Chapter 2
Waves in Media

2.1 Introduction  The propagation of waves in a medium depends on the magnetic permeability and electric permittivity functions $\mu(\omega)$ and $\varepsilon(\omega)$ for the medium. We will generally deal with systems in which $\mu(\omega)$ can be approximated as having the value 1. A general form often used to approximate the electric permittivity is the Sellmeier equation

$$\varepsilon(\omega) = 1 + \sum_m \frac{A_m}{\omega_m^2 - i\gamma_m \omega - \omega^2}.$$  

(2.122)

This equation is based on the assumption that the system behaves as a set of resonances. This will be observed in the two systems we will examine in this chapter. First we will consider electromagnetic waves in ionic crystals. The discussion will be restricted to frequencies below that of the electronic interband transitions. In this frequency range the dielectric function is determined by the ionic displacements and the polarizabilities of the ions. The second system will be electrical conductors, either metals or plasmas. In these systems the frequency dependent conductivity determines the dielectric function.

The analysis of the models will be followed by a discussion of wave packets in dispersive materials. This discussion will use the permittivities obtained for the two models. The propagation of electromagnetic waves in a medium involves the index of refraction of the medium. From the deduced analytic properties of the complex index of refraction we will obtain the Kramer-Kronig relationship between the real and imaginary parts of the index of refraction.

We will close the chapter with a discussion of reflectance and transmittance of electromagnetic waves at the interface between media.

2.2 Dielectric constant for ionic crystals  Ionic crystals such as the alkali-halides are cubic crystals. Note that electromagnetic fields only interact with the ‘optic modes’ for the crystal. These are the modes in which the positive and negative ions move in opposite directions. The dielectric function of cubic crystals is independent of the direction of the electric field and can be characterized by three parameters, the static dielectric constant $\varepsilon_s$, the optical dielectric constant $\varepsilon_{opt}$, and a characteristic resonant frequency $\omega_T$. For reference typical resonant frequencies are in the range 0.02eV - 0.04eV, optical frequencies are in the range 2eV - 3eV, and the electronic interband transitions begin around 4 or 5eV. An empirical dielectric function for these crystals is

$$\varepsilon(\omega) = \varepsilon_{opt} + \omega_T^2 \frac{\varepsilon_{opt} - \varepsilon_s}{\omega^2 + i\gamma \omega - \omega_T^2}.$$  

(2.123)

We will find that $\omega_T$ can be identified as the transverse optical phonon frequency for the crystal. A damping factor has been included to indicate what a real dielectric function might look like. This is an approximation to the Sellmeier equation, given in Eq.2.122, in a frequency range near one resonant term and far from any other resonance. Figure 2. shows a typical dielectric function (KBr$^1$: $\varepsilon_s = 4.90$, $\varepsilon_{opt} = 2.34$, and $\omega_T = 2.26 \times 10^{13}$ rad/s, 14.3 meV) with no damping.

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1 Ashcroft and Mermin, Solid State Physics (Saunders College, 1976, page 553)
2.2.1 Longitudinal optical crystal vibrations

From the permittivity we can obtain the frequency for a longitudinal optical mode of crystal vibration. These modes are characterized by a longitudinal charge density wave (that is, a charge density disturbance which propagates along the direction of the wave). This wave would generate a longitudinal electric field wave when it generates a current density. The divergence of the current density gives the time rate of change of the charge density. If this is self-sustaining in the absence of externally applied fields it is a normal mode of oscillation of the system. The electric field set up by the microscopic charge density will satisfy \( \nabla \cdot \mathbf{E}(\mathbf{r}, \omega) = 4\pi \rho_m(\mathbf{r}, \omega) \). In the absence of an ‘external charge density’ we must have that \( \nabla \cdot \mathbf{D}(\mathbf{r}, \omega) = 0 \). Since \( \mathbf{D}(\mathbf{r}, \omega) = \varepsilon(\omega) \mathbf{E}(\mathbf{r}, \omega) \) and \( \nabla \cdot \mathbf{E}(\mathbf{r}, \omega) \neq 0 \) we must find that \( \varepsilon(\omega_L) = 0 \) where \( \omega_L \) is the frequency of the longitudinal dynamic mode. The solution is given by

\[
\omega_L = \sqrt{\frac{\varepsilon_s}{\varepsilon_{opt}} \omega_T}
\]  
(2.124)

(For reference\(^2\): KBr \( \omega_L = 3.14 \times 10^{13} \) rad/s and \( (\omega_L / \omega_T)^2 = 1.9 \)) This is known as the Lyddane-Sachs-Teller relation. In the first approximation the longitudinal optical modes of vibration of a crystal have a constant frequency independent of the wavelength of the charge density wave. The location of this frequency is noted in Fig. 2.

