

Chapter 2

Introduction to electrostatics

2.1 Coulomb and Gauss' Laws

We will restrict our discussion to the case of static electric and magnetic fields in a homogeneous, isotropic medium. In this case the electric field satisfies the two equations, Eq. 1.59a with a time independent charge density and Eq. 1.77 with a time independent magnetic flux density,

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_0(\mathbf{r}), \quad (1.59a)$$

$$\nabla \times \mathbf{E}(\mathbf{r}) = \mathbf{0}. \quad (1.77)$$

Because we are working with static fields in a homogeneous, isotropic medium the constituent equation is

$$D(\mathbf{r}) = \varepsilon E(\mathbf{r}). \quad (1.78)$$

Note : **D is sometimes written :** (1.78b)

$$D = \epsilon_0 E + P \dots \text{SI units}$$

$$D = E + 4\pi P \quad \text{in Gaussian units}$$

$$\text{in these cases } \varepsilon = [1 + 4\pi P/E] \text{ Gaussian}$$

The solution of Eq. 1.59 is

$$\mathbf{D}(\mathbf{r}) = \frac{1}{4\pi} \iiint \frac{\rho_0(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' + \mathbf{D}_0(\mathbf{r}), \text{ SI units} \quad (1.79)$$

with $\nabla \cdot \mathbf{D}_0(\mathbf{r}) = 0$

If we are seeking the contribution of the charge density, $\rho_0(\mathbf{r})$, to the electric displacement vector then $\mathbf{D}_0(\mathbf{r}) = \mathbf{0}$. The given charge density generates the electric field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon} \iiint \frac{\rho_0(\mathbf{r}')(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' \text{ SI units} \quad (1.80)$$

2.2 The electric or scalar potential

Faraday's law with static fields, Eq. 1.77, is automatically satisfied by any electric field $\mathbf{E}(\mathbf{r})$ which is given by

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \quad (1.81)$$

The function $\phi(\mathbf{r})$ is the scalar potential for the electric field. It is also possible to obtain the difference in the values of the scalar potential at two points by integrating the tangent component of the electric field along any path connecting the two points

$$\begin{aligned} -\int_{\mathbf{r}_a \rightarrow \mathbf{r}_b}^{\text{path}} \mathbf{E}(\mathbf{r}) \cdot d\ell &= \int_{\mathbf{r}_a \rightarrow \mathbf{r}_b}^{\text{path}} \nabla\phi(\mathbf{r}) \cdot d\ell \\ &= \int_{\mathbf{r}_a \rightarrow \mathbf{r}_b}^{\text{path}} \left[dx \frac{\partial\phi(\mathbf{r})}{\partial x} + dy \frac{\partial\phi(\mathbf{r})}{\partial y} + dz \frac{\partial\phi(\mathbf{r})}{\partial z} \right] \\ &= \int_{\mathbf{r}_a \rightarrow \mathbf{r}_b}^{\text{path}} d\phi(\mathbf{r}) = \phi(\mathbf{r}_b) - \phi(\mathbf{r}_a) \end{aligned} \quad (1.82)$$

The result obtained in Eq. 1.82 is independent of the path taken between the points \mathbf{r}_a and \mathbf{r}_b . It follows that the integral of the tangential component along a closed path is zero,

$$\oint \mathbf{E}(\mathbf{r}) \cdot d\ell = \oint d\phi(\mathbf{r}) = 0. \quad (1.83)$$

This last result actually follows from the requirement that $\nabla \times \mathbf{E}(\mathbf{r}) = \mathbf{0}$ and the application of Stoke's theorem.

To obtain the scalar potential *due to the charge density* $\rho_0(\mathbf{r})$ we note that

$$\begin{aligned} &\nabla \left[\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \right] \\ &= -\frac{(x-x')\mathbf{i} + (y-y')\mathbf{j} + (z-z')\mathbf{k}}{\left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{3/2}} \\ &= -\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}. \end{aligned} \quad (1.84)$$

Comparing the expression on the right hand side of Eq. 1.84 to the integrand in Eq. 1.80 we find that can write that the scalar potential due to the charge density $\rho_o(\mathbf{r})$ is

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \iiint \frac{\rho_o(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r' + \phi_0, \quad (1.85)$$

where ϕ_0 is a constant which fixes the (arbitrary) location of the zero for the scalar potential. Since the observed quantity is the electric force and, therefore, the electric field, only the difference in the values of the scalar potential at any two different points is significant. (See Eq. 1.82)

2.3 Surface charges and charge dipoles A surface charge is a charge density which is ‘restricted to lie on a surface’. It is characterized by the equation defining the surface, say $F(\mathbf{r}) = 0$, (or $F(x, y, z) = 0$) and a surface charge density, $\sigma(\mathbf{r})$, with dimensions of Coulombs/m² in SI units and statcoulombs per square centimeter in gaussian units.

Restricting a volume integral to a surface with a delta function in the integrand

The charge density associated with a surface charge will be the product of two terms. One term will contain a delta function which restricts the density to the surface, $\delta(F(\mathbf{r})) * f(\mathbf{r})$. This term should have dimensions of inverse length, m^{-1} in our units. and the other will be the surface charge density, $\sigma(\mathbf{r})$. The function $f(\mathbf{r})$ should be determined such that

$$\iiint H(\mathbf{r}) \delta(F(\mathbf{r})) * f(\mathbf{r}) d^3r = \iint_{F(\mathbf{r})=0} H(\mathbf{r}) dS(\mathbf{r}) \quad (1.86)$$

where $H(\mathbf{r})$ is any ‘smooth’ function and the surface integral on the right hand side is restricted to the surface $F(\mathbf{r}) = 0$.

Before attacking this problem we should note that, in one dimension,

$$\begin{aligned} \int_{-\infty}^{\infty} a(x) \delta(b(x)) |b'(x)| dx &= \int a(x) \delta(b(x)) db(x) \\ &= a(x_0), \text{ with } b(x_0) = 0 \end{aligned} \quad (1.87)$$

Since the delta function will restrict the integral to the surface we need only consider the region close to the surface. In this

region let the coordinates system consist of a coordinate axis perpendicular to the surface at each point and two orthogonal axes which are tangent to the surface at each point. The unit vector which is perpendicular to the surface at each point is either $\hat{\mathbf{n}} = +\nabla F(\mathbf{r}) / |\nabla F(\mathbf{r})|$ or $\hat{\mathbf{n}} = -\nabla F(\mathbf{r}) / |\nabla F(\mathbf{r})|$.

If we let $\xi \hat{\mathbf{n}}$ be the displacement of a point from the surface at \mathbf{r}' then $d^3r = [dS(\mathbf{r}') \hat{\mathbf{n}}] \cdot d\xi \hat{\mathbf{n}}$. In addition if \mathbf{r}' is a point on the surface then

$$\nabla F(\mathbf{r}') = \left[\frac{\partial F(\mathbf{r}' + \xi \hat{\mathbf{n}})}{\partial \xi} \right]_{\xi=0}. \quad (1.88)$$

When performing the volume integration we carry out the integration perpendicular to the surface at each point \mathbf{r}' and then multiply the result by the differential surface element at \mathbf{r}' , $dS(\mathbf{r}')$. We have in this way reduced our problem to a set of one dimensional problems, one for each point \mathbf{r}' on the surface. Viewing the problem in this way suggests that the appropriate expression for the function $f(\mathbf{r})$ is

$$\begin{aligned} f(\mathbf{r}' + \xi \hat{\mathbf{n}}) &= \hat{\mathbf{n}} \cdot \nabla' F(\mathbf{r}' + \xi \hat{\mathbf{n}}) \\ &= |\nabla' F(\mathbf{r}' + \xi \hat{\mathbf{n}})| \end{aligned} \quad (1.89)$$

It follows then that the volume charge density associated with a surface charge density, $\sigma(\mathbf{r})$, on the surface defined by $F(\mathbf{r}) = 0$ is

$$\rho(\mathbf{r}) = \sigma(\mathbf{r}) \delta(F(\mathbf{r})) |\nabla F(\mathbf{r})| \quad (1.90)$$

Note: A simpler way to obtain this result is as follows, where $\int \delta(F(\mathbf{r}))dF = 1$:

$$\begin{aligned}
 \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) dS(\mathbf{r}) &= \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) \left[\int' \delta(F(\mathbf{r}))dF \right] dS(\mathbf{r}) \\
 &= \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) \left[\int' \delta(F(\mathbf{r}))d\mathbf{r}' \cdot \nabla F \right] dS(\mathbf{r}) \\
 &= \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) \left[\int' \delta(F(\mathbf{r}))d\mathbf{r}' \cdot \frac{\nabla F}{|\nabla F|} |\nabla F| \right] dS(\mathbf{r}) \\
 &= \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) \int' \delta(F(\mathbf{r}))|\nabla F| d\mathbf{r}' \cdot \hat{\mathbf{n}}_s dS(\mathbf{r}) \\
 &= \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) \int' \delta(F(\mathbf{r}))|\nabla F| d\mathbf{r}' \cdot d\mathbf{S}(\mathbf{r})
 \end{aligned} \tag{1.90b}$$

where the primed integration symbol indicates that the integration must be taken along a direction, $d\mathbf{r}'$ which is along the normal to the surface.

$$= \iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) \int' \delta(F(\mathbf{r}))|\nabla F| dV' \tag{1.90c}$$

$$\iint_{F(\mathbf{r})=0} \sigma(\mathbf{r}) dS(\mathbf{r}) = \iiint_{V \text{ contains } F(\mathbf{r})=0} \sigma(\mathbf{r}) \delta(F(\mathbf{r})) |\nabla F| dV \tag{1.90d}$$

Note that the final volume integral must be over a volume which contains all of the surface, Using this form for the charge density allows one to manipulate the variables of integration more freely

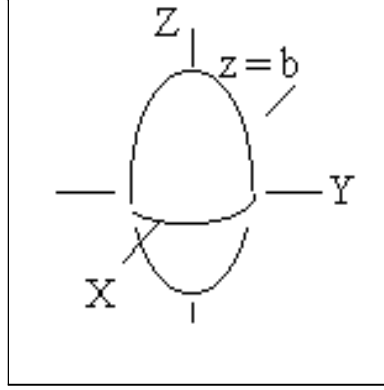
2.3.1 Example: uniformly charged ellipsoidal surface

The system is a uniformly charged ellipsoidal surface (Fig. ??) defined by $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} - 1 = 0$. The surface charge density is σ_0 . Find the potential on the z axis.

Solution: From the class notes, Eq. 1.90, the volume charge density is given by

$$\begin{aligned}
 \rho(\mathbf{r}) &= \sigma_0 \delta \left(\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} - 1 \right) \left| \nabla \left(\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} - 1 \right) \right| \\
 &= \sigma_0 \delta \left(\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} - 1 \right) 2 \left(\frac{x^2+y^2}{a^4} + \frac{z^2}{b^4} \right)^{1/2}
 \end{aligned} \tag{1.91}$$

Because of the cylindrical symmetry of the charge distribution it is convenient to work in cylindrical coordinates, $x = \xi \cos \varphi$ and $y = \xi \sin \varphi$. In these coordinate the charge density is



.Fig. 3

$$\rho(\mathbf{r}) = 2\sigma_0 \delta\left(\frac{\xi^2}{a^2} + \frac{z^2}{b^2} - 1\right) \sqrt{\frac{\xi^2}{a^4} + \frac{z^2}{b^4}}$$

Taking the zero of potential at infinity, the potential on the z axis due to the surface charge is

$$\phi(z\hat{\mathbf{z}}) = \frac{2\sigma_0}{4\pi\epsilon} \iiint \frac{\delta\left(\frac{\xi^2}{a^2} + \frac{z'^2}{b^2} - 1\right) \sqrt{\frac{\xi^2}{a^4} + \frac{z'^2}{b^4}}}{\sqrt{\xi'^2 + (z - z')^2}} d^3r' \quad \text{SI units} \quad (1.92)$$

with $d^3r' = \xi' d\xi' d\varphi' dz'$. The integrand is independent of φ' and the angle integration can be performed giving 2π . The

integrand is also only a function of ξ'^2 . This suggests that we define an $\eta = a^{-2}\xi'^2$. In this case $\xi' d\xi' = 0.5 a^2 d\eta$. With these changes the potential is given by

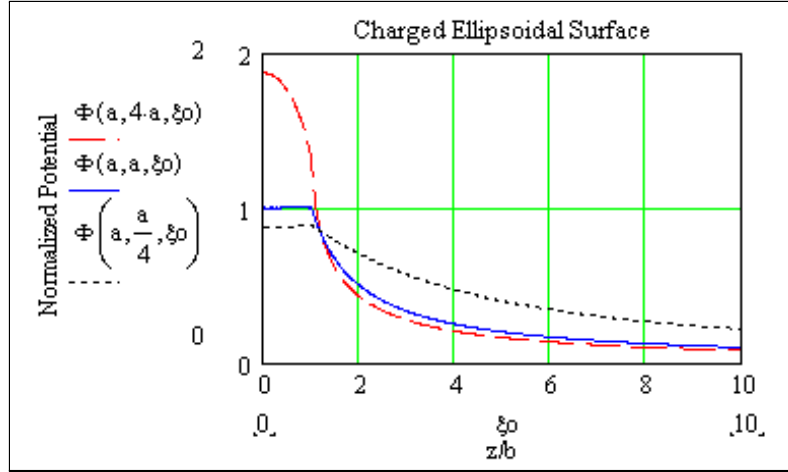
$$\phi(z\hat{\mathbf{z}}) = \frac{2\pi\sigma_0 a^2}{4\pi\epsilon} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\delta(\eta + b^{-2}z'^2 - 1) \sqrt{a^{-2}\eta + b^{-4}z'^2}}{\sqrt{a^2\eta + (z - z')^2}} d\eta dz' \quad (1.93)$$

The delta function allows the η integration to be carried out yielding

$$\phi(z\hat{\mathbf{z}}) = \frac{\sigma_0 a}{2\epsilon} \int_{-b}^b \left[\frac{a^{-2} + b^{-2}z'^2 (b^{-2} - a^{-2})}{(1 - b^{-2}z'^2) + a^{-2}(z - z')^2} \right]^{1/2} dz' \quad (1.94)$$

It is convenient to express the z coordinates in units of b , $z = b\xi_0$ and $z' = b\xi'$, then, with $\beta = (b/a)^2$,

$$\phi(\xi_0 b\hat{\mathbf{z}}) = \frac{\sigma_0 a}{2\epsilon} \int_{-1}^1 \left[\frac{\beta + \xi'^2 (1 - \beta)}{(1 - \xi'^2) + \beta (\xi_0 - \xi')^2} \right]^{1/2} d\xi' \quad (1.95)$$



.Fig. 4

The integral in Eq. 1.95 can be done numerically and the normalized potential, $\Phi(a, b, z\hat{\mathbf{z}}) = \frac{\phi(z\hat{\mathbf{z}})}{\sigma_0 a/\epsilon}$, is plotted below in Fig. 4 for three cases. $b = 4a, a$ and $\frac{a}{4}$. Note that when $a = b$ the ellipsoid of revolution becomes a charged sphere.

