^{-7}c^2$	10^{-7}	1	1	"	"	"	1C $= c_o 10^9$ statcou

system/ units	k_1	k_2	α	k_3
esu	1	$cm^{-2}s^2$	1	1
emu	$cm^2 s^{-2}$	1	1	1
Gausian	1	$cm^{-2}s^2$	cm/s	cm/s
Heaviside-Lorentz	1	$cm^{-2}s^2$	cm/s	cm/s
SI	kg-$m^3 s^{-4} Amp^{-2}$	kg-$m s^{-2} Amp^{-2}$	1	1

quantity		
Ampere	$\frac{dq}{dt} Coul.s^{-1}$	V/ohm = V/Ω
$\epsilon_o =$	$\frac{10^7}{4\pi c^2}$	8.854 187 10^{-12} Farad/m (SI)
$\mu_o =$	$4\pi 10^{-7}$	1.256 637 10^{-6} Henry/m
$\sqrt{\frac{\mu_o}{\epsilon_o}}$	376.730 ohms (Ω)	376.730 $\cdot 10^5/c^2$
$\hbar$	1.0546 $10^{-34} Js$	6.582 10^{-16}eVs
e	1.602 $10^{-19} Coulomb$	4.80 10^{16}statcoul $(esu, Gaussian)$
$\frac{e^2}{4\pi\epsilon_o c\hbar}$ $(SI) =$	$\frac{1}{137.04}$	$\frac{e^2}{c\hbar}$ $(esu, Gaussian)$
Bohr radius, r_H	0.529177 Å	0.529177 10^{-10} m
$\frac{e^2}{4\pi\epsilon_o (2r_H)}$ $(SI) =$	13.605 eV	$\frac{e^2}{2r_H}$ $(esu, Gaussian)$

1.2 Maxwell's equations in a medium

1.2.1 Decomposition of the sources and fields In the last section we introduced Maxwell's equations in a vacuum *with (charge density and current density) sources*.

$$\begin{aligned}\nabla\cdot\mathbf{E}_o(\mathbf{r}) &= \rho_o(\mathbf{r})/\epsilon_o \\ \nabla\times\mathbf{E}_o(\mathbf{r},t) &= -\frac{\partial\mathbf{B}_o(\mathbf{r},t)}{\partial t} \\ \nabla\cdot\mathbf{B}_o(\mathbf{r},t) &= 0 \\ \nabla\times\mathbf{B}_o(\mathbf{r},t) &= \mu_o\mathbf{J}_o(\mathbf{r},t)+\frac{1}{c^2}\frac{\partial\mathbf{E}_o(\mathbf{r},t)}{\partial t}\end{aligned} \tag{1.34}$$

where the subscript o implies that the fields are due to specified charge and charge current densities$\rho_o(\mathbf{r})$ and. $\mathbf{J}_o(\mathbf{r},t)$, respectively. However the Lorentz force law implies that the fields can affect these densities. In this section we will examine how the equations can be modified if the system of charges are permitted to respond to the fields.

We investigate here a system (gas, liquid, or solid) which is in a region where there are electric and magnetic fields, $\mathbf{E}_o$ and $\mathbf{B}_o$ due to fixed charge and charge current densities. The classical Maxwell's equations are explicitly linear in the fields. This suggests that we consider the electric field in the system, **E,** to be the sum of the original electric field, $\mathbf{E}_o$, and a 'response' electric field, $\Delta\mathbf{E}$ due to the response of the system to the applied fields. Similarly the magnetic field, **B**, will be the sum of $\mathbf{B}_o$ and a 'response' field $\Delta\mathbf{B}$.

Maxwell's equations can be written for the $\Delta\mathbf{E}$ and $\Delta\mathbf{B}$

$$\begin{aligned}\nabla\cdot\Delta\mathbf{E}(\mathbf{r}) &= \Delta\rho(\mathbf{r},t)/\epsilon_o \\ \nabla\cdot\Delta\mathbf{B}(\mathbf{r}) &= 0 \\ \nabla\times\Delta\mathbf{E}(\mathbf{r},t) &= -\frac{\partial\Delta\mathbf{B}(\mathbf{r},t)}{\partial t} \\ \nabla\times\Delta\mathbf{B}(\mathbf{r},t) &= \mu_o\Delta\mathbf{J}(\mathbf{r},t)+\frac{1}{c^2}\frac{\partial\Delta\mathbf{E}(\mathbf{r},t)}{\partial t}\end{aligned} \tag{1.35}$$

1.2.2 The material properties

For this analysis we'll neglect the dependence of the current density on the magnetic field and assume that the electrical conductivity function for the system, σ, is known. In this case the current density would be given by

$$\Delta J_i(\mathbf{r},t)=\sum_{j=1}^{3}\iiiint\sigma_{ij}(\mathbf{r}-\mathbf{r}',t-t')\ E_j(\mathbf{r}',t')\ d^3r'\,dt'. \tag{1.36}$$

This integral involves four'convolutions'. The convolution, C(x) , of two functions f(x) and g(x) is defined as

$$C(x)=\int f(x-x')\ g(x')\ dx'.$$

Defining the $\mathbf{k}-$frequency Fourier transform of $J_i(\mathbf{r},t)$ as $j_i(\mathbf{k},\omega)$,

$$\Delta J_i(\mathbf{r},t)=\frac{1}{(2\pi)^2}\iiiint e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\Delta\, j_i(\mathbf{k}',\omega')\ d^3k'\,d\omega' \tag{1.37}$$

and making use of the identity

$$\delta\left(\mathbf{k}-\mathbf{k}'\right)\delta\left(\omega-\omega'\right)=\frac{1}{(2\pi)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\left[\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)-\left(\omega-\omega'\right)t\right.}d^3r dt. \tag{1.38}$$

we obtain

$$\begin{aligned}&\iiiint e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Delta J_i\left(\mathbf{r},t\right)\,d^3r\,dt \qquad (1.40)\\ =\;& \frac{1}{(2\pi)^2}\iiiint\Delta\, j_i\left(\mathbf{k}',\omega'\right)\,d^3k'\,d\omega'\;\left[\iiiint e^{-i\left[\mathbf{r}\cdot\left(\mathbf{k}-\mathbf{k}'\right)-\left(\omega-\omega'\right)t\right.}\,d^3r\,dt\right]\\ =\;& (2\pi)^2\iiiint\delta\left(\mathbf{k}-\mathbf{k}'\right)\delta\left(\omega-\omega'\right).\Delta\, j_i\left(\mathbf{k}',\omega'\right)\,d^3k'\,d\omega'\end{aligned}$$

and the space-time Fourier transforms of the current density:

$$\Delta j_i\left(\mathbf{k},\omega\right)\equiv\frac{1}{(2\pi)^2}\iiiint e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\Delta J_i\left(\mathbf{r},t\right)\,d^3r\,dt \tag{1.41}$$