2.2.2 Transverse electromagnetic waves in the crystal

The transverse optical modes do not generate a charge density. They do involve current densities but the electromagnetic fields generated by these currents do not contribute an appreciable amount to the forces involved in the crystal vibrations. However an electromagnetic wave propagating through the material can couple to the transverse optical mode of vibration.

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These modes have approximately constant frequency and wave vectors, $k = \frac{2\pi}{\lambda_{TO}}$, ranging from zero to of order $1 \text{ rad/nm}$. That is, $\lambda_{TO} \geq 2\pi \text{ nm}$. Examples of the dispersion curves for longitudinal and transverse optical and acoustic phonon modes are shown in Fig. 3. Compare the range of phonon wave vectors to a typical wave vector for an $\omega = 10^{13} \text{ rad/s}$ electromagnetic wave: $k = \frac{\omega}{c} = \frac{10^{13}\text{ rad/s}}{3 \times 10^8 \cdot 10^9 \text{ nm/s}} \approx 3 \times 10^{-5} \text{ rad/nm}$. In the $\omega$ versus $k$ curves shown in Fig. 3, the $\omega$ versus $k$ of an electromagnetic wave would lie on the right hand vertical axis. To search for transverse modes we consider the propagation of a transverse electromagnetic wave in the crystal. In the following we assume for convenience that $\mu(\omega) = 1$. Since we seek transverse modes we only consider electric fields satisfying $\nabla \cdot \mathbf{E}(\mathbf{r}, \omega) = 0$.

\[ \nabla \times (\nabla \times \mathbf{E}(\mathbf{r}, \omega)) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \]
\[ -\nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial}{\partial t}(\nabla \times \mathbf{B}) \]
\[ -\nabla^2 \mathbf{E} = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{D} + \frac{4\pi}{c} \frac{\partial}{\partial t} \mathbf{J}_o \]

In this case the wave equation for the electric field, (with no current sources) is given by

\[ -\nabla^2 \mathbf{E}(\mathbf{r}, \omega) = -\varepsilon(\omega) \frac{[-i\omega]^2}{c^2} \mathbf{E}(\mathbf{r}, \omega) \]
\[ = \varepsilon(\omega) \frac{\omega^2}{c^2} \mathbf{E}(\mathbf{r}, \omega) \]

or, in wave vector space,

\[ k^2 \mathbf{E}(\mathbf{k}, \omega) = \varepsilon(\omega) \frac{\omega^2}{c^2} \mathbf{E}(\mathbf{k}, \omega) \]

(Figure 3.)
so that

\[(k^2 - \varepsilon(\omega) \frac{\omega^2}{c^2})E(k, \omega) = 0\]

Thus, for a transverse electromagnetic wave to propagate in the medium the wave vector and angular frequency must satisfy the dispersion relation

\[k^2 = \varepsilon(\omega) \frac{\omega^2}{c^2}\]  \hspace{1cm} (2.127)

or, using the dielectric function without damping,

\[\varepsilon(\omega) = \varepsilon_{\text{opt}} + \omega^2 \frac{\varepsilon_{\text{opt}} - \varepsilon_s}{\omega^2 - \omega_0^2} = \frac{\varepsilon_{\text{opt}} \omega^2 - \omega_0^2 \varepsilon_s}{\omega^2 - \omega_0^2}\]

or, letting \(x = \omega^2/\omega_0^2\),

\[\varepsilon_{\text{opt}} x^2 - (\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2)x + \left(\frac{kc}{\omega_0}\right)^2 = 0\]