The potential due to a spherical shell of radius a with total charge, Q , (and uniform surface charge density, $\sigma_0 = \frac{Q}{4\pi a^2}$) can be obtained using Gauss' law. Because of the spherical symmetry \mathbf{E} has a constant magnitude on a sphere of radius r

$$\begin{aligned} \iiint \nabla \cdot \mathbf{E} dV &= \iiint \rho/\epsilon dV \\ \iint \mathbf{E} \cdot d\mathbf{S} &= Q_{\text{enclosed}}/\epsilon \\ |\mathbf{E}|4\pi r^2 &= 4\pi a^2 \sigma_0/\epsilon \quad \text{if } r \geq a \\ &= 0 \quad \text{if } r < a \end{aligned}$$

Thus

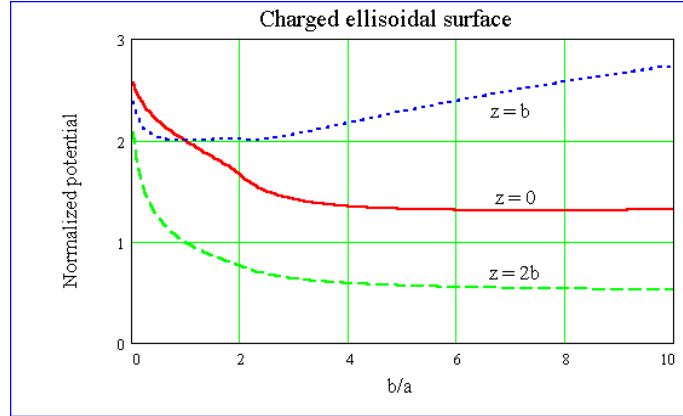
$$\begin{aligned} |\mathbf{E}| &= \frac{Q}{4\pi\epsilon r^2} \quad \text{if } r \geq a \\ &= 0 \quad \text{if } r < a \end{aligned}$$

From $\mathbf{E} = -\nabla\phi$ one finds

$$\begin{aligned} \phi_{\text{spherical shell}}(\mathbf{r}) &= \frac{Q}{4\pi\epsilon r} = \frac{\sigma_0 a^2}{\epsilon r} \quad \text{if } r \geq a \\ &= \frac{Q}{4\pi\epsilon a} = \frac{\sigma_0 a}{\epsilon} \quad \text{if } r \leq a \end{aligned}$$

Thus the potential is constant inside the uniformly charged spherical shell and the electric field is zero. But inside the ellipsoidal shell (see Fig. 4) one can not assume that \mathbf{E} has a constant magnitude on a spherical shell and Gauss' law is not particularly useful. As seen in Fig. 4 the potential inside the ellipsoidal shell is not constant, and the electric field is not zero.

For general information the potential, divided by $\sigma_0 b/4\epsilon$ (at $z = 0, b, 2b$) as a function of b/a is shown in Fig. 5. The small values of b/a correspond to a ‘pancake’ shaped ellipse and the potentials at the origin, the middle of the ‘pancake surface’, and just above the ‘pancake’ surface are nearly the same. Large values of b/a give a ‘needle’ shaped ellipse. (Note that the curves have some strange behavior since the normalization potential is proportional to b .)



.Fig. 5

2.3.2 The potential and fields due to a surface charge density

It is possible to have physical systems with

charge distributions which lie approximately along a surface. Unless one probes within the distribution one can approximate the fields and potentials by those of a surface charge density. The potential due to a surface charge density, $\sigma(\mathbf{r})$, is formally given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \iint_{\text{surface}} \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS' + \phi_0. \quad \text{SI units} \quad (1.96)$$

For finite surface charge densities the potential will be continuous at the surface.

2.3.3 Example: uniformly charged plane

Let the system be a uniform plane sheet of charge, surface charge density σ_0 , lying in the $z = 0$ plane. Determine the value of the potential at (x, y, z) (also denoted by \mathbf{r}) with the condition that the potential vanish at the surface.

Solution: The differential surface area for this problem is $dS' = dx'dy'$. This can be used in Eq. 1.96 to obtain the

required integral

$$\begin{aligned}
 \phi(x, y, z) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sigma_0/4\pi\epsilon}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} dx' dy' + \phi_0 \\
 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sigma_0/4\pi\epsilon}{\sqrt{(x-x')^2 + (y-y')^2 + z^2}} d(x-x') d(y-y') + \phi_0 \\
 &= \frac{\sigma_0}{4\pi\epsilon} \int_0^{\infty} [r^2 + z^2]^{-1/2} 2\pi r dr + \phi_0.
 \end{aligned} \tag{1.97}$$

The integral, which has the potential equal to zero at infinity, is divergent. To handle this difficulty we note that the potential is independent of x and y . We will assume that the surface has a finite radius R centered on the z axis. Then the potential at $x = y = 0$ is

$$\begin{aligned}
 \phi(0, 0, z) &= \frac{\sigma_0}{2\epsilon} \int_0^R [r^2 + z^2]^{-1/2} r dr + \phi_0(R). \\
 &= \frac{\sigma_0}{2\epsilon} \left[\sqrt{R^2 + z^2} - |z| \right] + \phi_0(R)
 \end{aligned} \tag{1.98}$$

The condition that the potential is zero at $z = 0$ requires that $\phi_0(R) = -\frac{\sigma_0}{2\epsilon} R$. With this value of $\phi_0(R)$ we can evaluate the potential in the limit that R goes to infinity,

$$\begin{aligned}
 \phi(0, 0, z) &= \lim_{R \rightarrow \infty} \frac{\sigma_0}{2\epsilon} \left[\sqrt{R^2 + z^2} - |z| - R \right] \\
 &= \frac{-\sigma_0}{2\epsilon} |z| = \phi(x, y, z)
 \end{aligned} \tag{1.99}$$

This potential is normally obtained with less effort by starting with the electric field. Note that $\phi(x, y, z) < 0$ and that the resulting electric field is along the z axis, as expected.

$$\mathbf{E}(z) = -\nabla\phi = \hat{\mathbf{z}} \frac{\sigma_0}{2\epsilon} = \hat{\mathbf{n}} \frac{\sigma_0}{2\epsilon} = \mathbf{E}_1(z) \tag{1.99b}$$

On the other side of the surface charge, along $-\hat{\mathbf{n}}$

$$\mathbf{E}_2(z) = -\nabla\phi_2 = -\hat{\mathbf{z}} \frac{\sigma_0}{2\epsilon} = -\hat{\mathbf{n}} \frac{\sigma_0}{2\epsilon} \tag{1.99c}$$

This gives a surface discontinuity of the electric field shown in Eq. 1.102 below.

+

The field due to an arbitrary surface charge density is formally given by

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) = \frac{1}{4\pi\epsilon} \iint \frac{\sigma(\mathbf{r}') [\mathbf{r} - \mathbf{r}']}{|\mathbf{r} - \mathbf{r}'|^3} dS. \tag{1.100}$$

Gauss' law, Eq. 1.59, requires the component of the electric displacement vector which is perpendicular to the surface to be discontinuous. The discontinuity is given by

$$\hat{\mathbf{n}}(\mathbf{r}) \cdot (\mathbf{D}_1(\mathbf{r}) - \mathbf{D}_2(\mathbf{r})) = \sigma(\mathbf{r}) \quad (1.101)$$

where $\hat{\mathbf{n}}(\mathbf{r})$ (the unit normal to the surface at \mathbf{r}) is directed towards the side with the field $\mathbf{D}_1(\mathbf{r})$. Since the electric field at \mathbf{r} is related to the displacement vector by $\mathbf{E}(\mathbf{r}) = \mathbf{D}(\mathbf{r})/\epsilon$ the electric fields at the surface have a discontinuity given by

$$\hat{\mathbf{n}}(\mathbf{r}) \cdot (\epsilon_1 \mathbf{E}_1(\mathbf{r}) - \epsilon_2 \mathbf{E}_2(\mathbf{r})) = \sigma(\mathbf{r}) \text{ for surface charge layer} \quad (1.102)$$

Fields and potential due to a surface electric dipole layer

A surface electric dipole layer is a neutral charge layer with an electric dipole moment per unit area directed perpendicular to the surface. It can be modeled as two surface charge layers, $\sigma(\mathbf{r}, \delta)$ and $-\sigma(\mathbf{r}, \delta)$, lying on each side of the surface defined by $F(\mathbf{r}) = 0$. The unit vector $\hat{\mathbf{n}} = \nabla F(\mathbf{r})/|\nabla F(\mathbf{r})|$ is directed from the negative surface charge density to the positive surface charge density (it's sufficient to replace $F(\mathbf{r})$ by $-F(\mathbf{r})$ in order to adjust the sense of $\hat{\mathbf{n}}$). The charge layers lie on the surfaces $F(\mathbf{r} \pm \hat{\mathbf{n}}\delta/2)$ and the surface dipole moment density is

$$\begin{aligned} \boldsymbol{\pi}_d(\mathbf{r}) &= \lim_{\delta \rightarrow 0} \hat{\mathbf{n}} \sigma(\mathbf{r}, \delta) \delta = \pi_d(\mathbf{r}) \hat{\mathbf{n}} & 1.103 \\ \lim_{\delta \rightarrow 0} \sigma(\mathbf{r}, \delta) &= 0 \\ \lim_{\delta \rightarrow 0} \sigma(\mathbf{r}, \delta) \delta &= \pi_d(\mathbf{r}) \end{aligned}$$

Since the surface is neutral (total charge = 0), in the limit that $\delta \rightarrow 0$ with $\sigma(\mathbf{r}, \delta)\delta$ fixed

$$\hat{\mathbf{n}}(\mathbf{r}) \cdot (\varepsilon_1 \mathbf{E}_1(\mathbf{r}) - \varepsilon_2 \mathbf{E}_2(\mathbf{r})) = 0 \text{ for electric dipole surface layer} \quad 1.104$$

From Gauss' law one can determine the electric field contributions, \mathbf{E}_+ and \mathbf{E}_- , and from Eq. (1.99) the potential field contributions, ϕ_+ and ϕ_- , from each individual layer. The latter contributions are shown in the schematics below. The total \mathbf{E}_1 and ϕ_1 (above the dipole layer) and total \mathbf{E}_2 and ϕ_2 (below the dipole layer) are given by $\mathbf{E}_+ + \mathbf{E}_-$ and $\phi_+ + \phi_-$ in each region

Schematic of the \mathbf{E} field contributions

$$\begin{array}{c} \sigma(\mathbf{r}, \delta) \hat{\mathbf{n}}_1 / 2\varepsilon \quad \uparrow \\ \text{++++++++} \\ -\sigma(\mathbf{r}, \delta) \hat{\mathbf{n}}_1 / 2\varepsilon \quad \downarrow \\ -\sigma(\mathbf{r}, \delta) \hat{\mathbf{n}}_2 / 2\varepsilon \quad \downarrow \\ \text{-----} \\ \sigma(\mathbf{r}, \delta) \hat{\mathbf{n}}_2 / 2\varepsilon \quad \uparrow \end{array}$$

where $\hat{\mathbf{n}}_1 = \hat{\mathbf{n}}_2$ and the negative sign on the field due to the lower layer comes from the negative charge "surface layer".

Schematic of the ϕ field contributions

$$\begin{aligned}
 & -|z|\sigma(\mathbf{r}, \delta)/2\epsilon \\
 & ++++++++ \\
 & -|z|\sigma(\mathbf{r}, \delta)/2\epsilon \\
 & |z|\sigma(\mathbf{r}, \delta)/2\epsilon \\
 & ----- \\
 & |z|\sigma(\mathbf{r}, \delta)/2\epsilon
 \end{aligned}$$

Thus the total electric fields are:

$$\mathbf{E}_{1,total}(\mathbf{r}, \delta) = \sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_1/2\epsilon - \sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_2/2\epsilon = 0 \quad \text{above the dipole layer}$$

$$\mathbf{E}_{2,total}(\mathbf{r}, \delta) = -\sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_1/2\epsilon + \sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_2/2\epsilon = 0 \quad \text{below the dipole layer}$$

giving

$$\hat{\mathbf{n}}_1 \cdot [\mathbf{E}_{1,total} - \mathbf{E}_{2,total}] = 0$$

Between the two layers for finite δ the electric field is constant, directed downward and equal to,

$$\mathbf{E}_{3,total}(\mathbf{r}, \delta) = -\sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_1/2\epsilon - \sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_2/2\epsilon = -\sigma(\mathbf{r}, \delta)\hat{\mathbf{n}}_1/\epsilon.$$

The total potential above the dipole layer (where $|z_2| = \delta + |z_1|$) is

$$\phi_{1,total}(\mathbf{r}, \delta) = \phi_+(\mathbf{r}, \delta) + \phi_-(\mathbf{r}, \delta) = -|z_1|\sigma(\mathbf{r}, \delta)/2\epsilon + |z_2|\sigma(\mathbf{r}, \delta)/2\epsilon = \delta\sigma(\mathbf{r}, \delta)/2\epsilon$$

Below the dipole layer (where $|z_1| = \delta + |z_2|$)

$$\phi_{2,total}(\mathbf{r}, \delta) = |z_2|\sigma(\mathbf{r}, \delta)/2\epsilon - |z_1|\sigma(\mathbf{r}, \delta)/2\epsilon = -\delta\sigma(\mathbf{r}, \delta)/2\epsilon,$$

and for finite δ between the two charge layers where $|z_2| = \delta - |z_1|$ the total potential is

$$\begin{aligned}
 \phi_{3,total}(\mathbf{r}, \delta) &= -|z_1|\sigma(\mathbf{r}, \delta)/2\epsilon + |z_2|\sigma(\mathbf{r}, \delta)/2\epsilon = \delta\sigma(\mathbf{r}, \delta)/2\epsilon - |z_1|\sigma(\mathbf{r}, \delta)/\epsilon \\
 &= \delta\sigma(\mathbf{r}, \delta)/2\epsilon + z_1\sigma(\mathbf{r}, \delta)/\epsilon.
 \end{aligned}$$

Note that ϕ is continuous at each individual planar charged layer:

$$\phi_{3,total} = \delta\sigma(\mathbf{r}, \delta)/2\epsilon = \phi_{1,total} \quad \text{at the + layer when } z_1 = 0 \quad \#$$

$$\phi_{3,total} = -\delta\sigma(\mathbf{r}, \delta)/2\epsilon = \phi_{2,total} \quad \text{at the - layer when } z_1 = -\delta \quad \#$$

Finally the discontinuity in the potential above and below the surface dipole layer is

$$\phi_1(\mathbf{r}) - \phi_2(\mathbf{r}) = \lim_{\delta \rightarrow 0} [\delta\sigma(\mathbf{r}, \delta)/2\epsilon - (-\delta\sigma(\mathbf{r}, \delta)/2\epsilon)] = \lim_{\delta \rightarrow 0} \frac{1}{\epsilon} \delta\sigma(\mathbf{r}, \delta) = \frac{1}{\epsilon} \pi_d(\mathbf{r}) \quad 1.106$$