Simlarly the space-time Fourier transform of the electric field is

$$e_i\left(\mathbf{k},\omega\right)\equiv\frac{1}{(2\pi)^2}\iiiint e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}E_i\left(\mathbf{r},t\right)\,d^3r\,dt. \tag{1.42}$$

We want the relationship between $j_i\left(\mathbf{k},\omega\right)$ and $e_i\left(\mathbf{k},\omega\right)$:

$$\begin{aligned}\Delta j_i\left(\mathbf{k},\omega\right)\;&\equiv\;\frac{1}{(2\pi)^2}\iiiint e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left[\sum_{j=1}^{3}\iiiint\sigma_{ij}\left(\mathbf{r}-\mathbf{r}',t-t'\right)\,E_j\left(\mathbf{r}',t'\right)\,d^3r'\,dt'\right]\,d^3r\,dt\\ &=\;\frac{1}{(2\pi)^2}\iiiint e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\sum_{j=1}^{3}\iiiint\sigma_{ij}\left(\mathbf{r}-\mathbf{r}',t-t'\right)\;\cdot\\ &\quad\left[\frac{1}{(2\pi)^2}\iiiint e^{i\left(\mathbf{k}'\cdot\mathbf{r}'-\omega' t'\right)}\,e_j\left(\mathbf{k}',\omega'\right)\,d^3k'\,d\omega'\right]\,d^3r'\,dt'\,d^3r\,dt.\end{aligned}$$

(Now insert $e^{-i\left(\mathbf{k}'\cdot\mathbf{r}-\varpi' t\right)}\;e^{i\left(\mathbf{k}'\cdot\mathbf{r}-\varpi' t\right)}$ and rearrange terms.) $\varpi' = \omega'$

$$\begin{aligned}\Delta j_i\left(\mathbf{k},\omega\right)\;&=\;\frac{1}{(2\pi)^4}\iiiint\left[\iiiint e^{-i\left(\left(\mathbf{k}-\mathbf{k}'\right)\cdot\mathbf{r}-\left(\omega-\varpi'\right)t\right)}\,d^3r\,dt\right]e_j\left(\mathbf{k}',\omega'\right)\,d^3k'\,d\omega'\;\cdot\\ &\quad\left[\sum_{j=1}^{3}\iiiint\sigma_{ij}\left(\mathbf{r}-\mathbf{r}',t-t'\right)\,e^{-i\left(\mathbf{k}'\cdot\left(\mathbf{r}-\mathbf{r}'\right)-\omega'\left(t-t'\right)\right)}\;d^3r'\,dt'\right]\end{aligned}$$

Finally,with $\mathbf{r}''=\mathbf{r}-\mathbf{r}'$, $\widetilde{\sigma_{ij}}\left(\mathbf{k},\omega\right)$ defined by

$$\widetilde{\sigma_{ij}}\left(\mathbf{k},\omega\right)\equiv\iiiint\sigma_{ij}\left(\mathbf{r}'',t''\right)\,e^{-i\left(\mathbf{k}\cdot\mathbf{r}''-\omega t''\right)}\,(-1)^4d^3r''\,dt'', \tag{1.43}$$

we obtain,

$$\Delta j_i(\mathbf{k},\omega) = \frac{1}{(2\pi)^4} \iiiint (2\pi)^4 \, \delta(\mathbf{k}-\mathbf{k}') \, \delta(\omega-\omega') \, e_j(\mathbf{k}',\omega') \; \widetilde{\sigma_{ij}}(\mathbf{k},\omega) \, d^3k' \, d\omega'.$$

Note that Eq. 1.38 gives the $\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega')$ factor and enables the simple integration over $d^3k'\,d\omega'$:

$$\Delta j_i(\mathbf{k},\omega) = \sum_{j=1}^{3} \widetilde{\sigma_{ij}}(\mathbf{k},\omega) \; e_j(\mathbf{k},\omega) \tag{1.45}$$

The charge conservation law will relate the Fourier transform of the charge current density to the Fourier transform of the charge density. We define the Fourier transform of the charge density by

$$\widetilde{\Delta\rho}(\mathbf{k},\omega) \equiv \frac{1}{(2\pi)^2} \iiiint \exp\left[-i(\mathbf{k}\cdot\mathbf{r}-\omega t)\right] \Delta\rho(\mathbf{r},t) \; d^3r \, dt. \tag{1.46}$$

Charge conservation, $\boldsymbol{\nabla}\cdot\boldsymbol{\Delta}\mathbf{J}(\mathbf{r},t) + \frac{\partial\Delta\rho(\mathbf{r},t)}{\partial t} = 0$ will then require that

$$\begin{aligned} \boldsymbol{\nabla}\cdot\frac{1}{(2\pi)^2}\iiiint e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\boldsymbol{\Delta}\mathbf{j}(\mathbf{k},\omega)\;d^3k\,d\omega &= -\frac{\partial}{\partial t}\frac{1}{(2\pi)^2}\iiiint e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\widetilde{\Delta\rho}(\mathbf{k}',\omega')\;d^3k'\,d\omega' \\ \iiiint i\mathbf{k}\cdot e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\boldsymbol{\Delta}\mathbf{j}(\mathbf{k},\omega)\;d^3k\,d\omega &= \iiiint i\omega' e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\;\widetilde{\Delta\rho}(\mathbf{k}',\omega')\;d^3k'\,d\omega' \\ \iiiint e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\left[\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{j}(\mathbf{k},\omega)-\omega\,\widetilde{\Delta\rho}(\mathbf{k},\omega)\right]\;d^3k\,d\omega &= 0 \end{aligned}$$