\[\omega^2 = \frac{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2}{2\varepsilon_{\text{opt}}} \pm \left\{ \left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2 - 4\varepsilon_{\text{opt}} \left(\frac{kc}{\omega_0}\right)^2 \right\}^{1/2}\]  \hspace{1cm} (2.128)

\[\omega^2 = \frac{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2}{2\varepsilon_{\text{opt}}} \left[ 1 \pm \left\{ 1 - \frac{4\varepsilon_{\text{opt}} \left(\frac{kc}{\omega_0}\right)^2}{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2} \right\}^{1/2} \right]\]

\[\approx \frac{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2}{2\varepsilon_{\text{opt}}} \left[ 1 \pm \frac{1}{2} \frac{4\varepsilon_{\text{opt}} \left(\frac{kc}{\omega_0}\right)^2}{\varepsilon_s^2} \right]\]  \hspace{1cm} \text{when} \ \frac{kc}{\omega_0} \rightarrow 0

\[\approx \frac{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2}{2\varepsilon_{\text{opt}}} \left[ 1 \pm \frac{1}{2} \frac{4\varepsilon_{\text{opt}} \left(\frac{kc}{\omega_0}\right)^2}{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2} \right]\]  \hspace{1cm} \text{when} \ \frac{kc}{\omega_0} \rightarrow \infty

As \(\frac{kc}{\omega_0} \rightarrow 0\)

\[\frac{\omega^2}{\omega_0^2_{\text{upper branch}}} \approx \frac{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2}{2\varepsilon_{\text{opt}}} \frac{\varepsilon_s}{\varepsilon_{\text{opt}}} \rightarrow 0\]

\[\frac{\omega^2}{\omega_0^2_{\text{lower branch}}} \approx \frac{\left(\varepsilon_s + \left(\frac{kc}{\omega_0}\right)^2\right)^2}{2\varepsilon_{\text{opt}}} \frac{4\varepsilon_{\text{opt}} \left(\frac{kc}{\omega_0}\right)^2}{\varepsilon_s^2} \rightarrow 0\]

As \(\frac{kc}{\omega_0} \rightarrow \infty\)
These two dispersion relations for the electromagnetic waves are shown in Fig. 4 with the minus sign giving the lower branch and the plus sign the upper. We note that an electromagnetic wave in the frequency range \( \omega_T < \omega < \omega_L \) (or \( 1 < \frac{\omega}{\omega_T} < \sqrt{\frac{\varepsilon_s}{\varepsilon_{opt}}} = 1.39 \)) can not propagate in the crystal, where \( \omega_L = \sqrt{\frac{\varepsilon_s}{\varepsilon_{opt}}} \omega_T \). The transverse optical modes of the crystal will lie in this range of frequencies.

The lower branch does not appear to be photon-like (maybe phonon-like?). In contrast the dispersion relation for the upper branch approaches \( \omega = \frac{k c}{\sqrt{\varepsilon_{opt}}} \) for large \( k \), exhibiting the standard photon-like behavior in the range of optical frequencies. This behavior of the dispersion relation for the electromagnetic wave implies that \( \omega_T \) is a frequency for a transverse vibrational mode for the crystal. A wave entering the crystal with a frequency near the resonant frequency of the transverse optical mode of the crystal would be strongly coupled to the crystal mode. The result would be a propagating wave with a mixture of the characteristics of the mechanical and electromagnetic waves. The gap in the range of frequencies of propagating waves is due to this coupling. The transverse optical vibration at \( \omega_T \) depresses the frequency of the electromagnetic wave. The long wavelength electromagnetic wave oscillating at the frequency of the longitudinal optical
vibration would provide the electric field that is generated in the longitudinal mode. The \( \omega_L \) therefore will provide the other end of the frequency band for the 'strongly coupled' mechanical-electromagnetic mode.

Finally, we note that the frequency difference between the longitudinal and transverse optical modes, \( \omega_L - \omega_T \), can be attributed to the electric field which is generated by the longitudinal mode. Even in the long wavelength region this provides an extra restoring force for the displaced ions thereby making the longitudinal mode frequency greater than that of the transverse mode.