To formally obtain the potential due to a surface dipole layer we follow the standard approach. The potential will be the sum of the potential of the positively charged surface layer and the potential of the negatively charged surface layer as follows. After the limit $\delta \rightarrow 0$ is taken a point \mathbf{r}' on the planar dipole layer will be on the surface S' .

$$\phi(\mathbf{r}) = \lim_{\delta \rightarrow 0} \left[\frac{1}{4\pi\epsilon} \iint \frac{\sigma(\mathbf{r}' + \frac{1}{2}\delta\hat{\mathbf{n}}, \delta)}{|\mathbf{r} - (\mathbf{r}' + \frac{1}{2}\delta\hat{\mathbf{n}})|} dS' - \frac{1}{4\pi\epsilon} \iint \frac{\sigma(\mathbf{r}' - \frac{1}{2}\delta\hat{\mathbf{n}}, \delta)}{|\mathbf{r} - (\mathbf{r}' - \frac{1}{2}\delta\hat{\mathbf{n}})|} dS' \right]$$

recall : $df(\mathbf{r}') = d\mathbf{r}' \cdot \nabla' f = f(\mathbf{r}' + d\mathbf{r}') - f(\mathbf{r}')$

$$\begin{aligned} \phi(\mathbf{r}) &= \lim_{\delta \rightarrow 0} \left[\frac{1}{4\pi\epsilon} \iint \left[\frac{\sigma(\mathbf{r}', \delta)}{|\mathbf{r} - \mathbf{r}'|} + \left(\frac{1}{2}\delta\hat{\mathbf{n}} \right) \cdot \nabla' \left[\frac{\sigma(\mathbf{r}', \delta)}{|\mathbf{r} - \mathbf{r}'|} \right] - \frac{\sigma(\mathbf{r}', \delta)}{|\mathbf{r} - \mathbf{r}'|} - \left(\frac{-1}{2}\delta\hat{\mathbf{n}} \right) \cdot \nabla' \left[\frac{\sigma(\mathbf{r}', \delta)}{|\mathbf{r} - \mathbf{r}'|} \right] \right] dS' \\ &= \lim_{\delta \rightarrow 0} \frac{1}{4\pi\epsilon} \iint \sigma(\mathbf{r}', \delta) \delta\hat{\mathbf{n}} \cdot \nabla' \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] dS'; \quad \sigma(\mathbf{r}', \delta) \text{ is a constant, independent of } \mathbf{r}' \\ &= \frac{1}{4\pi\epsilon} \iint \pi_d(\mathbf{r}') \nabla' \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] \cdot \hat{\mathbf{n}} dS' \quad \text{Eq. 1.107} \\ &= \frac{-1}{4\pi\epsilon} \iint \pi_d(\mathbf{r}') \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \cdot \hat{\mathbf{n}} dS'; \quad \text{since } \nabla' \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] = -\nabla' \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] = \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \\ &= \frac{-1}{4\pi\epsilon} \iint \pi_d(\mathbf{r}') d\Omega' \quad \text{Eq. 1.107} \end{aligned}$$

$$\phi(\mathbf{r}) = \frac{-1}{4\pi\epsilon} \iint \pi_d(\mathbf{r}') d\Omega'_s. \quad 1.108$$

In this relationship $d\Omega_s$ is the solid angle subtended by the differential surface element of the dipole layer at \mathbf{r}' , specifically as viewed from the observation point \mathbf{r} when in the configuration shown in Fig. 6 below. Formally, the solid angle subtended by a surface, S , is the projection of the surface onto a unit sphere centered at the observation point, \mathbf{r} :

$$\Omega_S = \iint_{S_{\text{exposed}}} \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \cdot \hat{\mathbf{n}}_s dS' = \iint_{S_{\text{exposed}}} d\Omega_s. \quad 1.109$$

S is defined by $f(x, y, z) = 0$ and has normals, $\hat{\mathbf{n}} = \pm \nabla f / |\nabla f|$. The $\hat{\mathbf{n}}_s = \hat{\mathbf{n}}$ in the solid angle expression of Eq. 1.109 is always directed out of the volume enclosed by the surface and the unit sphere upon which the surface is being projected. Using this prescription Ω_S is always positive. When calculating solid angles one often only integrates over that part of the surface which is visible to the observer.

In Eq. 1.107, however, $\hat{\mathbf{n}}$ always points from $-\sigma$ toward the σ charge layer (where depending on the problem σ could *later* be interpreted as negative) and one must integrate over the entire surface upon which the dipole surface charge density resides. Eq. 1.107 gives the correct sign for all \mathbf{r} independent of configuration. The discontinuity in $\phi(\mathbf{r})$ arises automatically from the change in sign of the differential $d\Omega'$ as one traverses the dipole layer. (In $d\Omega'_s$ one adjusts the sign artificially to give a positive solid angle for all configurations.)

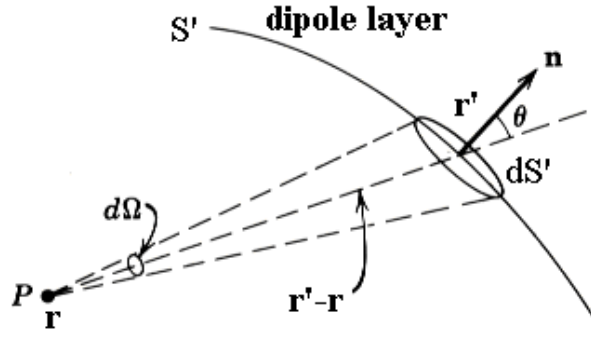


Fig. 6a

The relationship between the parameters is shown in the above figure.

Example 1: Find the electrostatic field inside and outside a constant dipole surface charge density, $\pi_d = q/a$, residing on a sphere of radius R . Let the sphere be centered at the origin. From Eq. 1.107 $\phi(\mathbf{r})$ is given by

$$\begin{aligned}\phi(\mathbf{r}) &= -\frac{1}{4\pi\epsilon} \iint \pi_d(\mathbf{r}') \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \cdot \hat{\mathbf{n}} dS' \\ &= -\frac{q}{4\pi a\epsilon} \iint_S \frac{(\mathbf{r}' - \mathbf{r})}{|\mathbf{r}' - \mathbf{r}|^3} \cdot \hat{\mathbf{n}} dS' \\ &= -\frac{q}{4\pi a\epsilon} \iint_S d\Omega'\end{aligned}$$

where the sphere is defined by $r' = R$. All solid angles viewed from inside the sphere are 4π and correspond to integrating over the entire sphere, so

$$\phi(\mathbf{r}) = -\frac{q}{a\epsilon} \quad \text{for } r < R$$

Outside the sphere, at $r > R$:

$$\begin{aligned}\phi(r\hat{\mathbf{z}}) &= \frac{q}{4\pi a\epsilon} \iint \hat{\mathbf{r}}' \cdot \nabla' \left[\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right] dS'; \text{ note that } r > r' \\ &= \frac{q}{4\pi a\epsilon} \iint \int \delta(r' - R) \hat{\mathbf{r}}' \cdot \nabla' \left[\sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r'^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) \right] r'^2 \sin\theta' d\theta' d\varphi' dr' \\ &= \frac{q}{4\pi a\epsilon} \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta, \varphi) \int \delta(r' - R) r'^2 \frac{\partial}{\partial r'} \frac{r'^l}{r'^{l+1}} dr' \iint \sqrt{4\pi} Y_{lm}^*(\theta', \varphi') Y_{00}(\theta', \varphi') \sin\theta' d\theta' d\varphi' \\ &= \frac{q}{4\pi a\epsilon} \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta, \varphi) \int \delta(r' - R) r'^2 \frac{lr'^{l-1}}{r'^{l+1}} dr' \sqrt{4\pi} \delta_{l0} \delta_{m0} \\ &= \frac{q}{4\pi a\epsilon} \frac{4\pi}{1} Y_{00}(\theta, \varphi) \int \delta(r' - R) r'^2 \frac{0}{r'^2} dr' \sqrt{4\pi} \\ &= 0 \quad \text{for } r > R\end{aligned}$$

Note that the above result of $\phi(r\hat{\mathbf{z}}) = 0$ is only valid if $r > r'$. [One could use the above procedure to derive the result for $r < r'$. In this case the partial derivative with respect to r'

gives an overall factor of $-(l+1)$ or -1 for $l=0$ and the final result is $\frac{-q}{a\epsilon}$. We will discuss these techniques later.] This result is consistent with spherical symmetry, Gauss's law and a total charge enclosed = 0. The electric field is zero everywhere. (Note that outside the sphere the solid angle approach is not helpful. When determining the solid angle from outside the sphere part of the sphere is obscured and one integrates over only the visible part of S . The potential calculation, on the other hand, calls for integrating over the entire sphere!)

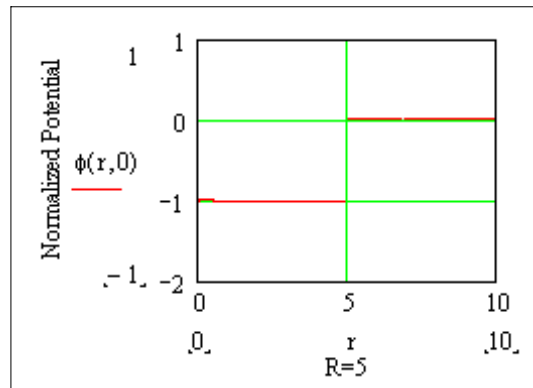


Fig. 6b Sphere with $\pi_d = q/a$

The normalized potential is divided by $\frac{q}{a\epsilon}$ and uses variables $(r, \theta = 0)$ where $\mathbf{r} = z\hat{\mathbf{z}} = r\cos\theta\hat{\mathbf{z}}$

Example 2: Find the electrostatic potential along the symmetry axis inside and outside a sphere of radius, R , centered at the origin, with dipole surface charge density, $\pi_d = [q/a]\cos\theta$ On the symmetry axis $\mathbf{r} = r\hat{\mathbf{z}}$, and

Use Eq. 1.107

$$\begin{aligned}\phi(r\hat{\mathbf{z}}) &= -\frac{q}{4\pi a\epsilon} \iint_S \frac{\cos\theta'(\mathbf{r}' - \mathbf{r}) \cdot \hat{\mathbf{r}}'}{|\mathbf{r}' - \mathbf{r}|^3} r'^2 \sin\theta' d\theta' d\phi'; \quad r' = R \\ &= -\frac{q}{2a\epsilon} \int_0^\pi \frac{\cos\theta'(\mathbf{r}' - \mathbf{r}) \cdot \hat{\mathbf{r}}'}{|\mathbf{r}' - \mathbf{r}|^3} r'^2 \sin\theta' d\theta' \\ &= -\frac{q}{2a\epsilon} \int_0^\pi \frac{\cos\theta'(R^3 - r\cos\theta'R^2)}{[R^2 + r^2 - 2r\cos\theta'R]^3} \sin\theta' d\theta'; \quad r = aR\end{aligned}$$

where $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = \cos\theta'$ and the symmetry gives 2π for the ϕ' integration. The electric field along $\hat{\mathbf{z}}$ is

$$\mathbf{E}(\mathbf{r}) \cdot \hat{\mathbf{z}} = -\frac{d\phi(\mathbf{r})}{dz}$$

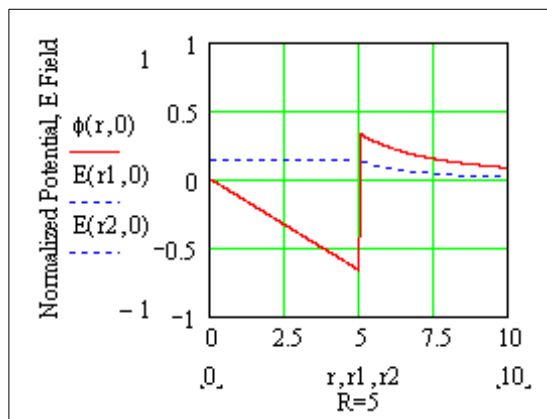


Fig. 6c Sphere with $\pi_d = q/a \cos\theta$

Note that ϕ is discontinuous and $\mathbf{E} \cdot \hat{\mathbf{n}}$ is continuous across the dipole layer. The normalized potential is divided by $\frac{q}{a\epsilon}$ and uses variables $(r, \theta = 0)$ where $\mathbf{r} = z\hat{\mathbf{z}} = r\cos\theta\hat{\mathbf{z}}$.

Example 3 An infinite plane with a constant dipole surface charge density, $\pi_d = q/a$

$$\phi(\mathbf{r}) = -\frac{q}{4\pi a\epsilon} \iint_S \frac{(\mathbf{r}' - \mathbf{r}) \cdot \hat{\mathbf{n}}}{|\mathbf{r}' - \mathbf{r}|^3} dS'$$

We let $z' = 0$ define the plane, and let $\mathbf{r} = z\hat{\mathbf{z}}$. Then $\mathbf{r} \cdot \mathbf{r}' = 0$ and $\hat{\mathbf{n}} = \hat{\mathbf{z}}$. Thus

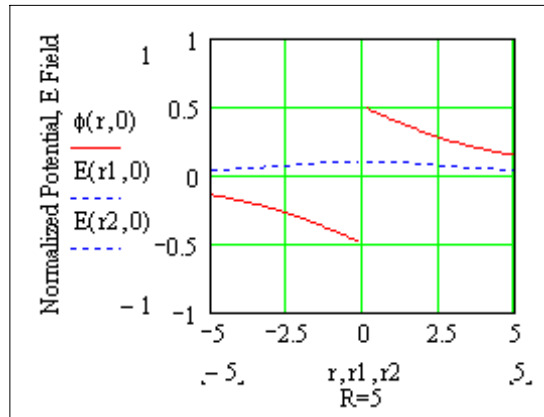
$$\begin{aligned} \phi(r\hat{\mathbf{z}}) &= -\frac{q}{4\pi a\epsilon} \iint_S \frac{-\mathbf{r} \cdot \hat{\mathbf{z}}}{|\mathbf{r}' - \mathbf{r}|^3} dx' dy' \\ &= \frac{qz}{4\pi a\epsilon} \iint_S \frac{1}{[\rho'^2 + z^2]^{3/2}} \rho' d\rho' d\phi' \\ &= \frac{qz2\pi}{4\pi a\epsilon} \int_0^\infty \frac{1}{[\rho'^2 + z^2]^{3/2}} \frac{1}{2} d(\rho'^2) \\ &= \frac{qz}{4a\epsilon} \int_0^\infty d(-2[\rho'^2 + z^2]^{-1/2}) \\ &= \frac{qz}{2a\epsilon} [z^2]^{-1/2} = \frac{q}{2a\epsilon} \frac{z}{|z|} \end{aligned}$$

The electric field is zero everywhere. This unique results occurs only when the plane is infinite as the next example indicates.