Since each $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ is a linearly independent functions of (x,y,z,t), the following is satisfied for all $(\mathbf{k},\omega)$

$$\omega\,\widetilde{\Delta\rho}(\mathbf{k},\omega) = \mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{j}(\mathbf{k},\omega) \tag{1.47}$$

or

$$\widetilde{\Delta\rho}(\mathbf{k},\omega) = \omega^{-1}\sum_{i,j=1}^{3} k_i\,\widetilde{\sigma_{ij}}(\mathbf{k},\omega)\; e_j(\mathbf{k},\omega). \tag{1.48}$$

We define a vector $d_i(\mathbf{k},\omega)$ by

$$d_i(\mathbf{k},\omega) = \epsilon_o \sum_{i,j=1}^{3}\left[\delta_{ij} + \frac{i}{\omega\epsilon_o}\widetilde{\sigma_{ij}}(\mathbf{k},\omega)\right] e_j(\mathbf{k},\omega) \tag{1.49}$$

so that:

$$\mathbf{d}(\mathbf{k},\omega) = \epsilon_o\mathbf{e}(\mathbf{k},\omega) + \frac{i}{\omega}\boldsymbol{\Delta}\mathbf{j}(\mathbf{k},\omega) \tag{1.50}$$

This vector is the transform of the **displacement vector,** $\mathbf{D}(\mathbf{r},t)$.

The Fourier transforms of the equations for the total electric and magnetic fields, $\mathbf{e}(\mathbf{k},\omega)$ and $\mathbf{b}(\mathbf{k},\omega)$ are defined by

$$\begin{aligned} \mathbf{e}(\mathbf{k},\omega) &= (2\pi)^{-2}\iiiint\left[\mathbf{E}_o(\mathbf{r},t)+\Delta\mathbf{E}(\mathbf{r},t)\right]e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d^3r\,dt, \\ \mathbf{b}(\mathbf{k},\omega) &= (2\pi)^{-2}\iiiint\left[\mathbf{B}_o(\mathbf{r},t)+\Delta\mathbf{B}(\mathbf{r},t)\right]e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}d^3r\,dt, \end{aligned} \tag{1.51}$$

where (from the $\boldsymbol{\nabla}\cdot\mathbf{E}_o$ Maxwell equation)

$$\boldsymbol{\nabla}\cdot\frac{1}{(2\pi)^2}\iiiint e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\mathbf{e}_o(\mathbf{k},\omega)\ d^3k\,d\omega=\frac{1}{\epsilon_o}\frac{1}{(2\pi)^2}\iiiint e^{i(\mathbf{k}'\cdot\mathbf{r}-\omega' t)}\widetilde{\rho}_o(\mathbf{k}',\omega')\ d^3k'\,d\omega' \tag{1.52}$$

$$\iiiint e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\ \left[i\mathbf{k}\cdot\mathbf{e}_o(\mathbf{k},\omega)-\frac{1}{\epsilon_o}\widetilde{\rho}_o(\mathbf{k}',\omega')\right]d^3k\,d\omega=0$$

so that

$$\mathbf{k}\cdot\mathbf{e}_o(\mathbf{k},\omega)=\frac{1}{i\epsilon_o}\widetilde{\rho}_o(\mathbf{k}',\omega')\,. \tag{1.53}$$

Similarly,

$$\mathbf{k}\cdot\boldsymbol{\Delta}\mathbf{e}(\mathbf{k},\omega)=\frac{1}{i\epsilon_o}\widetilde{\Delta\rho}(\mathbf{k}',\omega')\,.$$

Thus using Eqs. 1.49 and 1.50,

$$\begin{aligned}\mathbf{k}\cdot\mathbf{d}(\mathbf{k},\omega)&=\mathbf{k}\cdot\mathbf{e}(\mathbf{k},\omega)\,\epsilon_o+\mathbf{k}\cdot\frac{i}{\omega}\Delta\mathbf{j}(\mathbf{k},\omega)\\ \mathbf{k}\cdot\mathbf{d}(\mathbf{k},\omega)&=\mathbf{k}\cdot[\mathbf{e}_o(\mathbf{k},\omega)\,\epsilon_o+\Delta\mathbf{e}(\mathbf{k},\omega)\,\epsilon_o]+i\widetilde{\Delta\rho}(\mathbf{k},\omega)\\ &=\frac{1}{i}\left[\widetilde{\rho}_o(\mathbf{k},\omega)+\widetilde{\Delta\rho}(\mathbf{k},\omega)\right]-\frac{1}{i}\,\widetilde{\Delta\rho}(\mathbf{k},\omega)\\ &=-i\widetilde{\rho}_o(\mathbf{k},\omega)\end{aligned} \tag{1.54}$$

The fiirst three Maxwell's Eqs. (in terms of the Fourier transform of the displacement vector, $\mathbf{d}(\mathbf{k},\omega)$) become in SI units :,

$$\begin{aligned}\mathbf{k}\cdot\mathbf{d}(\mathbf{k},\omega)&=-i\,\widetilde{\rho}_o(\mathbf{k},\omega)\\ \mathbf{k}\cdot\mathbf{b}(\mathbf{k},\omega)&=0\\ \mathbf{k}\times\mathbf{e}(\mathbf{k},\omega)&=\omega\mathbf{B}(\mathbf{k},\omega)\end{aligned} \tag{1.55}$$

and for the last equation ($\boldsymbol{\nabla}\times\mathbf{B}$) using $\mathbf{J}(\mathbf{r},t)=\mathbf{J}_o(\mathbf{r},t)+\Delta\mathbf{J}(\mathbf{r},t)$ and Eq. 1.50 for $\mathbf{d}(\mathbf{k},\omega)$

$$\begin{aligned}i\mathbf{k}\times\mathbf{b}(\mathbf{k},\omega)&=\mu_o[\mathbf{j}_o(\mathbf{k},\omega)+\Delta\mathbf{j}(\mathbf{k},\omega)]-\frac{i\omega}{c^2}[\mathbf{e}(\mathbf{k},\omega)]\\ \mathbf{k}\times\mathbf{b}(\mathbf{k},\omega)&=-i\mu_o[\mathbf{j}_o(\mathbf{k},\omega)+\Delta\mathbf{j}(\mathbf{k},\omega)]-\frac{\omega}{c^2\epsilon_o}\left[\mathbf{d}(\mathbf{k},\omega)-\frac{i}{\omega}\Delta j(\mathbf{k},\omega)\right]\\ &=-i\mu_o\mathbf{j}_o(\mathbf{k},\omega)-\frac{\omega}{c^2\epsilon_o}\mathbf{d}(\mathbf{k},\omega)+\frac{i}{c^2\epsilon_o}\Delta j(\mathbf{k},\omega)-i\mu_o\Delta\mathbf{j}(\mathbf{k},\omega)\\ &=-i\mu_o\mathbf{j}_o(\mathbf{k},\omega)-\frac{\omega}{c^2\epsilon_o}\mathbf{d}(\mathbf{k},\omega)\qquad\text{in SI units}\end{aligned} \tag{1.56}$$