### 2.2.3 Index of refraction for an ionic crystal

Generally, when dealing with the propagation of an electromagnetic wave in a medium, the frequency of the wave is determined by the boundary conditions and sources.

![Image of Index of Refraction](image)

The relation between the wave vector and the angular frequency is given by

\[
k = k_r + ik_i \text{ with }
\]

\[
k_r (\omega) &= n(\omega) \frac{\omega}{c} = \text{Re } k \tag{2.129}
\]

\[
k_i (\omega) &= \kappa(\omega) \frac{\omega}{c} = \text{Im } k \tag{2.130}
\]

From Eq. 2.127 we find

\[
k^2 &= |\text{Re } \varepsilon (\omega) + i \text{Im } \varepsilon (\omega)| \frac{\omega^2}{c^2} \tag{2.131}
\]

\[
= [n^2 (\omega) + \kappa^2 (\omega)] \frac{\omega^2}{c^2}
\]

\[
= n^2 (\omega) - \kappa^2 (\omega) + 2in (\omega) \kappa (\omega) \frac{\omega^2}{c^2}
\]

\[
n^2 (\omega) - \kappa^2 (\omega) = \text{Re } \varepsilon (\omega) \tag{2.132}
\]

\[
2n (\omega) \kappa (\omega) = \text{Im } \varepsilon (\omega)
\]
and also

\[ n^2(\omega) + \kappa^2(\omega) = |\varepsilon(\omega)| \]

\[
n^2(\omega) = \frac{1}{2} [\text{Re}\, \varepsilon(\omega)] + |\varepsilon(\omega)|, \ n(\omega) > 0 \tag{2.133}
\]

\[
\kappa^2(\omega) = \frac{1}{2} [-\text{Re}\, \varepsilon(\omega)] + |\varepsilon(\omega)|, \ sgn[\kappa(\omega)] = sgn[\text{Im}\, \varepsilon(\omega)] \tag{2.134}
\]

For an ionic crystal

\[
\varepsilon(\omega) = \varepsilon_{\text{opt}} + \omega_T^2 \frac{\varepsilon_{\text{opt}} - \varepsilon_s}{\omega^2 + i\gamma \omega - \omega_T^2} \cdot \frac{\omega^2 - i\gamma \omega - \omega_T^2}{\omega^2 - i\gamma \omega - \omega_T^2}
\]

\[
= \varepsilon_{\text{opt}} + \omega_T^2 \frac{\varepsilon_{\text{opt}} - \varepsilon_s}{\omega^2 - \omega_T^2} \cdot \left[ \omega^2 - \omega_T^2 - i\gamma \omega \right] \tag{2.135}
\]

\[
\text{Re}\, \varepsilon(\omega) = \varepsilon_{\text{opt}} + \omega_T^2 \frac{\varepsilon_{\text{opt}} - \varepsilon_s}{\omega^2 - \omega_T^2} \cdot \left[ \omega^2 - \omega_T^2 + \gamma^2 \omega^2 \right] \tag{2.136}
\]

\[
\text{Im}\, \varepsilon(\omega) = -\omega_T^4 \frac{(\varepsilon_{\text{opt}} - \varepsilon_s) \gamma \omega}{(\omega^2 - \omega_T^2)^2 + \gamma^2 \omega^2} = -\omega_T^4 \frac{\varepsilon_{\text{opt}} (\omega_T^2 - \omega_T^2) \gamma \omega}{(\omega^2 - \omega_T^2)^2 + \gamma^2 \omega^2} \tag{2.137}
\]

The real and imaginary parts of the index of refraction for KBr are shown in Fig. 5.