Example 4 Assume the infinite plane is reduced to a finite disk of radius R and calculate the potential along the symmetry axis.

$$\begin{aligned}
 \phi(z\hat{\mathbf{z}}) &= -\frac{q}{4\pi a\epsilon} \iint_S \frac{-\mathbf{r} \cdot \hat{\mathbf{z}}}{|\mathbf{r}' - \mathbf{r}|^3} dx' dy' \\
 &= \frac{qz}{4\pi a\epsilon} \iint_S \frac{1}{[\rho'^2 + z^2]^{3/2}} \rho' d\rho' d\phi' \\
 &= \frac{qz2\pi}{4\pi a\epsilon} \int_0^{R^2} \frac{1}{[\rho'^2 + z^2]^{3/2}} \frac{1}{2} d(\rho'^2) \\
 &= \frac{qz}{4a\epsilon} \int_{\rho'=0}^{\rho'=R} d(-2[\rho'^2 + z^2]^{-1/2}) \\
 &= \frac{qz}{2a\epsilon} \left[\frac{1}{|z|} - \frac{1}{[R^2 + z^2]^{1/2}} \right]
 \end{aligned}$$

In the plot below the normalized potential is divided by $\frac{q}{a\epsilon}$ and uses variables $(r, \theta = 0)$ where $\mathbf{r} = z\hat{\mathbf{z}} = r\cos\theta\hat{\mathbf{z}}$



Just above disk's center the potential is the same as for an infinite plane, $\frac{q}{2a\epsilon} \frac{z}{|z|}$. The magnitude, however can be adjusted with an appropriate ϕ_o added to the potential.

2.4 Laplace and Poisson Equations

As noted, Faraday's law for static fields is satisfied by any electric field which can be written as

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}).$$

The function $\phi(\mathbf{r})$ has been identified as the electric or scalar potential associated with the electric field $\mathbf{E}(\mathbf{r})$. Since $\mathbf{D}(\mathbf{r}) = \epsilon\mathbf{E}(\mathbf{r})$, in order to satisfy Gauss' law for the displacement vector the scalar potential must be a solution to Poisson's Equation,

$$\nabla^2\phi(\mathbf{r}) = -\frac{1}{\epsilon}\rho(\mathbf{r}). \quad (1.106)$$

A special case of Poisson's Equation is obtained for $\rho(\mathbf{r}) = 0$. The result is known as Laplace's Equation,

$$\nabla^2\phi(\mathbf{r}) = 0. \quad (1.107)$$

This is the equation satisfied by the scalar potential in a charge free region. The general problem is to obtain the solution, $\phi(\mathbf{r})$, of Eq. 1.106 in a region of space with $\phi(\mathbf{r})$ satisfying specified boundary conditions on the boundary of the region

Green's functions

There is a class of solutions, $\{g(\mathbf{r}, \mathbf{r}')\}$, to Poisson's Equation with the source term given by $\delta^{(3)}(\mathbf{r} - \mathbf{r}')$. These solutions are called 'Green's functions'. One possible solution for $g(\mathbf{r}, \mathbf{r}')$ which we have already seen is

$$g_0(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}'|} \quad (1.108)$$

$$\nabla^2 g_0(\mathbf{r}, \mathbf{r}') = \delta^{(3)}(\mathbf{r} - \mathbf{r}'). \quad (1.109)$$

This $g_0(\mathbf{r}, \mathbf{r}')$ vanishes as $|\mathbf{r}| \rightarrow \infty$ for all finite $|\mathbf{r}'|$. If $g(\mathbf{r}, \mathbf{r}')$ is known (and has the correct boundary conditions) a formal solution of Eq. 1.106 is given by

$$\phi(\mathbf{r}) = \frac{1}{\epsilon} \iiint \rho(\mathbf{r}') g(\mathbf{r}, \mathbf{r}') d^3r'. \quad (1.110)$$

However, this solution will not generally satisfy the boundary conditions placed on $\phi(\mathbf{r})$. In the next section we derive a more formally correct solution which contains the expression in Eq. 1.110

2.4.1 Green's theorem and Green's function solution for $\phi(\mathbf{r})$

Green's theorem, uses the divergence theorem to relate solutions of Poisson's equation. Assume that we have two functions, $\phi(\mathbf{r})$ and $g(\mathbf{r}, \mathbf{r}_o)$, which satisfy Poisson equations as follows in volume, V , with surface, S :

$$\nabla^2 \phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon \quad (1.111)$$

$$\nabla^2 g(\mathbf{r}, \mathbf{r}_o) = \delta(\mathbf{r} - \mathbf{r}_o) \quad (1.112)$$

The divergence theorem states that

$$\iiint_{\text{volume } V} \nabla \cdot [\phi(\mathbf{r}) \nabla g(\mathbf{r}, \mathbf{r}_o) - g(\mathbf{r}, \mathbf{r}_o) \nabla \phi(\mathbf{r})] d^3r = \iint_{\text{bounding surfaces, } S} [\phi(\mathbf{r}) \nabla g(\mathbf{r}, \mathbf{r}_o) - g(\mathbf{r}, \mathbf{r}_o) \nabla \phi(\mathbf{r})] \cdot d\mathbf{S} \quad (1.113)$$

The left hand side can be written as follows:

$$\begin{aligned} & \iiint_{\text{volume}} [\phi(\mathbf{r}) \nabla^2 g(\mathbf{r}, \mathbf{r}_o) - g(\mathbf{r}, \mathbf{r}_o) \nabla^2 \phi(\mathbf{r})] d^3r + \iint_{\text{volume}} [\nabla \phi(\mathbf{r}) \cdot \nabla g(\mathbf{r}, \mathbf{r}_o) - \nabla g(\mathbf{r}, \mathbf{r}_o) \cdot \nabla \phi(\mathbf{r})] d^3r \\ = & \iiint_{\text{volume}} \phi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_o) d^3r - \iiint_{\text{volume}} g(\mathbf{r}, \mathbf{r}_o) [-\rho(\mathbf{r})] d^3r \quad \text{since the second term above is 0} \\ = & \phi(\mathbf{r}_o) - \iiint_{\text{volume}} g(\mathbf{r}, \mathbf{r}_o) [-\rho(\mathbf{r})/\varepsilon] d^3r \quad \text{if } \mathbf{r}_o \text{ is in the volume!} \end{aligned}$$

So if \mathbf{r}_o is in the volume, V , the left hand side substituted into Eq. 1.113 gives:

$$\phi(\mathbf{r}_o) = \iiint_{\text{volume}} g(\mathbf{r}, \mathbf{r}_o) [-\rho(\mathbf{r})/\varepsilon] d^3r + \iint_{\text{bounding surface}} [\phi(\mathbf{r}) \nabla g(\mathbf{r}, \mathbf{r}_o) - g(\mathbf{r}, \mathbf{r}_o) \nabla \phi(\mathbf{r})] \cdot \hat{\mathbf{n}}(\mathbf{r}) dS \quad (1.114)$$

This is a *Green's function* solution of Eq.1.111 . Outside the volume, where \mathbf{r}_o is **not** in V ,

$$\iiint_{\text{volume}} \phi(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}_o) d^3r = 0. \quad \text{for } \mathbf{r}_o \text{ not in } V$$

and we obtain no solution for $\phi(\mathbf{r}_o)$. The surface terms in Eq. 1.114 suggest the interpretation that $\nabla \phi(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r})$ at the surface is surface charge layer (because $\nabla \phi(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) = -\mathbf{E} \cdot \hat{\mathbf{n}}(\mathbf{r})$ and the surface might be replaced with an effective discontinuity in \mathbf{E}) and $\phi(\mathbf{r}) \hat{\mathbf{n}}(\mathbf{r})$ at the surface is a surface dipole layer (because the surface might be replaced with an effective discontinuity in $\phi(\mathbf{r})$). But the latter is not crucial to what we derive in this section. Here we have assumed that $\hat{\mathbf{n}}(\mathbf{r})$ is the bounding surface unit outward normal vector. In the next section we will find that specifying both the potential and the normal gradient of the potential on the bounding surface will 'over specify the problem'. In this case the problem will only have a solution when the specified potential and normal derivative are compatible.

2.4.2 Uniqueness of the solutions to Poisson's Equation Let $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ be two solutions to Eq. 1.111 and let both satisfy the same boundary conditions (still of an unspecified nature) on the surface bounding the volume. It follows that

$$\chi(\mathbf{r}) = \phi_1(\mathbf{r}) - \phi_2(\mathbf{r}) \quad (1.115)$$

satisfies Laplace's Equation,

$$\begin{aligned} \nabla^2 \chi(\mathbf{r}) &= \nabla^2 \phi_1(\mathbf{r}) - \nabla^2 \phi_2(\mathbf{r}) \\ &= \rho(r)/\varepsilon - \rho(r)/\varepsilon \\ &= 0 \end{aligned}$$

with 'zero boundary conditions'. If we now apply the divergence theorem to $\chi(\mathbf{r}) \nabla \chi(\mathbf{r})$ we obtain

$$\begin{aligned} \iiint_{\text{volume}} \nabla \cdot [\chi(\mathbf{r}) \nabla \chi(\mathbf{r})] d^3r &= \iint_{\text{bounding surface}} \chi(\mathbf{r}) \nabla \chi(\mathbf{r}) \cdot d\mathbf{S} \quad (1.116) \\ &= \iiint_{\text{volume}} [\chi(\mathbf{r}) \nabla^2 \chi(\mathbf{r}) + |\nabla \chi(\mathbf{r})|^2] d^3r = \iint_{\text{bounding surface}} \chi(\mathbf{r}) \nabla \chi(\mathbf{r}) \cdot d\mathbf{S} \\ &= \iiint_{\text{volume}} |\nabla \chi(\mathbf{r})|^2 d^3r = \iint_{\text{bounding surface}} \chi(\mathbf{r}) \nabla \chi(\mathbf{r}) \cdot d\mathbf{S} \end{aligned}$$

Using Eq. 1.116 we now investigate the boundary conditions satisfied by $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$..

(1) First, let us assume that the values of the potential, but not its normal derivative, are given on the bounding surface. In this case $\chi(\mathbf{r}) = 0$ on the boundary and the right hand side of **Eq. 1.116** vanishes. Since the integrand of the volume integral cannot be negative it must be identically zero or $\nabla \chi(\mathbf{r}) = 0$. This requires that $\phi_1(\mathbf{r}) - \phi_2(\mathbf{r})$ be constant. Since they are equal on the boundary the appropriate constant is zero and $\phi_1(\mathbf{r}) = \phi_2(\mathbf{r})$. Specifying the potential on the boundary gives a unique boundary value problem satisfied by only one potential function. This particular problem in which the function on the boundary is specified is called a '**Dirichlet problem**' and the boundary conditions are called '**Dirichlet boundary conditions**'.

(2) Next we assume that the values of the normal derivative of the potential on the boundary is given.. In this case the normal derivative of $\chi(\mathbf{r})$ will vanish on the boundary. Again the right hand side of Eq. 1.116 will vanish and therefore, as before, $\phi_1(\mathbf{r}) - \phi_2(\mathbf{r})$ will be constant. In this case it's only the normal derivative which vanishes on the boundary and the gradient of any constant is zero. It follows that $\phi_1(\mathbf{r}) = \phi_2(\mathbf{r}) + \phi_0$ and the solution to this boundary value problem is only unique up to an additive constant. The problems in which the normal derivative is given on the boundary are known as '**Neumann problems**' and the boundary conditions are called '**Neumann boundary conditions**'.

(3) One can also contemplate problems in which the value of the normal derivative of the potential is specified on some sections of the boundary and the value of the potential is specified on other, different, sections of the boundary. In this case the potential is said to satisfy '**mixed boundary conditions**'.

Since the Dirichlet problem has unique solutions, specifying the potential on the boundary fixes the normal derivative of the potential on the boundary. Similarly the solution of a Neumann problem will give the potential on the boundary (up to an additive constant). We find then that **Eq. 1.114**, which requires the values of the potential and its normal derivative on the bounding surface, does not provide a convenient solution to Poisson's equation unless one can eliminate one of the surface integrals by imposing boundary conditions on $g(\mathbf{r}, \mathbf{r}_0)$.

2.5 The Green's functions for Dirichlet and Neumann problems The difficulty with **Eq. 1.114** is that both the potential and its normal derivative appear in the integrand of the surface integral. However, we have not imposed any boundary conditions on the Green's function. If we could find a function $g(\mathbf{r}, \mathbf{r}_0)$, satisfying **Eq. 1.112** whose normal derivative on the bounding surface vanished, only the potential on the surface would be required. In the other case, if the function $g(\mathbf{r}, \mathbf{r}_0)$ vanished on the bounding surface, only the normal derivative of the potential on the surface would be required. The apparent difficulty in applying **Eq. 1.116** to obtain the solution of Poisson's Equation is circumvented by obtaining the appropriate $g(\mathbf{r}, \mathbf{r}_0)$.

One approach is to write $g(\mathbf{r}, \mathbf{r}_0)$ in the region of interest as the sum of a solution to the inhomogeneous equation (**Eq. 1.112**) and the homogeneous equation (Laplace's equation). That is,

$$g(\mathbf{r}, \mathbf{r}_0) = \frac{-1}{4\pi|\mathbf{r} - \mathbf{r}_0|} + F(\mathbf{r}, \mathbf{r}_0) \quad (1.117)$$

with

$$\nabla^2 F(\mathbf{r}, \mathbf{r}_0) = 0. \text{ inside } V \quad (1.118)$$

V is the region for which the solution, $\phi(\mathbf{r})$, is defined. The function $F(\mathbf{r}, \mathbf{r}_0)$ is determined so that $g(\mathbf{r}, \mathbf{r}_0)$ will satisfy the required boundary conditions.

2.5.1 The Dirichlet problem For Dirichlet problems we require that $g(\mathbf{r}, \mathbf{r}_0) = G_D(\mathbf{r}, \mathbf{r}_0)$ with

$$G_D(\mathbf{r}, \mathbf{r}_0) = 0 \text{ for } \mathbf{r} \text{ on the boundary.} \quad (1.119)$$

In this case we find that

$$\phi_D(\mathbf{r}_0) = \frac{-1}{\epsilon} \iiint_{\text{volume}} G_D(\mathbf{r}, \mathbf{r}_0) \rho(\mathbf{r}) d^3r + \iint_{\text{bounding surface}} \phi(\mathbf{r}_s) \nabla G_D(\mathbf{r}_s, \mathbf{r}_0) \cdot d\mathbf{S} \quad (1.120)$$

which only requires the values of the potential on the boundary. The function $G_D(\mathbf{r}, \mathbf{r}_0)$ is the '*Dirichlet green's function*'.

The function $F_D(\mathbf{r}, \mathbf{r}_0)$ satisfies

$$F_D(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} \text{ for } \mathbf{r} = \mathbf{r}_S \quad (1.121)$$

where \mathbf{r}_S is on the boundary of the volume. Note that this does not mean that $F_D(\mathbf{r}, \mathbf{r}_0) = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|}$. Rather $F_D(\mathbf{r}, \mathbf{r}_0)$ is a function which reduces to the latter expression on the surface. In particular, **Eq. 1.118** must be satisfied in V .