In the general units (using the k_1, k_2 and k_3 given in the table on page 10) the transformed Maxwell's Eqs. are

$$\begin{aligned}\mathbf{k}\cdot\mathbf{d}(\mathbf{k},\omega)&=-i\,\widetilde{\rho}_o(\mathbf{k},\omega)\\ \mathbf{k}\cdot\mathbf{b}(\mathbf{k},\omega)&=0\\ \mathbf{k}\times\mathbf{e}(\mathbf{k},\omega)&=k_3\omega\mathbf{B}(\mathbf{k},\omega)\,;\quad k_3=1\text{ in SI units }(c^{-1}\text{in Gaussian units})\end{aligned} \tag{1.57a}$$

$$\mathbf{k}\times\mathbf{b}\left(\mathbf{k},\omega\right)=-i4\pi\frac{k_{2}}{k_{3}}\mathbf{j}_{o}\left(\mathbf{k},\omega\right)-\omega\frac{k_{2}}{k_{1}k_{3}\epsilon_{o}}\mathbf{d}\left(\mathbf{k},\omega\right) \tag{1.57b}$$

If the magnetic forces were included, the magnetic flux density or magnetic induction, $\mathbf{b}\left(\mathbf{k},\omega\right)$, in Eq. 1.57b would be replaced by the magnetic field, $\mathbf{h}\left(\mathbf{k},\omega\right)\mu(\omega)$.

In general the current density, the displacement vector, and the magnetic field are functions of the electric field and the magnetic flux density**. The functions are a property of the material and are called the constitutive equations.**

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Because the response of a system to the applied fields is generally dependent on the frequency and sometimes the wavelength of the fields, the Fourier transform of Maxwell's equations in a medium is often a more useful starting point. Recognizing that the constitutive equations for the displacement vector and magnetic field can be complicated functionals of the electric field and the magnetic flux density (even non-linear functionals), Maxwell's equations for the fields in the region containing material are written as

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D}\left(\mathbf{r},t\right) &= 4\pi k_{1}\epsilon_{o}\rho_{0}\left(\mathbf{r},t\right) && (1.58a)\\
\boldsymbol{\nabla}\cdot\mathbf{B}\left(\mathbf{r},t\right) &= 0 && (1.32)\\
\boldsymbol{\nabla}\times\mathbf{E}\left(\mathbf{r},t\right) &= -k_{3}\frac{\partial\mathbf{B}\left(\mathbf{r},t\right)}{\partial t} && (1.23)\\
\boldsymbol{\nabla}\times\mathbf{H}\left(\mathbf{r},t\right) &= 4\pi k_{2}\frac{\alpha}{\mu_{o}}\mathbf{J}_{0}\left(\mathbf{r},t\right)+\frac{k_{2}}{k_{1}}\frac{\alpha}{\mu_{o}\epsilon_{o}}\frac{\partial\mathbf{D}\left(\mathbf{r},t\right)}{\partial t} && (1.27)
\end{aligned}$$

In SI units:

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{D}\left(\mathbf{r},t\right) &= \rho_{0}\left(\mathbf{r},t\right) && (1.59)\\
\boldsymbol{\nabla}\cdot\mathbf{B}\left(\mathbf{r},t\right) &= 0\\
\boldsymbol{\nabla}\times\mathbf{E}\left(\mathbf{r},t\right) &= -\frac{\partial\mathbf{B}\left(\mathbf{r},t\right)}{\partial t}\\
\boldsymbol{\nabla}\times\mathbf{H}\left(\mathbf{r},t\right) &= \frac{\partial\mathbf{D}\left(\mathbf{r},t\right)}{\partial t}+\mathbf{J}_{0}\left(\mathbf{r},t\right)
\end{aligned}$$

1.3 Boundary conditions on the fields at interfaces between materials

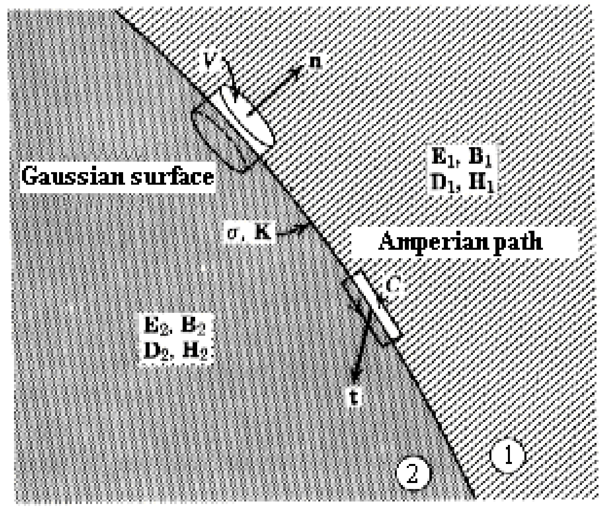
Maxwell's equations can be used to relate the components of the fields on each side of an interface between two materials.

Gauss' law for the magnetic flux density

The quick applications of either of Gauss' laws require that one choose an appropriate volume and integrate both sides of the equation over this volume. In the case of the magnetic flux density

$$\iiint_{vol}\boldsymbol{\nabla}\cdot\mathbf{B}\left(\mathbf{r},t\right)d^{3}r=0=\iint_{\substack{closed\\surface}}\mathbf{B}\left(\mathbf{r},t\right)\cdot d\mathbf{S}. \tag{1.60}$$

To examine the relationship between the components of the magnetic flux density on each side of a interface take a differential surface area, ΔS, which is tangent to the interface at $\mathbf{r}$.