The electric field for a plane electromagnetic wave travelling in the +z direction is given by

\[
E(z, t) = \int_{-\infty}^{\infty} A(\omega) \exp \left[ i k z - \omega t \right] d\omega
\]

\[
= \int_{-\infty}^{\infty} A(\omega) \exp \left[ i \frac{\omega}{c} (n(\omega) + i \kappa(\omega)) z - \omega t \right] d\omega
\]

\[
E(z, t) = \int_{-\infty}^{\infty} A(\omega) \exp \left[ i \frac{\omega}{c} (n(\omega) z - c t) \right] \exp \left[ -\frac{\omega}{c} \kappa(\omega) z \right] d\omega, \ A_3(\omega) = 0 \tag{2.138}
\]

In general \( \varepsilon_s > \varepsilon_{\text{opt}} \) and, therefore, \( \omega \kappa(\omega) \geq 0 \) for all \( \omega \). It follows that Eq. 2.138 describes a damped plane wave. The properties of this wave will be analyzed in a later section.
Assignment 6a: Jackson Problem 7.19 -
An approximately monochromatic plane wave packet in one dimension has the instantaneous form, \( u(x,0) = f(x) \exp(i\omega x) \) with \( f(x) \) the modulation envelope.

\[
\begin{align*}
(1) f(x) &= N_1 \exp\left(-\frac{\alpha |x|}{2}\right) \\
(2) f(x) &= N_2 \exp\left(-\alpha^2 x^2/4\right) \\
(3) f(x) &= N_3 (1 - \alpha |x|) \Theta(1 - \alpha^2 x^2) \\
(4) f(x) &= N_4 \Theta(1 - \alpha^2 x^2)
\end{align*}
\]

(a) Calculate the wave number spectrum \( |A(k)|^2 \) for each of the above forms for \( f(x) \).
(b) Explicitly evaluate the rms deviations from the mean, \( \Delta x \) and \( \Delta k \).
(c) Using graphs compare the functions \( |f(x)|^2 \) and the functions \( |A(k)| \)
. Include two graphs one comparing the functions \( f(x) \) and the other comparing the functions \( |A(k)|^2 \).

Assignment 6b: Jackson Problem 7.20 - A homogeneous, isotropic, nonpermeable dielectric is characterized by a complex index of refraction \( n(\omega) \). (a) Show that the general solution for plane waves in one dimension can be written

\[
u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \left[ A(\omega) e^{i\omega n(\omega)x/c} + B(\omega) e^{-i\omega n(\omega)x/c} \right] d\omega
\]

where \( u(x,t) \) is a component of \( \mathbf{E} \) or \( \mathbf{B} \). (b) If \( u(x,t) \) is real show that \( n(-\omega) = n(\omega)^* \). (c) Show that, if \( u(0,t) \) and \( \left[ \partial u(x,t) / \partial x \right]_{x=0} \) are the boundary values of \( u(x,t) \) at \( x = 0 \), the coefficients \( A(\omega) \) and \( B(\omega) \) are

\[
\left( \begin{array}{c} A(\omega) \\ B(\omega) \end{array} \right) = \frac{1}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \left[ u(0,t) + \frac{ie}{\omega n(\omega)} \left( \partial u(x,t) / \partial x \right)_{x=0} \right] dt
\]

2.3 Electronic plasmas

A plasma is an electrically charged gas consisting of ions and electrons. It is similar to a liquid conductor, but is generally distinguished from a liquid conductor by density. A liquid conductor has a number density of the order of \( 10^{22} \text{cm}^{-3} \) whereas a plasma has a number density of the order of \( 10^{18} \text{cm}^{-3} \). A plasma can be thought of as a gaseous conductor. [ See S. Gartenhaus, *Elements of Plasma Physics*, Holt, Rinehart and Winston, New York 1964 for an introduction to plasmas.]

We consider here a system composed of ‘free electrons’ and a uniform background of positive charge to provide charge neutrality for the system. In most cases the radiation emitted (lost) by the accelerated ions in the plasma is negligible, as are quantum effects (which are relevant only at high densities and low temperatures.) In a gas of density \( 10^{18} \text{cm}^{-3} \) the average separation distance is \( \approx 10^{-18}1/3 = 10^{-6} \text{cm} = 100 \text{Å} \). The ‘macroscopic’ treatment of electric and magnetic fields in a plasma assumes that properties are averaged over distances \( \geq 100 \text{Å} \). The electrons in the gas can also undergo close collisions with the ions and with other electrons. The collisions between electrons thermalize the electron ‘gas’ while the collisions with ions will reduce the momentum of the electron gas.