Symmetry of the Green's function

A characteristic of the Green's function is that $G_D(\mathbf{r}, \mathbf{r}') = G_D(\mathbf{r}', \mathbf{r})$. This result can be obtained by considering two Dirichlet Green's functions $G_D(\mathbf{r}, \mathbf{r}_0)$ and $G_D(\mathbf{r}, \mathbf{r}'_0)$ each satisfying the condition that they vanish for \mathbf{r} on the boundary

surface. If we now apply the divergence theorem to $G_D(\mathbf{r}, \mathbf{r}_0) \nabla G_D(\mathbf{r}, \mathbf{r}'_0) - G_D(\mathbf{r}, \mathbf{r}'_0) \nabla G_D(\mathbf{r}, \mathbf{r}_0)$ we find

$$\begin{aligned}
 & \iiint_{\text{volume}} \nabla \cdot [G_D(\mathbf{r}, \mathbf{r}_0) \nabla G_D(\mathbf{r}, \mathbf{r}'_0) - G_D(\mathbf{r}, \mathbf{r}'_0) \nabla G_D(\mathbf{r}, \mathbf{r}_0)] d^3r = \\
 & \iiint_{\text{volume}} G_D(\mathbf{r}, \mathbf{r}_0) \delta(\mathbf{r} - \mathbf{r}'_0) d^3r - \iiint_{\text{volume}} G_D(\mathbf{r}, \mathbf{r}'_0) \delta(\mathbf{r} - \mathbf{r}_0) d^3r \\
 & = G_D(\mathbf{r}'_0, \mathbf{r}_0) - G_D(\mathbf{r}_0, \mathbf{r}'_0) \\
 & = \iint_{\text{bounding surface}} [G_D(\mathbf{r}, \mathbf{r}_0) \nabla G_D(\mathbf{r}, \mathbf{r}'_0) - G_D(\mathbf{r}, \mathbf{r}'_0) \nabla G_D(\mathbf{r}, \mathbf{r}_0)] \cdot \hat{\mathbf{n}}(\mathbf{r}) dS \\
 & = 0
 \end{aligned}$$

and

$$\begin{aligned}
 G_D(\mathbf{r}'_0, \mathbf{r}_0) & = G_D(\mathbf{r}_0, \mathbf{r}'_0) \text{ . or} \\
 G_D(\mathbf{r}', \mathbf{r}) & = G_D(\mathbf{r}, \mathbf{r}') \text{ since } \mathbf{r}'_0, \mathbf{r}_0 \text{ are arbitrary variables.}
 \end{aligned} \tag{1.122}$$

2.5.2 The Neumann problem Since the solutions to the Neumann problems are only unique up to an additive constant the Green's function for the Neumann problem,

$$G_N(\mathbf{r}, \mathbf{r}_0) = g(\mathbf{r}, \mathbf{r}_0) + F_N(\mathbf{r}, \mathbf{r}_0)$$

with

$$\nabla^2 F_N(\mathbf{r}, \mathbf{r}_0) = 0 \text{ in } V.$$

satisfies a slightly more complicated boundary condition. If we apply the divergence theorem to

$$\begin{aligned}
 \nabla^2 G_N(\mathbf{r}, \mathbf{r}_0) & = \delta^{(3)}(\mathbf{r} - \mathbf{r}_0) \\
 \iiint \nabla \cdot \nabla G_N(\mathbf{r}, \mathbf{r}_0) d^3r & = \iiint \delta^{(3)}(\mathbf{r} - \mathbf{r}_0) d^3r
 \end{aligned} \tag{1.123}$$

we find the Neumann restriction:

$$\iint_{\text{bounding surface}} \nabla G_N(\mathbf{r}, \mathbf{r}_0) \cdot \hat{\mathbf{n}}(\mathbf{r}) dS = 1 \tag{1.124a}$$

It follows that we are not permitted to set $\nabla G_N(\mathbf{r}, \mathbf{r}_0) \cdot \hat{\mathbf{n}}(\mathbf{r}) = 0$ on the boundary of the volume. The next simplest choice

is to satisfy Eq.1.123 with a simple boundary condition on G_N :

$$\text{simple boundary condition: } \nabla G_N(\mathbf{r}, \mathbf{r}_0) \cdot \hat{\mathbf{n}}(\mathbf{r}) = \frac{1}{S_{\text{tot}}} \tag{1.124b}$$

The solution to the Neumann problem is then given by

$$\phi(\mathbf{r}_0) = \frac{1}{\epsilon} \iiint_{\text{volume}} G_N(\mathbf{r}, \mathbf{r}_0) [-\rho(\mathbf{r})] d^3r + \phi_{\text{avg}} + \iint_{\text{bounding surface}} G_N(\mathbf{r}, \mathbf{r}_0) \nabla \phi(\mathbf{r}) \cdot \hat{\mathbf{n}}(\mathbf{r}) dS \tag{1.125}$$

with ϕ_{avg} the average value of the potential on the bounding surface,

$$\phi_{avg} = \frac{\iint \phi(\mathbf{r}) dS}{\iint dS} \quad (1.126)$$

where S is the area of the boundary

$$S_{tot} = \iint_{\substack{\text{bounding} \\ \text{surface}}} dS.$$

Since the Neumann problem only defines the potential up to an additive constant we could arbitrarily take $\phi_{avg} = 0$ without

losing any information. The function $F_N(\mathbf{r}, \mathbf{r}_0)$ satisfies the condition that

$$[\nabla G_N(\mathbf{r}, \mathbf{r}_0)] \cdot \hat{\mathbf{n}}(\mathbf{r}) = \left[\nabla F_N(\mathbf{r}, \mathbf{r}_0) + \nabla \left(\frac{-1}{4\pi |\mathbf{r} - \mathbf{r}_0|} \right) \right] \cdot \hat{\mathbf{n}}(\mathbf{r}) = \frac{1}{S_{tot}}. \quad (1.127)$$

on S

To investigate the relationship between $G_N(\mathbf{r}, \mathbf{r}')$ and $G_N(\mathbf{r}', \mathbf{r})$ which we artificially label $G'_N(\mathbf{r}', \mathbf{r})$ with a prime and let

$$\nabla'^2 G'_N(\mathbf{r}', \mathbf{r}'_o) = \delta(\mathbf{r}' - \mathbf{r}'_o) \quad (1.128)$$

$G'_N(\mathbf{r}', \mathbf{r}'_o)$ is a solution to the Neumann problem, with $\rho(\mathbf{r}')/\varepsilon = \delta(\mathbf{r}' - \mathbf{r}'_o)$ and from Eq. 1.125,

$$\begin{aligned} G'_N(\mathbf{r}_0, \mathbf{r}'_o) &= \iiint_{\text{volume}} G_N(\mathbf{r}', \mathbf{r}_0) \delta(\mathbf{r}' - \mathbf{r}'_o) d^3 r' + \\ G'_{avg}(\mathbf{r}'_o) &+ \iint_{\substack{\text{bounding} \\ \text{surface}}} G_N(\mathbf{r}'_s, \mathbf{r}_0) \nabla' G'_N(\mathbf{r}'_s, \mathbf{r}'_o) \cdot d\mathbf{S}' \end{aligned} \quad (1.129)$$

The result is

$$G'_N(\mathbf{r}_0, \mathbf{r}'_o) = G_N(\mathbf{r}'_o, \mathbf{r}_0) + G'_{ave}(\mathbf{r}'_o) - G_{ave}(\mathbf{r}_0) \quad (1.130)$$

Dropping the primes on the G symbols,

$$G_N(\mathbf{r}_0, \mathbf{r}'_o) - \langle G_N(\cdot, \mathbf{r}'_o) \rangle_{avg} = G_N(\mathbf{r}'_o, \mathbf{r}_0) - \langle G_N(\cdot, \mathbf{r}_0) \rangle_{avg}. \quad (1.131)$$

Since the solutions of the Neumann problem are only unique up to an additive constant if we take any Neumann green's function and subtract its average value on the boundary we will obtain a green's function which is symmetric in the interchange of the parameters.

2.5.3 The functions $F(\mathbf{r}, \mathbf{r}_0)$ The simple green's function $g(\mathbf{r}, \mathbf{r}') = \frac{-1}{4\pi|\mathbf{r}-\mathbf{r}'|}$ can be identified as the potential due to a 'unit' negative point charge located at \mathbf{r}' . The function $F(\mathbf{r}, \mathbf{r}')$ must satisfy Laplace's equation in the volume, V , of interest. If we identify it as a potential its sources will be charges which lie outside our volume. We will find that viewing $F(\mathbf{r}, \mathbf{r}')$ as the potential due to charges outside the region of interest will provide us with a technique for generating this function.

2.6 Solutions $\nabla^2\Phi = 0$ using G_D and G_N for Poisson's Equation This section has introduced the Green's functions for Poisson's equation with Dirichlet or Neumann boundary conditions given. Since Laplace's equation is a special case of Poisson's equation with the source term, $\rho(\mathbf{r})$, equal to zero, the Green's functions can be used to provide a solution to the Dirichlet problems for Laplace's equation, $\phi_{L-D}(\mathbf{r})$,

$$\begin{aligned} \phi_{L-D}(\mathbf{r}_o) &= \iiint 0 \cdot G_D(\mathbf{r}', \mathbf{r}_0) d^3x' + & (1.132) \\ &\iint_{\text{bounding surface}} \phi(\mathbf{r}_s)_{L-D} \nabla G_D(\mathbf{r}_s, \mathbf{r}_0) \cdot d\mathbf{S} - \iint G_D(\mathbf{r}_s, \mathbf{r}_0) \nabla \phi(\mathbf{r}_s) \cdot d\mathbf{S} \\ &= \iint_{\text{bounding surface}} \phi(\mathbf{r}_s)_{L-D} \nabla G_D(\mathbf{r}_s, \mathbf{r}_0) \cdot d\mathbf{S} \end{aligned}$$

Given the solution on the boundary, S , one can find a Green's function solution to Laplace's equation.

where the second term and third terms are zero because $\nabla^2\Phi = 0$ and $G_D(\mathbf{r}_s, \mathbf{r}_0) = 0$, respectively. A solution to the Neumann problem for Laplace's equation, $\phi_{L-N}(\mathbf{r})$

$$\begin{aligned} \phi_{L-N}(\mathbf{r}) &= \iiint 0 \cdot G_N(\mathbf{r}', \mathbf{r}_0) d^3x' + & (1.133) \\ &\iint \phi(\mathbf{r}_s)_{L-N} \nabla G_N(\mathbf{r}_s, \mathbf{r}_0) \cdot d\mathbf{S} - \iint_{\text{bounding surface}} G_N(\mathbf{r}_s, \mathbf{r}_0) \nabla \phi_{L-N}(\mathbf{r}_s) \cdot d\mathbf{S} \\ &= - \iint_{\text{bounding surface}} G_N(\mathbf{r}_s, \mathbf{r}_0) \nabla \phi_{L-N}(\mathbf{r}_s) \cdot d\mathbf{S} + \phi_{avg} \\ &= - \iint_{\text{bounding surface}} G_N(\mathbf{r}_s, \mathbf{r}_0) \nabla \phi_{L-N}(\mathbf{r}_s) \cdot d\mathbf{S} \end{aligned}$$

as we can set $\phi_{avg} = 0$ on the bounding surface because the Neumann problem gives the solution to within a constant.

2.7 Electrostatic energy density, capacitance

2.7.1 The electrostatic energy density Starting with Maxwell's equations we can obtain the relationship

$$\begin{aligned}\nabla \cdot [\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)] &= -\mathbf{E}(\mathbf{r}, t) \cdot [\nabla \times \mathbf{H}(\mathbf{r}, t)] + \mathbf{H}(\mathbf{r}, t) \cdot [\nabla \times \mathbf{E}(\mathbf{r}, t)] \\ &= -\mathbf{E}(\mathbf{r}, t) \cdot [\mathbf{J}(\mathbf{r}, t) + \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t)] + \left[\mathbf{H}(\mathbf{r}, t) \cdot -\frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) \right].\end{aligned}\quad (1.134)$$

We can identify $\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{J}(\mathbf{r}, t)$ as the rate at which the E field does work on the mechanical system per unit area, or

equivalently, the power density transfer between the fields and the mechanical system. With this identification we are led to associate $\mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t)$ with $\mathbf{S}(\mathbf{r}, t)$, the energy current density carried in the electromagnetic field. $\mathbf{S}(\mathbf{r}, t)$ is called **the Poynting vector**.

$$\mathbf{S}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t) \times \mathbf{H}(\mathbf{r}, t) \quad (1.135)$$

units of S: energy/sec/area = power/area = Watts/square meter
--

and to associate the term

$$\frac{\partial u(\mathbf{r}, t)}{\partial t} = \left[\mathbf{H}(\mathbf{r}, t) \cdot \frac{\partial}{\partial t} \mathbf{B}(\mathbf{r}, t) + \mathbf{E}(\mathbf{r}, t) \cdot \frac{\partial}{\partial t} \mathbf{D}(\mathbf{r}, t) \right] \quad (1.136)$$

with the time rate of change of the energy density stored in the 'electromagnetic fields' where

$$u(\mathbf{r}, t) = \frac{1}{2} [\mathbf{E}(\mathbf{r}, t) \cdot \mathbf{D}(\mathbf{r}, t) + \mathbf{H}(\mathbf{r}, t) \cdot \mathbf{B}(\mathbf{r}, t)] \quad (1.137)$$

Then Eq. 1.134 gives

$$\nabla \cdot \mathbf{S}(\mathbf{r}, t) = -\frac{\partial u(\mathbf{r}, t)}{\partial t} - \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{J}(\mathbf{r}, t) \quad (1.138)$$

and

$$\iint \mathbf{S}(\mathbf{r}, t) \cdot d\mathbf{A} + \iiint \mathbf{E}(\mathbf{r}, t) \cdot \mathbf{J}(\mathbf{r}, t) d^3r = - \iiint \frac{\partial u(\mathbf{r}, t)}{\partial t} d^3r \quad (1.139)$$

Equation 1.139 states that the rate at which energy leaves a region plus the rate at which field energy is converted to

mechanical energy equals the rate at which energy is transferred to the fields and the material¹. We can expect that if the fields are varied 'adiabatically' we have a unique value for the energy stored by the electromagnetic fields. We will therefore ignore for the present some subtle or not so subtle problems which arise due to frequency dependent relationships between \mathbf{E} and \mathbf{D} . We will claim that the energy density stored in the electrostatic electric field is

¹ We note that there is an ad hoc separation of the charge currents. Part of the current is explicitly given by $\mathbf{J}(\mathbf{r}, t)$ while $\frac{\partial u}{\partial t}$ has a contribution from an implicit current. In addition the magnetic field energy includes an implicit energy current density which transmits energy across the boundary of the region.

$$w(\mathbf{r}) = \frac{\epsilon}{2} |\nabla\phi(\mathbf{r})|^2 \quad SI\text{units} \quad (1.140)$$