.Fig. 2. Interface

Label the parameters of the system on one side of the interface with the subscript '1' and on the other side of the interface with the subscript '2'. Let $\mathbf{n}$ be a unit vector which is normal to the surface at $\mathbf{r}$ and points from the side labelled '2' to the side labelled '1'. The volume used in Gauss' law is the infinitesimal cylindrical volume which is swept out by displacing the disk-like surface (with area $\pi(d\mathbf{r}\cdot d\mathbf{r})$) along $\pm\delta\,\mathbf{n}$. The $d\mathbf{r}$ are tangent to the interface at $\mathbf{r}$ .See the Gaussian surface shown in Figure 2. Eq.1.60 can be used to show that the component of the magnetic flux density which is perpendicular to the surface is continuous. In particular, as $\delta \longrightarrow \mathbf{0}$

$$\iint_{\substack{closed\\surface}} \mathbf{B}(\mathbf{r},t)\cdot d\mathbf{S} = 0 \tag{1.61}$$

$$0 = [\mathbf{B}_1(\mathbf{r},t) - \mathbf{B}_2(\mathbf{r},t)]\cdot\mathbf{n}\,\pi(dr)^2 + \lim_{\delta\longrightarrow\mathbf{0}} \mathbf{B}(\mathbf{r},t)\cdot 2\delta 2\pi d\mathbf{r} \tag{1.62}$$

so that

$$[\mathbf{B}_1(\mathbf{r},t) - \mathbf{B}_2(\mathbf{r},t)]\cdot\mathbf{n} = 0. \tag{1.63}$$

1.3.1 Gauss' Law for the electric displacement vector

Gauss' law for the electric displacement vector is used to obtain the integral expression

$$\iint_{\substack{closed\\surface}} \mathbf{D}(\mathbf{r},t)\cdot d\mathbf{S} = 4\pi k_1\epsilon_o \iiint_{\substack{bdd\\vol}} \rho(\mathbf{r},t)\; d^3r = \iiint_{\substack{bdd\\vol}} \rho(\mathbf{r},t)\; d^3r \text{ in SI units} \tag{1.64}$$

where we now drop the subscript on $\rho_o(\mathbf{r},t)$ and use $\rho(\mathbf{r},t)$. Unlike the corresponding expression for the magnetic flux density the right hand side of this equation is not automatically zero. To investigate the constraints placed on the components of the displacement at an interface we use the volume and unit normal vector defined above for Eq. 1.60. We can define a surface charge density $\sigma(\mathbf{r},t) = dQ/dA$ by the integration of $\rho(\mathbf{r},t) = dQ/dV$ over the height, 2δ, of a cylindrical volume with radius dr

$$\sigma(\mathbf{r},t) = \lim_{\delta\longrightarrow\mathbf{0}} \int_{-\delta}^{\delta} \rho\left(\mathbf{r} + \boldsymbol{\delta}'\mathbf{n}, t\right)\; d\delta' \tag{1.65}$$

where $\mathbf{n}$ is a unit length vector which is normal to the interface at $\mathbf{r}$ and is directed from material '2' to material '1'. Using this volume for Eq. 1.64 we obtain

$$[\mathbf{D}_1(\mathbf{r},t) - \mathbf{D}_2(\mathbf{r},t)] \cdot \mathbf{n}\,\pi(dr)^2 + \lim_{\delta \longrightarrow \mathbf{0}} \mathbf{D}(\mathbf{r},t) \cdot 2\delta 2\pi d\mathbf{r} = \lim_{\delta \longrightarrow \mathbf{0}} \pi(dr)^2 \int_{-\delta}^{\delta} \rho\left(\mathbf{r} + \boldsymbol{\delta}'\mathbf{n}, t\right)\, d\delta' \tag{1.66}$$

$$[\mathbf{D}_1(\mathbf{r},t) - \mathbf{D}_2(\mathbf{r},t)] \cdot \mathbf{n} = \sigma(\mathbf{r},t) \quad \text{in SI units} \tag{1.67}$$

If $\sigma(\mathbf{r},t) = 0$ then the component of the displacement vector which is perpendicular to the surface is continuous.

1.3.2 Faraday's law for electric fields

Faraday's law for the electric field can be written in the integral form, and using Stokes' theorem we find

$$\iint_{\substack{open\ bounded \\ surface}} (\boldsymbol{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} = \oint_{\substack{closed \\ path}} \mathbf{E} \cdot \mathbf{d}\ell = -\frac{d}{dt} \iint_{\substack{open\ bounded \\ surface}} \mathbf{B} \cdot \mathbf{dS}. \tag{1.68}$$

To use this law we must define a path and the surface bounded by the path. We begin with the normal to the surface. As above let $\mathbf{n}$ be the **unit normal** to the interface at $\mathbf{r}$which is directed from the material labelled '2' towards the material labelled '1', as shown in Fig. 2**.** Let $\mathbf{t}$ be any **unit vector** which is tangent to the interface at $\mathbf{r}$. Take a directed straight line segment $\boldsymbol{\Delta}\mathbf{l} = \Delta l\ (\mathbf{t} \times \mathbf{n})$ which is tangent to the surface at **r.** Translate this line by $\pm\delta\,\mathbf{n}$ to obtain a directed line on each side of the surface. Connect the corresponding ends of the two lines to form the rectangular path labelled as an Amperean path in Figure 2. This rectangle will be the path used in Faraday's law.

The magnetic flux passing through the rectangular area bounded by the Amperian path is

$$\Delta\Phi_m\,(\text{surface, } \mathbf{r}) = \mathbf{t} \cdot \mathbf{B}(\mathbf{r},t)\ 2\,|\Delta\mathbf{l}|\ \delta. \tag{1.69}$$

The flux along with its time derivative will vanish as $\delta \rightarrow 0$. The contributions to $\mathbf{E} \cdot \mathbf{d}\ell$ along the ends of the rectangle, cancel exactly since the paths are oppositely directed. It follows then that (as $\delta \rightarrow 0$),

$$\begin{aligned} (\mathbf{E}_1(\mathbf{r},t) - \mathbf{E}_2(\mathbf{r},t)) \cdot \Delta l\ (\mathbf{t} \times \mathbf{n}) &= \mathbf{t} \cdot \mathbf{B}(\mathbf{r},t)\ 2\,|\Delta\mathbf{l}|\ \delta. = 0, \text{ or} \\ \mathbf{n} \times (\mathbf{E}_1(\mathbf{r},t) - \mathbf{E}_2(\mathbf{r},t)) \cdot \mathbf{t} &= 0 \end{aligned} \tag{1.70}$$

for any $\mathbf{t}$ which is tangent to the surface at **r.** It follows that (since $\mathbf{t}$ is arbitrary)

$$\mathbf{n} \times (\mathbf{E}_1(\mathbf{r},t) - \mathbf{E}_2(\mathbf{r},t)) = 0 \tag{1.71}$$

