Consequently the coefficients in (4.1) are:

\[ q_{lm} = \int Y_{lm}^m(\theta', \phi') r' \rho(x') \, d^3 x' \]  \hspace{1cm} (4.3)

These coefficients are called multipole moments. To see the physical interpretation of them we exhibit the first few explicitly in terms of Cartesian coordinates:

\[ q_{00} = \frac{1}{\sqrt{4\pi}} \int \rho(x') \, d^3 x' = \frac{1}{\sqrt{4\pi}} q \]  \hspace{1cm} (4.4)

\[ q_{11} = -\frac{3}{\sqrt{8\pi}} \int (x' - iy') \rho(x') \, d^3 x' = -\frac{3}{\sqrt{8\pi}} (p_x - ip_y) \]  \hspace{1cm} (4.5)

\[ q_{10} = \frac{3}{\sqrt{4\pi}} \int z' \rho(x') \, d^3 x' = \frac{3}{\sqrt{4\pi}} p_z \]

\[ q_{22} = \frac{15}{4 \sqrt{2\pi}} \int (x' - iy')^2 \rho(x') \, d^3 x' = \frac{15}{12 \sqrt{2\pi}} (Q_{11} - 2tQ_{12} - Q_{22}) \]  \hspace{1cm} (4.6)

\[ q_{21} = -\frac{15}{8\pi} \int z'(x' - iy') \rho(x') \, d^3 x' = -\frac{15}{8\pi} (Q_{13} - iQ_{23}) \]

\[ q_{20} = \frac{5}{\sqrt{4\pi}} \int (3z' r^2 - r'^2) \rho(x') \, d^3 x' = \frac{5}{2 \sqrt{4\pi}} Q_{23} \]

Only the moments with \( m \geq 0 \) have been given, since (3.54) shows that for a real charge density the moments with \( m < 0 \) are related through

\[ q_{l,-m} = (-1)^m q_{lm} \]  \hspace{1cm} (4.7)

In equations (4.4)–(4.6), \( q \) is the total charge, or monopole moment, \( p \) is the electric dipole moment:

\[ p = \int x' \rho(x') \, d^3 x' \]  \hspace{1cm} (4.8)

and \( Q_{ij} \) is the traceless quadrupole moment tensor:

\[ Q_{ij} = \int (3x'i'x'_j - r'^2 \delta_{ij}) \rho(x') \, d^3 x' \]  \hspace{1cm} (4.9)

We see that the \( l \)th multipole coefficients [(2\( l \) + 1) in number] are linear combinations of the corresponding multipoles expressed in rectangular coordinates. The expansion of \( \Phi(x) \) in rectangular coordinates

\[ \Phi(x) = \frac{1}{4\pi \epsilon_0} \left[ \frac{q}{r} + \frac{p \cdot x}{r^3} + \frac{1}{2} \sum_{ij} Q_{ij} \frac{x_i x_j}{r^5} + \cdots \right] \]  \hspace{1cm} (4.10)

by direct Taylor series expansion of \( 1/|x - x'| \) will be left as an exercise for the reader. It becomes increasingly cumbersome to continue the expansion in (4.10) beyond the quadrupole terms.

The electric field components for a given multipole can be expressed most
The added delta function does not contribute to the field away from the site of the dipole. Its purpose is to yield the required volume integral (4.18), with the convention that the spherically symmetric (around \( \mathbf{x}_0 \)) volume integral of the first term is zero (from angular integration), the singularity at \( \mathbf{x} = \mathbf{x}_0 \) causing an otherwise ambiguous result. Equation (4.20) and its magnetic dipole counterpart (5.64), when handled carefully, can be employed as if the dipoles were idealized point dipoles, the delta function terms carrying the essential information about the actually finite distributions of charge and current.

### 4.2 Multipole Expansion of the Energy of a Charge Distribution in an External Field

If a localized charge distribution described by \( \rho(\mathbf{x}) \) is placed in an external potential \( \Phi(\mathbf{x}) \), the electrostatic energy of the system is:

\[
W = \int \rho(\mathbf{x}) \Phi(\mathbf{x}) \, d^3x
\]

If the potential \( \Phi \) is slowly varying over the region where \( \rho(\mathbf{x}) \) is nonnegligible, then it can be expanded in a Taylor series around a suitably chosen origin:

\[
\Phi(\mathbf{x}) = \Phi(0) + \mathbf{x} \cdot \nabla \Phi(0) + \frac{1}{2} \sum_i \sum_j x_i x_j \frac{\partial^2 \Phi}{\partial x_i \partial x_j}(0) + \cdots
\]

Utilizing the definition of the electric field \( \mathbf{E} = -\nabla \Phi \), the last two terms can be rewritten. Then (4.22) becomes:

\[
\Phi(\mathbf{x}) = \Phi(0) - \mathbf{x} \cdot \mathbf{E}(0) - \frac{1}{2} \sum_i \sum_j x_i x_j \frac{\partial E_j}{\partial x_i}(0) + \cdots
\]

Since \( \nabla \cdot \mathbf{E} = 0 \) for the external field, we can subtract \( \frac{1}{6} r^2 \nabla \cdot \mathbf{E}(0) \) from the last term to obtain finally the expansion:

\[
\Phi(\mathbf{x}) = \Phi(0) - \mathbf{x} \cdot \mathbf{E}(0) - \frac{1}{6} \sum_i \sum_j (3x_i x_j - r^2 \delta_{ij}) \frac{\partial E_j}{\partial x_i}(0) + \cdots
\]

When this is inserted into (4.21) and the definitions of total charge, dipole moment (4.8), and quadrupole moment (4.9) are employed, the energy takes the form:

\[
W = q \Phi(0) - p \cdot \mathbf{E}(0) - \frac{1}{6} \sum_i \sum_j Q_{ij} \frac{\partial E_j}{\partial x_i}(0) + \cdots
\]

This expansion shows the characteristic way in which the various multipoles interact with an external field—the charge with the potential, the dipole with the electric field, the quadrupole with the field gradient, and so on.

In nuclear physics the quadrupole interaction is of particular interest. Atomic nuclei can possess electric quadrupole moments, and their magnitudes and signs reflect the nature of the forces between neutrons and protons, as well as the