The electromagnetic energy of a system is given by integrating the energy density over the volume of the system

$$W = \frac{\epsilon}{2} \iiint_{\text{volume}} |\nabla\phi(\mathbf{r})|^2 d^3r \geq 0. \quad (1.141)$$

Let $\rho(\mathbf{r})$ denote the charge density for the system. If the charge density is composed of charge densities for fixed elements,

charged point particles, charged surfaces, etc., then it is convenient to express the total charge density as a sum of the individual charge densities

$$\rho(\mathbf{r}) = \sum_i \rho_i(\mathbf{r}). \quad (1.142)$$

Each charge density will contribute, $-\nabla\phi_i(\mathbf{r})$, to the total electric field at each point where $\nabla^2\phi_i(\mathbf{r}) = -\rho_i(\mathbf{r})/\epsilon$. In this

case

$$-\nabla\phi(\mathbf{r}) = -\sum_i \nabla\phi_i(\mathbf{r}) \quad (1.143)$$

In terms of the fields of the individual charge elements the electromagnetic energy of the system is

$$\begin{aligned} W &= \frac{\epsilon}{2} \iiint_{\text{volume}} \sum_i \nabla\phi_i(\mathbf{r}) \cdot \sum_j \nabla\phi_j(\mathbf{r}) d^3r \\ &= \frac{\epsilon}{2} \sum_i \iiint_{\text{volume}} |\nabla\phi_i(\mathbf{r})|^2 d^3r \\ &\quad + \frac{\epsilon}{2} \sum_{\substack{i,j \\ i < j}} \iiint_{\text{volume}} \nabla\phi_i(\mathbf{r}) \cdot \nabla\phi_j(\mathbf{r}) d^3r \end{aligned} \quad (1.144)$$

The term

$$W_i = \frac{\epsilon}{2} \iiint_{\text{volume}} |\nabla\phi_i(\mathbf{r})|^2 d^3r \quad \text{self energy} \quad (1.145)$$

is the electromagnetic ‘self energy’ of the i^{th} charge element. As long as the charge density of the element does not change

this ‘self energy’ will be fixed. The second term,

$$W_{ij} = \epsilon \iiint_{\text{volume}} \nabla\phi_i(\mathbf{r}) \cdot \nabla\phi_j(\mathbf{r}) d^3r, \quad \text{interaction energy} \quad (1.146)$$

with $i \neq j$ provides the interaction energy between the charged elements. The significance of the various terms is illustrated by the following example.

Example

Consider two identical non-conducting spherical shells. Let each have a radius a and let their centers be separated by a distance $2a$. The surface charge of one shell is σ_0 and the surface charge of the other is $-\sigma_0$. (a) What is the self energy of each spherical shell? (b) What is the interaction energy of the spherical shells?

Solution: (a) The magnitude of the charge on each shell is $q = 4\pi a^2 \sigma_0$. The electric field of spherical shell vanishes inside the shell while outside the shell the electric field is

$$\begin{aligned} \mathbf{E}(\mathbf{r} - \mathbf{r}_{\pm}) &= \frac{\pm q (\mathbf{r} - \mathbf{r}_{\pm})}{4\pi\epsilon |\mathbf{r} - \mathbf{r}_{\pm}|^3} \text{ SI units} \\ &= \frac{\pm q (\mathbf{r} - \mathbf{r}_{\pm})}{\epsilon |\mathbf{r} - \mathbf{r}_{\pm}|^3} \text{ in Gaussian units} \\ &= -\nabla\phi_{\pm}(\mathbf{r}). \end{aligned}$$

r_+ and r_- locate the center of the spherical shells.

To calculate the self energy of each shell we evaluate, Eq.1.145 ,

$$W_0 = \frac{\epsilon q^2}{2(4\pi\epsilon)^2} \iiint_{|\mathbf{r}-\mathbf{r}_{\pm}|>a} \frac{d^3r}{|\mathbf{r} - \mathbf{r}_{\pm}|^4}.$$

For each shell we can let $\mathbf{r}' = \mathbf{r} - \mathbf{r}_{\pm}$ and the self energy of each spherical shell is found to be

$$\begin{aligned} W_0 &= \frac{\epsilon q^2}{2(4\pi\epsilon)^2} \iiint_{|\mathbf{r}'|>a} \frac{d^3r'}{|\mathbf{r}'|^4} = \frac{4\pi\epsilon q^2}{2(4\pi\epsilon)^2} \int_a^{\infty} \frac{1}{r'^2} dr' \\ &= \frac{q^2}{2 \cdot 4\pi\epsilon} \left[\frac{-1}{r'} \right]_a^{\infty} \end{aligned}$$

Why does one integrate from a to infinity, rather than zero to infinity?

$$\begin{aligned} W_0 &= \frac{q^2}{2 \cdot 4\pi\epsilon a}. \quad \text{Self energy of each shell in SI units} \\ &= \frac{q^2}{2\epsilon a} \quad \text{Gaussian units} \end{aligned} \tag{1.147}$$

Aside: the charge radius of the electron

Equation 1.147 can be used to introduce a classic physics problem which persists in all models. Suppose we consider the electron to be a spherical charged shell. In this case the electromagnetic energy stored in the electric field of the electron would be ($\epsilon = 1$)

$$W_e = \frac{e^2}{2a} \text{ in Gaussian units}$$

The total energy of an electron at rest is $m_e c^2$. The question now is what would be the radius of the electron if all its energy was that stored by the electric field? The answer is

$$\begin{aligned} a &= \frac{e^2}{2m_e c^2} = \frac{(4.8 \times 10^{-10} \text{ statC})^2}{2 \times (9.1 \times 10^{-28} \text{ g}) \times (3 \times 10^{10} \text{ cm/s})^2} \\ &= 1.4 \times 10^{-13} \text{ cm} !! \end{aligned}$$

If this were correct the electron radius would be the same order of magnitude as the *observed* proton radius. But the upper limit on the radius of the electron is several orders of magnitude below the radius of the proton. **In fact the results of all experiments at this time are consistent with the electron being a point particle!** What happened to its electromagnetic energy?

Solution(b) The potential due to a uniform spherical shell of charge centered at the origin is

$$\begin{aligned} \phi(\mathbf{r}) &= \frac{q}{4\pi\epsilon r} \text{ for } r > a \\ \phi(\mathbf{r}) &= \frac{q}{4\pi\epsilon a} \text{ for } r \leq a \end{aligned}$$

where q is the total charge of the shell and a is the radius of the shell. In terms of this potential the interaction energy stored

by the electric fields is, Eq. 1.146,

$$W_{int} = \epsilon \iiint \nabla\phi(\mathbf{r} - \mathbf{r}_+) \cdot \nabla\phi(\mathbf{r} - \mathbf{r}_-) d^3r$$

Using the identity,

$$\nabla \cdot [\phi(\mathbf{r} - \mathbf{r}_+) \nabla\phi(\mathbf{r} - \mathbf{r}_-)] = \nabla\phi(\mathbf{r} - \mathbf{r}_+) \cdot \nabla\phi(\mathbf{r} - \mathbf{r}_-) + \phi(\mathbf{r} - \mathbf{r}_+) \nabla^2\phi(\mathbf{r} - \mathbf{r}_-)$$

$$W_{int} = -\epsilon \iiint_{\text{all space}} \phi(\mathbf{r} - \mathbf{r}_+) \nabla^2\phi(\mathbf{r} - \mathbf{r}_-) d^3r + \epsilon \iint_{\text{surface at } \infty} \phi(\mathbf{r} - \mathbf{r}_+) \text{charge density for shell of charge}$$

Now let $\mathbf{r}' = \mathbf{r} - \mathbf{r}_-$ **Note: \mathbf{r}' is at the center of the shell.**

$$= -\epsilon \iiint_{\text{all space}} \phi(\mathbf{r}' - \mathbf{r}_+ + \mathbf{r}_-) \nabla'^2\phi(\mathbf{r}') d^3r' + \epsilon \iint_{\text{surface at } \mathbf{r}'_s \rightarrow \infty} \phi(\mathbf{r}'_s - \mathbf{r}_+ + \mathbf{r}_-) \frac{1}{r'} \nabla'\phi(\mathbf{r}'_s) \cdot d\mathbf{S}' \frac{1}{r'^2} r'^2 d\Omega'$$

Since the product of the field and the potential vanish at infinity as $r'_s{}^{-3}$ whereas $d\mathbf{S}'$ only increases like $r'_s{}^2$ the surface integral does not contribute to the interaction energy. To evaluate the spatial integration we note that $\nabla'^2\phi(\mathbf{r}')$ refers to the negative shell and (see Jackson problem 1.3a):

$$\nabla'^2\phi(\mathbf{r}') = -\frac{\sigma_0}{\epsilon} \delta(r' - a). \quad \text{SI units}$$

In the example the distance between the centers of the spheres is greater than the diameter of the spheres. Therefore the

spheres do not overlap and the interaction energy is given by

$$\begin{aligned} W_{int} &= -\varepsilon \iiint_{\text{all space}} \phi(\mathbf{r}' - \mathbf{r}_+ + \mathbf{r}_-) \nabla^2 \phi(\mathbf{r}') d^3 r' \\ &= -\varepsilon \iiint_{\text{all space}} \frac{q}{4\pi\varepsilon|\mathbf{r}' - 2\mathbf{a}|} \left[-\frac{\sigma_0}{\varepsilon} \delta(r' - a)\right] d^3 r' \end{aligned}$$

Letting $2\mathbf{a} = R\hat{\mathbf{z}}$

$$W_{int} = -\frac{q\sigma_0}{\varepsilon} \frac{1}{4\pi} \int_0^\pi \int_0^\infty \int_0^{2\pi} \frac{\delta(r - a)}{[r^2 + R^2 - 2rR \cos \theta]^{1/2}} d\gamma r^2 dr \sin \theta d\theta$$

The integration over the angle γ and the radial direction r are readily carried out leaving ($\xi = \cos \theta$)

$$\begin{aligned} W_{int} &= -\frac{q\sigma_0}{\varepsilon} \frac{2\pi}{4\pi} \int_0^\pi \frac{a^2}{[a^2 + R^2 - 2aR \cos \theta]^{1/2}} \sin \theta d\theta \\ W_{int} &= -\frac{q \left[\frac{q}{4\pi a^2}\right] a^2}{2\varepsilon} \int_{-1}^{+1} \frac{d\xi}{[a^2 + R^2 - 2aR\xi]^{1/2}} \\ &= -\frac{q^2}{2 \cdot 4\pi\varepsilon} \int_{\xi=-1}^{\xi=+1} -\frac{1}{aR} d[a^2 + R^2 - 2aR\xi]^{1/2} \\ &= \frac{q^2}{2 \cdot 4\pi\varepsilon} \frac{[R - a] - [R + a]}{aR} \\ &= -\frac{q^2}{4\pi\varepsilon R} \quad \text{interaction energy of two shells} \end{aligned}$$

The interaction energy is equal to that of two charged point particles, with charges $\pm q$, separated by a distance R . The total energy stored in the electrostatic field is

$$W = \frac{q^2}{4\pi\varepsilon a} - \frac{q^2}{4\pi\varepsilon R} > 0 \quad \text{total energy for the two shells}$$

As R decreases the stored energy decreases.

$$dW = -|dR| \frac{\partial}{\partial R} \left[-\frac{q^2}{4\pi\varepsilon R} \right] = -\frac{q^2}{4\pi\varepsilon R^2} |dR| = -F |dR|$$

Therefore the force between the shells is attractive with a magnitude of $q^2/[4\pi\varepsilon R^2]$ and does work dW

$$W + dW = \frac{q^2}{4\pi\varepsilon a} - \frac{q^2}{4\pi\varepsilon R} \left[1 + \frac{dR}{R} \right]$$

2.8 Conductors and Capacitance

A commonly encountered system consists of a set of N isolated charged

conductors. (In electrostatics the potential inside and at the surface of a conductor is constant.). Let the i^{th} conductor have a charge Q_i and let all other conductors have zero net charge. The charge on the i^{th} conductor will be distributed over its surface with a surface charge density $\sigma_{ii}(\mathbf{r})$ which is proportional to Q_i . It also will induce a surface charge density on each of the other conductors $\sigma_{ji}(\mathbf{r})$ which will be proportional to Q_i . Of course

$$\begin{aligned} Q_i &= \iint_i \sigma_{ii}(\mathbf{r}) dS \\ 0 &= \iint_j \sigma_{ji}(\mathbf{r}) dS, \quad j \neq i \end{aligned} \quad (1.148)$$

If all the conductors are charged then the surface charge density on a conductor will be the sum of the surface charge densities which would be on the conductor with only one conductor charged. [This follows from the uniqueness of the solution to the boundary value problem. In this case the potential of each conductor can not vary over the conductor. This is satisfied if either the i^{th} conductor has a charge Q_i or the j^{th} conductor has a charge Q_j . If the potentials obtained with each of these conditions are added the result will also satisfy the boundary conditions].

$$\sigma_i(\mathbf{r}) = \sum_j \sigma_{ij}(\mathbf{r}).$$

The j^{th} term in the sum is proportional to Q_j . The potential of the i^{th} conductor is given by

$$V_i = \sum_{k=1}^N \iint_k \frac{\sigma_k(\mathbf{r})}{|\mathbf{r}_i - \mathbf{r}|} dS \quad (1.149)$$

where \mathbf{r}_i is any point on the i^{th} conductor. Because the $\{\sigma_k(\mathbf{r})\}$, are linear functions of the $\{Q_i\}$, it follows that

$$V_i = \sum_{j=1}^N p_{ij} Q_j. \quad (1.150)$$

The proportionality constants $\{p_{ij}\}$ will depend on the geometry of the system of conductors but not the charges on the conductors. (Note that $p_{ij} \neq \frac{1}{Q_j} \iint_j \frac{\sigma_j(\mathbf{r})}{|\mathbf{r}_i - \mathbf{r}|} dS$ since the charge distribution on the surface of each conductor depends linearly on the charges of the other conductors.) The linear relationship in Eq. 1.150 can be inverted to obtain the charges on the conductors in terms of the potentials of the conductors

$$Q_i = \sum_{j=1}^N C_{ij} V_j \quad (1.151)$$

The coefficient C_{ii} is the capacitance of the i^{th} conductor in the system
while the $\{C_{ij}, \text{ with } i \neq j\}$ are the coefficients of induction.

The capacitance of the i^{th} conductor is obtained by ‘grounding’ all the other conductors (forcing $V_j = 0$ for $j \neq i$) and measuring the ratio

$$C_{ii} = \frac{Q_i}{V_i}, \quad V_j = 0 \text{ for } j \neq i. \quad (1.152)$$

The potential energy for a system of charged conductors is

$$W = \frac{1}{2} \sum_{i=1}^N Q_i V_i = \frac{1}{2} \sum_{i,j=1}^N C_{ij} V_i V_j \quad (1.153)$$

This equation must give the energy stored by the electric fields of the system and therefore $W \geq 0$ for all choices of $\{V_i\}$. It

follows that $C_{ii} > 0$ for all i and $\det [C_{ij}] > 0$. (We have tacitly assumed that the inverse of the matrix C_{ij} exists. In fact it is the matrix p_{ij} . The elements of C_{ij} form a non-singular matrix and therefore $\det [C_{ij}] \neq 0$.)