But $\mathbf{n} \times \mathbf{E}(\mathbf{r},t)$ is the component of the electric field which is tangent to the surface at $\mathbf{r}$. Therefore Faraday's law requires that the tangential component of the electric field be continuous at interfaces

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1.3.3 Ampere-Maxwell law for the magnetic field

The integral form of the Ampere-Maxwell law is (in SI units)

$$\iint_{\substack{open\ bounded \\ surface}} (\boldsymbol{\nabla} \times \mathbf{H}) \cdot d\mathbf{S} = \oint_{\substack{closed \\ path}} \mathbf{H}(\mathbf{r},t) \cdot \mathbf{d}\ell = \iint_{\substack{bounded \\ surface}} \left[\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \mathbf{J}_o(\mathbf{r},t)\right] \cdot \mathbf{dS}. \tag{1.72}$$

In applying the Ampere-Maxwell law to obtain the boundary conditions at an interface we use the same path used in Faraday's law. The displacement current for the bounded surface is

$$\Delta\Phi_e\,(surface,\, \mathbf{r}) = (\mathbf{D}(\mathbf{r},t) \cdot \mathbf{t})\ \ (2\,|\Delta\mathbf{l}|\ \delta)\,. \tag{1.73}$$

This (and the time derivative) will vanish as $\delta \to 0$. The current for the bounded surface is

$$I\left(surface,\, \mathbf{r}\right) = |\Delta \mathbf{l}| \int_{-\delta}^{\delta} \mathbf{J}_o\left(\mathbf{r} + \boldsymbol{\delta}'\mathbf{n}, t\right) \cdot \mathbf{t}\, d\delta'. \tag{1.74}$$

It is often convenient to let this be finite as $\delta \to 0$. In this case we define a surface current density by

$$\boldsymbol{\Sigma}\left(\mathbf{r}, t\right) = \int_{-\delta}^{\delta} \mathbf{J}_o\left(\mathbf{r} + \boldsymbol{\delta}'\mathbf{n}, t\right)\; d\delta'. \tag{1.75}$$

The magnetic field at the interface will satisfy

$$\left(\mathbf{H}_1\left(\mathbf{r}, t\right) - \mathbf{H}_2\left(\mathbf{r}, t\right)\right) \cdot \Delta l\; \left(\mathbf{t} \times \mathbf{n}\right) = \Delta l\, \boldsymbol{\Sigma}\left(\mathbf{r}, t\right) \cdot \mathbf{t} \tag{1.76}$$

for any tangent vector **t**. It follows then that

$$\mathbf{n}\times\left(\mathbf{H}_1\left(\mathbf{r}, t\right) - \mathbf{H}_2\left(\mathbf{r}, t\right)\right) = \boldsymbol{\Sigma}\left(\mathbf{r}, t\right). \tag{1.77}$$

A surface current density will result in a discontinuity in the component of the magnetic field which is tangent to the interface

1.3.4 Recap of the boundary conditions at interfaces between materials

The electric displacement vector, the electric field, the magnetic flux density, and the magnetic field satisfy the following relationships at interfaces between materials:

$$\begin{aligned} \left[\mathbf{B}_1\left(\mathbf{r}, t\right) - \mathbf{B}_2\left(\mathbf{r}, t\right)\right] \cdot \mathbf{n} &= 0, \\ \mathbf{n} \times \left(\mathbf{H}_1\left(\mathbf{r}, t\right) - \mathbf{H}_2\left(\mathbf{r}, t\right)\right) &= \boldsymbol{\Sigma}\left(\mathbf{r}, t\right), \\ \left[\mathbf{D}_1\left(\mathbf{r}, t\right) - \mathbf{D}_2\left(\mathbf{r}, t\right)\right] \cdot \mathbf{n} &= \sigma\left(\mathbf{r}, t\right), \\ \mathbf{n}\times\left(\mathbf{E}_1\left(\mathbf{r}, t\right) - \mathbf{E}_2\left(\mathbf{r}, t\right)\right) &= \mathbf{0}. \end{aligned} \tag{1.78}$$

These equations, together with the generally adequate connections between frequency dependent fields

$$\begin{aligned} \widetilde{\mathbf{B}}\left(\mathbf{r}, \omega\right) &= \mu\left(\omega\right) \widetilde{\mathbf{H}}\left(\mathbf{r},\omega\right) \\ \widetilde{\mathbf{D}}\left(\mathbf{r}, \omega\right) &= \varepsilon\left(\omega\right) \widetilde{\mathbf{E}}\left(\mathbf{r},\omega\right) \end{aligned} \tag{1.79}$$

with $\mu\left(\omega\right)$ and $\varepsilon\left(\omega\right)$ the frequency dependent permeability and permittivity of the material, provide the connection between the fields in different, adjacent material.

Appendix A
Special problems

1. (a) By using Gauss' law (Eq. 1.05) and the definition of the Dirac delta function show that if

$$\mathbf{E}(\mathbf{r}) = k_1 \frac{q}{r^3} \mathbf{r}$$

then the charge density is

$$\rho(\mathbf{r}) = q\, \delta^{(3)}(\mathbf{r}) .$$

(b) Show that if

$$\rho(\mathbf{r}) = \sum_{i=1}^{N} q_i\, \delta^{(3)}(\mathbf{r} - \mathbf{r}_i)$$

and

$$\mathbf{E}(\mathbf{r}) = k_1 \int_{\substack{all \\ space}} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}')$$

then

$$\mathbf{E}(\mathbf{r}) = k_1 \sum_{i=1}^{N} \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|^3} (\mathbf{r} - \mathbf{r}_i) .$$

2. The charge density for a line charge is given by

$$\begin{aligned} \rho(\mathbf{r}) &= (5\,\text{statC/cm})\ \delta(x)\ \delta(z) \ \text{ for } |y| < 3 \text{ cm} \\ \rho(\mathbf{r}) &= 0\ \text{statC/cm}^3 \text{ otherwise} \end{aligned}$$

(a) evaluate, exactly, the electric field along the straight line $y = +1$ cm, $z = 0$ cm. (b) Imagine cutting the line charge into N (N = 1, 4, 8, and 12) equal length segments. Replace each segment by a point particle having the charge of the segment and located at the center of the segment. Using a spreadsheet evaluate the value of the components of the electric field at points along the straight line between $(0.2\,cm,\ 1\,cm,\ 0\,cm)$ and $(15\,cm,\ 1\,cm,\ 0\,cm)$ for the each value of N. (c) Plot the difference between these values and those obtained from the in part (a). Use enough points to obtain a smooth curve.

3. Show that the curl of the electric field due to a point charge,

$$\mathbf{E}(\mathbf{r}) = k_1 \frac{q}{r^3} \mathbf{r}$$

vanishes. Be sure to check that it is not a delta function.

4. A spherical object is solid except for an interior spherical hole. The sphere has a radius R and the hole has a radius R/3. The object is composed of a material with a charge volume density ρ. Place the center of the spherical object at the origin of a xyz coordinate system and the center of the hole on the z axis at z = R/4. Using the integral form of Gauss' law and the superposition principle obtain the equations for and plot the values of the non-zero components of the electric field at points along (a) the z axis with $|z| < 2R$; (b) the y axis with $|y| < 2R$. The field will be in units of $\rho R/3\varepsilon_0$ and the coordinate in units of R.

The Dirac Delta Function, $\delta(x-x_o)$

Dirac Delta Function

In one dimension, $\delta(x-x_o)$ is defined to be such that:

$$\int_{a \text{ to } b} f(x)\,\delta(x-x_o)dx \quad \equiv \quad \begin{cases} 0 & \text{if } x_o \text{ is not in } [a,b]. \\ \tfrac{1}{2}f(x_o) & \text{if } x_o = a \text{ or } b; \\ f(x_o) & \text{if } x_o\ \varepsilon\ (a,b). \end{cases}$$

Properties of $\delta(x-x_o)$: (you should know those marked with *)

*1. $\delta(x-x_o) = 0 \quad$ if $x \neq x_o$

*2. $\int_{-\infty \text{ to } +\infty} \delta(x)dx \quad = \ 1$

3. $\delta(ax) = \delta(x)/|a|$

*4. $\delta(-x) = \delta(x)$

5. $\delta(x^2-a^2) = [\delta(x-a) + \delta(x+a)]/(2a); \quad a \geq 0$

6. $\int_{-\infty \text{ to } +\infty} \delta(x-a)\delta(x-b)dx \ = \ \delta(a-b)$

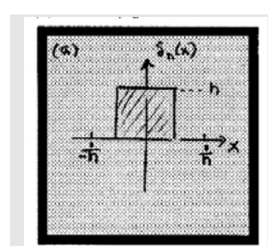
-| *7. $\delta(g(x)) = \sum_i \delta(x-x_{oi})/|dg/dx|_{x=xoi}$ where $g(x_{oi}) = 0$ and dg/dx exists at and in a region around x_{oi}.

-

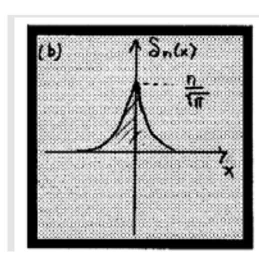
*8. $f(x)\delta(x-a) \ = \ f(a)\delta(x-a)$

9. $\delta(x)$ is a "symbolic" function which provides convenient notation for many mathematical expressions. Often one "uses" $\delta(x)$ in expressions which are not integrated over. However, it is understood that eventually these expressions will be integrated over so that the definition of δ (box above) applies.

10. No ordinary function having exactly the properties of $\delta(x)$ exists. However, one can **approximate** $\delta(x)$ by the limit of a sequence of (non-unique) functions, $\delta_n(x)$. Some examples of $\delta_n(x)$ which work are given below. In all these cases, $\int_{-\infty \text{ to } +\infty} \delta_n(x)dx = 1\ \forall$ n and $\text{limit}_{n \to \infty} \int_{-\infty \text{ to } +\infty} \delta_n(x-x_o)f(x)dx = f(x_o).\ \forall$ n.



(a) $\delta_n(x) \equiv \begin{cases} 0 & \text{for } x < -1/(2n) \\ n & \text{for } -1/(2n) \leq x \leq 1/(2n) \\ 0 & \text{for } x > 1/(2n) \end{cases}$



(b) $\delta_n(x) \equiv n/\sqrt{\pi}\ \exp[-n^2x^2]$

(c) $\delta_n(x) \equiv (n/\sqrt{\pi})\cdot 1/(1 + n^2x^2)$

(d) $\delta_n(x) \equiv \sin(nx)/\pi x \quad = \quad [1/(2\pi)]\int_{-n \text{ to } n} \exp(ixt)dt$

11. $\int_{a\text{ to }b} f(x)\, d^r/dx^r\, \delta(x-x_o)\, dx = \begin{cases} (-1)^r\, d^rf/dx^r|_{xo} & \text{if } x_o\ \varepsilon\ (a,b) \\ \tfrac{1}{2}[(-1)^r d^rf/dx^r|_{xo} & \text{if } x_o = a \text{ or } b \\ 0 & \text{otherwise} \end{cases}$

f(x) is arbitrary, continuous function at $x = x_o$

12. $\int_{a\text{ to }b} x^r\, d^r/dx^r\, \delta(x-x_o)dx = \int_{a\text{ to }b} (-1)^r r!\, \delta(x-x_o)\, dx$ where $x_o\ \varepsilon\ (a,b)$.

important expressions involving $\delta(x-x_o)$

13. $\delta(x-x_o) = [1/2\pi] \int_{-\infty\text{ to }+\infty} e^{ik(x-xo)}dk$

14. $\delta(\mathbf{r}-\mathbf{r}_o) \equiv \delta(x-x_o)\delta(y-y_o)\delta(z-z_o) = [1/2\pi]^3 \int\int\int_{\text{all k space}} \exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_o)]dk_xdk_ydk_z$

15. $\delta(g(x)) = \sum_n \delta(x-\beta_n)/|dg/dx|_{x=\beta n}$ where $g(\beta_n) = 0$.

16. $\delta(\mathbf{r}-\mathbf{r}_o) = \delta(q^1-q^1{}_o)\delta(q^2-q^2{}_o)\delta(q^3-q^3{}_o)/\sqrt{g}$ in general system.