If we know the Dirichlet green’s function for the region of space excluding the conductors we are able to generate the C_{ij} . (The surface charge density on a conductor in electrostatics equals the normal derivative of the potential at the surface.) It is also possible by using a variational technique to obtain estimates of the C_{ij} .

2.9 Variational calculations

2.9.1 A variational principle The behavior or properties of a system can generally be described by one or more functions. In classical mechanics, for example, we seek $\mathbf{r}_k(t)$ for a system of particles. A common technique is to minimize the action integral, $\mathbf{I} = \int \mathbf{L}(\mathbf{r}_k(t), \dot{\mathbf{r}}_k(t)) dt$. Recall that this leads to the Euler-Lagrange equations. If constraints are imposed on the system one introduces Lagrange multipliers. In electrostatics we apply a similar technique to equations such as

$$I(\phi) = \frac{1}{2} \iiint_{\text{volume}} \nabla \phi \cdot \nabla \phi \, d^3 r.$$

subject to constraints. This works particularly well when the electrical potential ϕ is due to a system of charged conductors.

More formally, we vary the functional form for the ‘system function’, $\psi(\mathbf{r})$, via a ‘functional’, I , defined as

$$I[\psi, \nabla \psi] = \iiint F(\mathbf{r}, \psi(\mathbf{r}), \nabla \psi(\mathbf{r})) d^3 r \quad (1.154)$$

which has an extremum in for the correct functional form for $\psi(\mathbf{r})$. We shall use the following definitions:

$$\begin{aligned} \delta \psi &= \text{‘the variation in the functional form for } \psi \\ \delta \nabla \psi &= \nabla[\delta \psi] \quad \text{and} \\ \delta I &= I[\psi, \nabla \psi] - I[\psi + \delta \psi, \nabla \psi + \delta \nabla \psi]. \end{aligned}$$

Using an expansion similar to a three dimensional Taylor series expansion,

$$f(\mathbf{r} + \Delta\mathbf{r}) = e^{\Delta\mathbf{r} \cdot \nabla} f = [1 + \Delta\mathbf{r} \cdot \nabla + [\Delta\mathbf{r} \cdot \nabla]^2/2! + \dots [\Delta\mathbf{r} \cdot \nabla]^n/n! + \dots]f \quad (1.155)$$

one can obtain,

$$\begin{aligned} I[\psi + \delta\psi, \nabla\psi + \delta\nabla\psi] &= \exp\left[\delta\psi \frac{\delta}{\delta\psi} + \delta\nabla\psi \cdot \frac{\delta}{\delta\nabla\psi}\right] I[\psi, \nabla\psi] \\ &= [1 + [\delta\psi \frac{\delta}{\delta\psi} + \delta\nabla\psi \cdot \frac{\delta}{\delta\nabla\psi}] + [\delta\psi \frac{\delta}{\delta\psi} + \delta\nabla\psi \cdot \frac{\delta}{\delta\nabla\psi}]^2/2! + \dots] I + \dots \\ &= I[\psi, \nabla\psi] + [\delta\psi \frac{\delta}{\delta\psi} + \delta\nabla\psi \cdot \frac{\delta}{\delta\nabla\psi}]I + [\delta\psi \frac{\delta}{\delta\psi} + \delta\nabla\psi \cdot \frac{\delta}{\delta\nabla\psi}]^2 I/2! + \dots \end{aligned} \quad (1.156)$$

To first order in $\delta\psi$ we have:

$$\delta I = \iiint_{vol} \left[\frac{\delta F}{\delta\psi} \delta\psi(\mathbf{r}) + \nabla[\delta\psi] \cdot \frac{\delta F}{\delta\nabla\psi} \right] d^3r + \text{higher order terms} \dots \quad (1.157)$$

where

$$\nabla[\delta\psi] \cdot \frac{\delta F}{\delta\nabla\psi} \equiv \sum_{i=1}^3 \frac{\partial}{\partial x_i} \delta\psi \frac{\delta F}{\delta[\partial\psi/x_i]}.$$

We can replace $\frac{\delta F}{\delta\psi}$ by $\frac{\partial F}{\partial\psi}$ when F is a continuous function of ψ in the volume.

$$\begin{aligned} \delta I &= \iiint_{vol} \left[\frac{\partial F}{\partial\psi} \delta\psi(\mathbf{r}) + \nabla[\delta\psi] \cdot \frac{\partial F}{\partial\nabla\psi} \right] d^3r + \text{higher order terms} \dots \\ &= \iiint_{vol} \left[\frac{\partial F}{\partial\psi} \delta\psi(\mathbf{r}) + \nabla \cdot \left(\delta\psi \frac{\partial F}{\partial\nabla\psi} \right) - \delta\psi \nabla \cdot \frac{\partial F}{\partial\nabla\psi} \right] d^3r + \text{higher order terms} \dots \end{aligned} \quad (1.158)$$

Using the divergence theorem,

$$\begin{aligned} \delta I &= \iiint_{vol} \left[\frac{\partial F}{\partial\psi} - \nabla \cdot \frac{\partial F}{\partial\nabla\psi} \right] \delta\psi d^3r + \iiint_{vol} \nabla \cdot \left(\delta\psi \frac{\partial F}{\partial\nabla\psi} \right) d^3r + \text{higher order terms} \\ &= \iiint_{vol} \left[\frac{\partial F}{\partial\psi} - \nabla \cdot \frac{\partial F}{\partial\nabla\psi} \right] \delta\psi d^3r + \iint_{\text{bounding surface}} \delta\psi \frac{\partial F}{\partial\nabla\psi} \cdot \hat{\mathbf{n}} dS + \text{higher order terms} \dots \end{aligned} \quad (1.159)$$

The functional I will have an extremum when $\delta I = 0$ or,

$$\iiint_{vol} \left[\frac{\partial F}{\partial\psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\partial F(\mathbf{r}, \psi(\mathbf{r}), \nabla\psi(\mathbf{r}))}{\partial(\partial\psi/\partial x_i)} \right] \delta\psi d^3r + \iint_{\text{bounding surface}} \left[\sum_{i=1}^3 \frac{\delta F}{\delta(\partial\psi/\partial x_i)} \hat{\mathbf{n}} \cdot \hat{\mathbf{x}}_i \right] \delta\psi dS = 0 \quad (1.160)$$

for all ‘permissible’ $\delta\psi(\mathbf{r})$. Generally the values of the function $\psi(\mathbf{r})$ are specified on the boundary of the volume. In this

case only $\delta\psi(\mathbf{r})$ which vanish on the boundary would be permitted and the surface integral would be zero. If that is the only restriction on the variation in $\psi(\mathbf{r})$ then the requirement that the first order term vanishes constrains $\psi(\mathbf{r})$ to be a solution of the following "Euler-Lagrange" equation.

$$\frac{\partial F}{\partial \psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\partial F(\mathbf{r}, \psi(\mathbf{r}), \partial\psi/\partial x_i)}{\partial (\partial\psi/\partial x_i)} = 0 \quad (1.161)$$

To determine whether the extremum is a minimum, maximum, or stationary point the second order term in $\delta\psi$ must be calculated.

2.9.2 Variations with equality constraints and fixed boundary conditions In an extension of the variational calculations the function $\psi(\mathbf{r})$ is required to satisfy one or more constraints in addition to the boundary conditions. Generally these have a form,

$$C_\alpha[\psi] = \iiint K_\alpha(\mathbf{r}, \psi(\mathbf{r}), \nabla\psi(\mathbf{r})) d^3r = D_\alpha \quad (1.162)$$

where $\{D_\alpha, \alpha = 1, \dots, N\}$ is a set of N constants. This places restrictions on the $\delta\psi(\mathbf{r})$ used in the variation of $\psi(\mathbf{r})$.in

Eqs.1.156-1.160. Proceeding as before to set the $\delta C_\alpha = 0$, we have $\alpha = 1, 2, 3, \dots N$. equations

$$\iiint_{vol} \left[\frac{\delta K_\alpha}{\delta \psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\delta K_\alpha}{\delta (\partial\psi/\partial x_i)} \right] \delta\psi d^3r + \iint_{\text{bounding surface}} \sum_{i=1}^3 \left[\frac{\delta K_\alpha}{\delta (\partial\psi/\partial x_i)} \hat{\mathbf{x}}_i \right] \cdot \hat{\mathbf{n}} \delta\psi dS = 0 \quad (1.163)$$

Because $\psi(\mathbf{r})$ must satisfy fixed boundary conditions $\delta\psi(\mathbf{r})$ is zero on the surface and the surface integral vanishes. The equations which must be satisfied are

$$\iiint_{vol} \left[\frac{\delta F(\mathbf{r}, \psi(\mathbf{r}), \nabla\psi(\mathbf{r}))}{\delta \psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\delta F(\mathbf{r}, \psi(\mathbf{r}), \nabla\psi(\mathbf{r}))}{\delta (\partial\psi/\partial x_i)} \right] \delta\psi(\mathbf{r}) d^3r = 0 \quad (1.164a)$$

and

$$\iiint_{vol} \left[\frac{\delta K_\alpha(\mathbf{r}, \psi(\mathbf{r}), \nabla\psi(\mathbf{r}))}{\delta \psi} - \sum_{i=1}^3 \frac{\partial}{\partial r_i} \frac{\delta K_\alpha(\mathbf{r}, \psi(\mathbf{r}), \nabla\psi(\mathbf{r}))}{\delta (\partial\psi/\partial r_i)} \right] \delta\psi(\mathbf{r}) d^3r = 0. \quad (1.164b)$$

We can consider the

$$u_\alpha(\mathbf{r}) = \frac{\delta K_\alpha}{\delta \psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\delta K}{\delta (\partial \psi / \partial x_i)} \quad (1.165)$$

to be a set of ‘vectors’ in a function space. In this case the restriction on $\delta \psi(\mathbf{r})$ is that it lie in the subspace which is orthogonal to the $u_\alpha(\mathbf{r})$. (Note that Eq. 1.164b looks like an inner product condition, $(u_\alpha, \delta \psi) = 0$ as does Eq. 1.164a, $(\frac{\delta F}{\delta \psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\delta F}{\delta (\partial \psi / \partial x_i)}, \delta \psi) = 0$.) In this case Eq. 1.164a requires that the function multiplying $\delta \psi(\mathbf{r})$ lie in the subspace spanned by the $u_\alpha(\mathbf{r})$ or

$$\frac{\delta F}{\delta \psi} - \sum_{i=1}^3 \frac{\partial}{\partial x_i} \frac{\delta F}{\delta (\partial \psi / \partial x_i)} = \sum_\alpha \lambda_\alpha u_\alpha(\mathbf{r}) \quad (1.166)$$

The ‘expansion coefficients’ λ_α are constants, called Lagrange multipliers, which are determined such that the constraint equations are satisfied.

2.9.3 Approximation using the functional If the function describing the property of the system provides an extremum for Eq. 1.159 an approximation to the function can be generated. In the simplest case let $\psi(\mathbf{r}) = w(\beta, \mathbf{r})$ where β is a free parameter and the $w(\beta, \mathbf{r})$ is chosen to satisfy the boundary conditions on $\psi(\mathbf{r})$ but is otherwise arbitrary. The functional equation is now a function of β

$$W(\beta) = \iiint F(\mathbf{r}, w(\beta, \mathbf{r}), \nabla w(\beta, \mathbf{r})) d^3r. \quad (1.167)$$

Assume further that,

$$W(\beta) = \frac{\varepsilon}{2} \iiint |\nabla w(\beta, \mathbf{r})|^2 d^3r.$$

To find the extremum of $W(\beta)$ ($\delta W(\beta) = 0$) subject to $w(\beta, \mathbf{r}_{s1}) = 1$, and $w(\beta, \mathbf{r}_{s2}) = 0$ we find the value of β , say b_0 , for which $W'(\beta) = 0$ restricted to functions of the form $w(\beta, \mathbf{r})$ (See also Eq. 1.166.). If $W''(b_0)$ is consistent with the type of extremum required, $w(b_0, \mathbf{r})$ is an approximation to $\psi(\mathbf{r})$.

2.9.4 Estimation of capacitance using a variational approach In an electrostatic problem the potential of each conductor is fixed. One then guesses a trial potential which satisfies the boundary conditions. This trial potential should depend on one or more parameters which are varied so as to obtain the minimum value of the energy for the system. In the case that the trial potential =1 on the i^{th} conductor and zero on all other conductors the capacitance, C_{ii} , is approximated by the estimated energy. That is, for two conductors,

$$\begin{aligned} W &= \frac{1}{2} \sum_{i,j=1}^2 C_{ij} V_i V_j \\ &= \frac{1}{2} C_{11} V_1^2 = \frac{1}{2} C_{11} && \text{if } V_1 = 1 \text{ and } V_2 = 0 \\ &= \frac{1}{2} C_{22} V_2^2 = \frac{1}{2} C_{22} && \text{if } V_2 = 1 \text{ and } V_1 = 0 \end{aligned}$$

Exercise:

Show that the electrostatic energy stored in the field of a charged conductor (with charge Q) and potential V is $W = \frac{1}{2}QV$. (Hint: solve the boundary value problem for $\phi(\mathbf{r})$ everywhere and calculate W from Eq. 1.141)

Example of simple variational method: coaxial cylinders

Consider two coaxial cylinders, one with a radius $b = 0.2\text{cm}$ and the other with a radius $c = 0.8\text{cm} = 4b$. Let $(x^2 + y^2) = \rho^2$ and let the z axis be along the cylindrical axis. Using the potential form,

$$w(\beta, \rho) = \beta \left[1 - \frac{\rho}{2b} + \frac{\rho^2}{2b} \right] + \frac{2\rho}{3c} - \frac{5\rho^2}{3c}$$

estimate the capacitance per unit length of the combination and compare the result to the actual value.