Dirac Delta Function in 3 Dimensions: $\delta(\mathbf{r} - \mathbf{r}_o) \equiv \delta(x-x_o)\delta(y-y_o)\delta(z-z_o)$

17. $\delta(\mathbf{k} - \mathbf{k}_o) = [1/2\pi]^3 \int_{-\infty\text{ to }+\infty}\int_{-\infty\text{ to }+\infty}\int_{-\infty\text{ to }+\infty} \exp[i\mathbf{r}\cdot(\mathbf{k} - \mathbf{k}_o)]\, dxdydz$

18. $\delta(\mathbf{r} - \mathbf{r}_o) = \delta(q^1-q^1{}_o)\delta(q^2-q^2{}_o)\delta(q^3-q^3{}_o)/\sqrt{g}$

derivation: $\int\int\int \delta(\mathbf{r} - \mathbf{r}_o)f(\mathbf{r})\, d^3x = f(\mathbf{r}_o) = \int\int\int \delta(q^1-q^1{}_o)\delta(q^2-q^2{}_o)\delta(q^3-q^3{}_o)/\sqrt{g}\cdot f(\mathbf{r}(q^i))\sqrt{g}dq^1dq^2dq^3$

19. $\delta(\mathbf{r} - \mathbf{r}_o)_{\text{spherical coordinates}} = \delta(r-r_o)\delta(\varphi-\varphi_o)\delta(\theta-\theta_o)/(r^2\sin\theta)$

20. $\tfrac{1}{2}[\delta(x-a)+\delta(x+a)] = [1/\pi] \int_{0\text{ to }\infty} \cos(ka)\cos(kx)\, dk$

Exercises: Evaluate the following integrals.

a) $\int_{-1\text{ to }5} \delta(2x-\pi) \exp[\sin^3(x-\pi)]\, dx$

b) $\int\int\int_{\text{all space}} \delta(\mathbf{r}\cdot\mathbf{r}-a^2)\delta(\cos\theta-1/\sqrt{2})\delta(\sin\varphi-\tfrac{1}{2})\exp[i\mathbf{k}\cdot\mathbf{r}]d^3x$

Helmholtz Theorem

If

(a) $\nabla\cdot\mathbf{F}(\mathbf{r}) = \rho(\mathbf{r})$ everywhere for finite r;
(b) $\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{J}(\mathbf{r})$ "
(c) $\text{limit}_{r\to\infty}\rho(\mathbf{r}) = 0$;
(d) $\text{limit}_{r\to\infty}|\mathbf{J}(\mathbf{r})| = 0$;

then

$\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) + \nabla \times \mathbf{A}(\mathbf{r})$

where Φ and $\mathbf{A}$ are determined from ρ and $\mathbf{J}$ as shown below.

proof:

1. Define Φ and $\mathbf{A}$ as follows:

$$\Phi(\mathbf{r}) = [1/4\pi]\iiint_{V=\text{all space}} \rho(\mathbf{r'})/|\mathbf{r} - \mathbf{r'}|\, d^3x' + \Phi_o(\mathbf{r}), \quad \text{where } \nabla^2\Phi_o(\mathbf{r}) = 0;$$

$$\mathbf{A}(\mathbf{r}) = [1/4\pi]\iiint_{V=\text{all space}} \mathbf{J}(\mathbf{r'})/|\mathbf{r} - \mathbf{r'}|\, d^3x' + \mathbf{A}_o(\mathbf{r}), \quad \text{where } \nabla\text{x}(\nabla\text{x}\mathbf{A}_o(\mathbf{r})) = 0;$$

2. Let $\mathbf{F}(\mathbf{r}) = -\nabla\Phi(\mathbf{r}) + \nabla\text{x}\mathbf{A}(\mathbf{r})$; we shall show that this $\mathbf{F}$ satisfies the conditions (a) and (b) if (c) and (d) hold:

$$\nabla\cdot\mathbf{F} = \nabla\cdot[-\nabla\Phi(\mathbf{r}) + \nabla\text{x}\mathbf{A}(\mathbf{r})] = -\nabla^2\Phi(\mathbf{r})$$
$$\nabla \times \mathbf{F} = \nabla\text{x}[-\nabla\Phi(\mathbf{r}) + \nabla\text{x}\mathbf{A}(\mathbf{r})] = \nabla\text{x}[\nabla\text{x}\mathbf{A}(\mathbf{r})].$$

3. $\nabla\cdot\mathbf{F} = -\nabla^2\Phi(\mathbf{r}) = -[1/4\pi]\iiint_{V=\text{all space}} \rho(\mathbf{r'})\nabla^2[1/|\mathbf{r} - \mathbf{r'}|]\, d^3x' + 0$

4. But $\nabla^2[1/|\mathbf{r} - \mathbf{r'}|]$
$= \nabla\cdot\nabla[1/|\mathbf{r} - \mathbf{r'}|] = \nabla\cdot[-(\mathbf{r} - \mathbf{r'})/|\mathbf{r} - \mathbf{r'}|^3]$
$=-[3/|\mathbf{r} - \mathbf{r'}|^3 + [-3/|\mathbf{r} - \mathbf{r'}|^4](\mathbf{r} - \mathbf{r'})\cdot(\mathbf{r} - \mathbf{r'})/|\mathbf{r} - \mathbf{r'}|\ ]$
$=-[3/|\mathbf{r} - \mathbf{r'}|^3 + [-3/|\mathbf{r} - \mathbf{r'}|^3]\ ]$
$= 0 \quad \text{if } \mathbf{r} \neq \mathbf{r'}$

5 What happens if $\mathbf{r} = \mathbf{r'}$? We shall see that the expression --> ∞, but with a crucial additional property!

$$CLAIM: \quad \nabla^2 \frac{1}{|r-r'|} = -4\pi\,\delta(r-r')$$

Derivation:
(a) Consider the following integral, $\iiint_V \nabla\cdot\nabla[1/r]\, d^3x = \iiint_V \nabla^2[1/r]\, d^3x$
where V = all space and V' = all space except a sphere of radius δ centered on the origin and a "funnel" extending from $\mathbf{r} = 0$ to r = ∞. See figure below.

We shall show that this integral = -4π if $\mathbf{r} = 0$ is in V.
Note: V contains $\mathbf{r} = 0$ and V' does not.