Solution :

Note that we want the function, $\phi(\beta, \rho)$, to be 1 at $\rho = b$ and 0 at $\rho = c$. This can be done by normalizing $w(\rho, z)$:

$$\begin{aligned} \phi(\beta, \rho) &= aw(\beta, \rho) - b' \quad \text{where } a \text{ and } b' \text{ are constants to be determined after we find } \beta \\ &= [w(\beta, \rho) - w(\beta, c)] / [w(\beta, b) - w(\beta, c)] \end{aligned}$$

Since $V_b = \phi(\beta, b) = 1$, and $V_c = \phi(\beta, c) = 0$ we will obtain $W = \frac{1}{2}C_{bb}$. The estimated energy of the coaxial system per unit length, L , in Gaussian units is

$$\begin{aligned} W(\beta) &= \frac{\epsilon_0}{8\pi L} \int \int \int |\nabla\phi|^2 \rho d\rho d\varphi dz \quad \text{! / 4\pi \epsilon_0 = 1 in Gaussian units} \\ &= \frac{1}{8\pi L} \int \int \int \left[\hat{\rho} \frac{\partial\phi}{\partial\rho} + \hat{\varphi} \frac{1}{\rho} \frac{\partial\phi}{\partial\varphi} + \hat{z} \frac{\partial\phi}{\partial z} \right] \cdot \left[\hat{\rho} \frac{\partial\phi}{\partial\rho} + \hat{\varphi} \frac{1}{\rho} \frac{\partial\phi}{\partial\varphi} + \hat{z} \frac{\partial\phi}{\partial z} \right] \rho d\rho d\varphi dz \\ &= \frac{1}{8\pi L} \int \left| \frac{\partial\phi}{\partial\rho} \right|^2 \rho d\rho \cdot 2\pi L \end{aligned}$$

$$W(\beta) = \frac{a^2}{4} \int_b^c \left| \frac{\beta}{2b} [-1 + 2\rho] + \frac{2}{3c} (1 - 5\rho) \right|^2 \rho d\rho$$

Note that b' disappears and a is a constant determined from the normalization.

This has an extremum (minimum) for

$$W'(\beta) = \frac{a^2}{2} \int_b^c \left| \frac{\beta}{2b} [-1 + 2\rho] + \frac{2}{3c} (1 - 5\rho) \right| \frac{1}{2b} [-1 + 2\rho] \rho d\rho = 0$$

$$\beta \int_b^c \left| \frac{1}{2b} [-1 + 2\rho] \right|^2 \rho d\rho = - \int_b^c \left| \frac{1}{2b} [-1 + 2\rho] \right| \frac{2}{3c} (1 - 5\rho) \rho d\rho$$

$$\beta = - \frac{\int_b^c \left| \frac{1}{2b} [-1 + 2\rho] \right| \frac{2}{3c} (1 - 5\rho) \rho d\rho}{\int_b^c \left| \frac{1}{2b} [-1 + 2\rho] \right|^2 \rho d\rho}$$

$$\beta = - \frac{4b}{3c} \frac{\int_{0.2\text{cm}}^{0.8\text{cm}} [-1\text{cm} + 2\rho] [2\text{cm} - 5\rho] \rho d\rho}{\int_{0.2\text{cm}}^{0.8\text{cm}} [|-1\text{cm} + 2\rho|^2] \rho d\rho} = 1.35$$

This gives an approximation to the capacitance per unit length equal to

$$\begin{aligned}
 2W(1.6) &= C = \frac{a^2}{2} \int_b^c \left| \frac{\beta}{2c} [-1 + 2\rho] + \frac{2}{3b} (1 - 5\rho) \right|^2 \rho d\rho \\
 &= 0.367 \qquad \beta = 1.35
 \end{aligned}$$

The plot in Fig. 7 shows the $W(\beta)$ (using the normalized potential) generated via mathcad. The β from this numerical approach is approximately 1.35 (see two.cylinder.example.mcd on the web.). The value of $W(1.35) = 0.183$ and $C = 2 \cdot .183 = 0.367$. The capacitance appears to be not too dependent on β . Figure 8 is a plot of the normalized potential as a function of ρ .

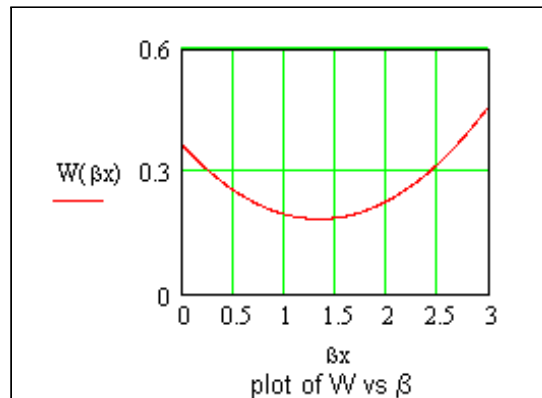


Fig. 7

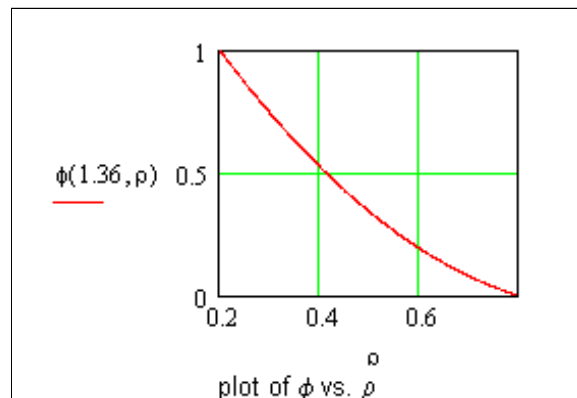


Fig. 8

The exact result is $C = \frac{1}{2 \ln |4|} = 0.361$. So the variational results differ by less than 2%. We shall determine the "exact" result for $L \rightarrow \infty$ later in the notes.

The exact result for the coaxial cylinders is non-trivial:

The exact result for the capacitance of two long cylindrical shells is non-trivial. But we can begin by letting the outer conductor be at 0 potential and the inner potential at V then $C = \frac{Q}{V}$. The length, L , of the cylinders satisfies $\frac{c}{L} \ll 1$.

$$\Phi(\rho, \varphi, z) = \frac{-1}{4\pi\epsilon} \iiint \sigma_0 \delta(\rho' - b) |\nabla(\rho' - b)| \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho' d\rho' d\varphi' dz' + \Phi_0(b, c)$$

where $\Phi_0(b, c)$ is a constant and we shall let z be along the cylindrical axis, with $\rho^2 = x^2 + y^2$. Furthermore, we shall evaluate the potential on the x axis so that $r = x = \rho \cos \varphi = \rho$, $z = 0$ and $\mathbf{r} \cdot \mathbf{r}' = \rho \rho' \cos \varphi'$

$$\begin{aligned} \Phi(\rho, 0, 0) &= \frac{-1}{4\pi\epsilon} \iiint \sigma_0 \delta(\rho' - b) |\nabla(\rho' - b)| \frac{1}{|\mathbf{r} - \mathbf{r}'|} \rho' d\rho' d\varphi' dz' + \Phi_0(b, c) \\ \Phi(\rho, 0, 0) &= \frac{-Q}{4\pi\epsilon(2\pi bL)} \iint \delta(\rho' - b) \frac{1}{[\rho^2 + \rho'^2 + z'^2 - 2\mathbf{r} \cdot \mathbf{r}']^{1/2}} \rho' d\rho' d\varphi' + \Phi_0(b, c) \\ &\approx \frac{-Q}{4\pi\epsilon(2\pi bL)} \int_{-L/2}^{L/2} \int_0^{2\pi} \frac{bd\varphi' dz''}{[\rho^2 + b^2 + z'^2 - 2\rho[b^2 + z'^2]^{1/2} \cos \varphi']^{1/2}} + \Phi_0(b, c) \\ &= \frac{-Q}{4\pi\epsilon(2\pi L)} \int_{-L/2}^{L/2} \int_0^{2\pi} \frac{d\varphi' dz'}{[\rho^2 + a^2 - 2\rho a \cos \varphi']^{1/2}} + \Phi_0(b, c) \quad \text{where } a^2 = [b^2 + z'^2] \end{aligned}$$

Unfortunately this reduces to an elliptic integral which is unmanageable. In the next chapter we shall have some methods for handling this problem.

Using Gauss' law one can obtain the exact result:

$$C = \frac{4\pi\epsilon_0}{2 \ln\left[\frac{b}{a}\right]}$$

Using the variational technique to find the upper and lower bounds for the capacitance:

The capacitance of a system of conductors depends on the geometry. However we often use the potential difference and the charges on the conductors to determine C . In Jackson problems 1.17 and 1.18 one is asked to show that upper and lower bounds can be placed on the capacitance by considering two different variations in the functional forms of C :

$$\left[\frac{Q^2[\sigma]}{2W[\sigma]} \right]_{\text{maximized}} \leq C_{\text{actual}} \leq 2W[\Psi]_{\text{minimized}} \quad (1.90)$$

where

$$\left[\frac{Q^2[\sigma]}{2W[\sigma]} \right]_{\text{maximized}} = \left[\frac{\left[\oint_{S_1} \sigma(\mathbf{r}) dS_1 \right]^2}{\oint_{S_1} \oint_{S_1} \sigma(\mathbf{r}) G(\mathbf{r}, \mathbf{r}') \sigma(\mathbf{r}') dS_1 dS_1'} \right]_{\text{max}} \quad (1.91)$$

and is more commonly written:

$$\left[\frac{1}{C[\sigma]} \right]_{\text{min}} = \left[\frac{\oint_{S_1} \oint_{S_1} \sigma(\mathbf{r}) G(\mathbf{r}, \mathbf{r}') \sigma(\mathbf{r}') dS_1 dS_1'}{\left[\oint_{S_1} \sigma(\mathbf{r}) dS_1 \right]^2} \right]_{\text{min}} \quad (1.92)$$

The right hand side of Eq. 1.90 is relatively easy to use and is often employed to determine C . The left hand condition of Eq. 1.90 is in contrast often intractable because of the need to find $G(\mathbf{r}, \mathbf{r}')$. The Green's function in Eqs. 1.91 and 1.92 must = 0: on all conductors except S_1 . The solution to Problem 1.18 provides the derivation of Eq 1.92. In this solution one considers the system for which all Q_i are fixed and all $\Phi_i = 0$ but Φ_1 . Subject to this condition, the surface charge densities are allowed to adjust to minimize $\left[\frac{W[\sigma]}{Q_1^2[\sigma]} \right]$. For two conductors, this means $\Phi_2 = 0$ (with fixed Q_2) while Φ_1 varies as $\sigma_1(\mathbf{r}_{S_1})$ is adjusted to minimize $\frac{1}{C[\sigma]}$. Problem 1.19 provides an example of how the actual C is always less than the approximated $C[\Psi]$, and is only equal to $C[\Psi]$ when the correct potential function is used. The minimization of $\frac{1}{C[\sigma]}$ occurs when the charge distributed on S_1 puts all the other conductors at constant potential (0)

2.10 Linear charge density Since a line is the intersection of two surfaces we could extend the approach used for

surface charge densities to the case of linear charge densities. In this case $\sigma(\mathbf{r})$ would involve a delta function which would restrict the charge to also lie on a second surface, say $G(\mathbf{r}) = 0$. The result would be that

$$\sigma(\mathbf{r}) = \lambda(\mathbf{r}) \delta(G(\mathbf{r})) |\nabla G(\mathbf{r})|$$

with $\lambda(\mathbf{r})$ the linear charge density. However a linear charge density is often given along a parametrized curve, $\mathbf{r}(s) = (x(s), y(s), z(s))$ being the points along the curve. With this information we know that the charge at the point \mathbf{r}_0 (parameter s_0) on the curve is

$$dq = \lambda(\mathbf{r}_0) \left| \frac{d\mathbf{r}(s)}{ds} \right|_{\mathbf{r}_0} ds = \lambda(\mathbf{r}_0) \sqrt{x'(s_0)^2 + y'(s_0)^2 + z'(s_0)^2} ds$$

and the function which gives the variation in the electrostatic potential due to this linear charge density is

$$\phi(\mathbf{R}) = \int \frac{\lambda(\mathbf{r}(s))}{|\mathbf{R} - \mathbf{r}(s)|} \sqrt{x'(s)^2 + y'(s)^2 + z'(s)^2} ds$$

Appendix A

Special problems

1. A hyperboloid of revolution has a uniform surface charge density of 7 statcoul/cm^2 . The charged surface lies between $z = -2 \text{ cm}$ and $z = +2 \text{ cm}$ and is given by the equation

$$\frac{x^2 + y^2}{2} - z^2 = 1 \text{ cm}^2.$$

- (a) Determine the function which gives the volume charge density corresponding to this surface charge. (b) Taking the potential at the origin to be zero, evaluate and plot the potential along the positive x axis from $x = 0 \text{ cm}$ to $x = 4 \text{ cm}$.² (c) Evaluate and plot the components of the electric field along the positive x axis from $x = 0 \text{ cm}$ to $x = 4 \text{ cm}$.
2. A line charge lies along the parabola $z = x^2/(3 \text{ cm})$, $y = 0 \text{ cm}$ in the region $0 \text{ cm} \leq z \leq 3 \text{ cm}$. The linear charge density is given by $\lambda(z) = (5 \text{ statcoul/cm}^2) z$. Along the line $z = 3 \text{ cm}$, $x = 0 \text{ cm}$ for $0 \text{ cm} \leq y \leq 3 \text{ cm}$ evaluate and plot the (a) potential and (b) the components of the electric field.
3. In Example ?? we found that the energy stored by the electric field of a charged spherical shell is

$$W_0 = \frac{q^2}{2\epsilon a}$$

with q the total charge of the shell and a the radius of the shell. Since

$$\frac{\delta W_0}{\delta a} = -\frac{q^2}{2\epsilon a^2}$$

there is a force acting at each point on the shell which would cause the shell to expand. (a) Determine and explain the source of this force. (b) Demonstrate that the source is correct by calculating the force per unit area of the sphere and the total work done by the force if the radius of the sphere expands by δa .

4. A spherical capacitor consists of two concentric conducting spherical shells. The inner shell has a radius a and the outer a radius $2a$. Using the ‘test potential’

$$\phi(\beta, r) = \frac{\sin \left[\frac{2\pi\beta}{a} (r - a) \right]}{\sin(2\pi\beta)}, \quad a < r < b$$

$$\phi(\beta, r) = 0, \quad r < a; \quad \phi(\beta, r) = 1, \quad 2a < r$$

- (a) determine the value of β which gives the minimum value for the energy stored in the electric field. (Show work) (b) Compare the energy obtained by the minimization technique to the actual energy this system would have for the given parameters. (c) Estimate the capacitance of the system and (d) compare this estimated value to the actual value for the capacitance of this spherical capacitor. (e) In the range $a < r < 2a$ plot the difference between the actual potential function between the spheres and the potential function obtained with $\phi(\beta, r)$. (f) In the range $a < r < 2a$ plot the difference between the actual radial electric field between the spheres and the radial field obtained with $\phi(\beta, r)$.
5. A sphere of radius 1 cm sets 2 cm above the center of a large grounded conducting plane. As a model assume that the plane is infinite in extent. Choosing a coordinate system with the sphere centered at the origin and the plane at $z = -2 \text{ cm}$ use the function

$$\Phi(p; r, \theta, \phi) = \left(\frac{1 \text{ cm}}{r}\right)^p \frac{r \cos \theta + 2 \text{ cm}}{1 \text{ cm} \cos \theta + 2 \text{ cm}}$$

to obtain an upper bound on the capacitance of the sphere.

² Note: At best the potential can be expressed in terms of tabulated integrals. Because those with experience with these tabulated integrals can often describe the general properties of the potentials the ‘closed’ form for the integral is often obtained. This capability is associated with having solved similar problems many times and checking the plots in a handbook of functions. **In this course I will assume that the function is known if it is written as a convergent integral with a finite integrand.**