Our analysis will use the linearized Boltzmann equation for an electron density in phase space, \( \rho(r,v,t) \) (with units \( \text{cm}^{-3} (\text{cm/sec})^{-3} \)). The equilibrium electron density will be velocity dependent but not spatially dependent, i.e., \( \rho_0(v) \) with \( \int \rho_0(v) d^3v = \rho_0 \) and \( \int v \rho_0(v) d^3v = 0 \). (The \( \rho_0 \) has units \( \text{cm}^{-3} \).) The linearized Boltzmann equation assumes that the total time derivative of \( \rho(r,v,t) \) taken along a particle trajectory is proportional to \( \rho(r,v,t) - \rho_0(v) \), the deviation of the electron density from equilibrium. The proportionality constant is \( -1/\tau \) and is due to ‘collisions’ not included in the dynamical equations for the system. In the model considered here the excluded interactions are close collisions between electrons and collisions with the ions.
As an example of a collision term we consider the electron-electron collisions. The collision cross-section is a function of the relative speed of the particles and the scattering angle. We note that the center of mass motion of the particles (conservation of momentum) and their relative speed (conservation of energy) are not changed by the collision. The collision process transfers particles between the velocity pairs \{v, v'\} and \{u, u'\} and can be classified by \(v - v'\) as the z axis, the angle \((\theta, \phi)\) for \(u - u'\).

\[
\left[ \frac{d \rho (r, v, t)}{dt} \right]_{\text{collision}} = \int \int \sigma (|v - v'|; \theta, \phi) |v - v'| \rho (r, u, t) \rho (r, u', t) \ d^3 v' \ d \Omega - \int \int \sigma (|v - v'|; \theta, \phi) |v - v'| \rho (r, v, t) \rho (r, v', t) \ d^3 v' \ d \Omega \tag{2.139}
\]

If \(\rho (r, v, t)\) is the Boltzmann distribution \(\rho_0 (r, v, t)\), the collision term is zero. In the case that

\[
\delta \rho (r, v, t) = \rho (r, v, t) - \rho_0 (r, v, t)
\]

is ‘small’ then the collision term can be approximated by a linear function of \(\delta \rho (r, v, t)\) (essentially using the principle of detailed balance) to obtain

\[
\left[ \frac{d \rho (r, v, t)}{dt} \right]_{\text{collision}} = \int \int \sigma (|v - v'|; \theta, \phi) |v - v'| [\delta \rho (r, u, t) \rho_0 (r, u', t) + \rho_0 (r, u, t) \delta \rho (r, u', t)] \ d^3 v' \ d \Omega - \int \int \sigma (|v - v'|; \theta, \phi) |v - v'| [\delta \rho (r, v, t) \rho_0 (r, v', t) + \rho_0 (r, v, t) \delta \rho (r, v', t)] \ d^3 v' \ d \Omega \tag{2.140}
\]

\[
\sim - \frac{\delta \rho (r, v, t)}{\tau} \tag{2.141}
\]

This will provide a reasonable, semi-empirical expression for the relaxation of the distribution to equilibrium. The Boltzmann equation itself is a hybrid combination of dynamics and probability.

The total time derivative along a trajectory will involve the time dependence of \(r\) and \(v\) on the trajectory.

\[
\frac{d}{dt} \rho (r, v, t) = \frac{\partial}{\partial t} \rho (r, v, t) + v \cdot \nabla_{\!r} \rho (r, v, t) + a \cdot \nabla_{\!v} \rho (r, v, t) = -\frac{\rho (r, v, t) - \rho_0 (v)}{\tau}. \tag{2.142}
\]

The acceleration of the electrons is due to the electric field generated by the spatial variation in the electron density and by any externally generated electric and magnetic fields. In particular the latter will be the transverse electric and magnetic fields of an electromagnetic wave. The acceleration of an electron located at \(r\) and moving with velocity \(v\) at time \(t\) is

\[
a = -\frac{e}{m} \left[ E (r, t) + \frac{v}{c} \times B (r, t) \right] \tag{2.143}
\]

The fields satisfy Maxwell’s equations:

\[
\nabla \cdot E (r, t) = \varepsilon_{\infty}^{-1} (-4\pi e) \int [\rho (r, v, t) - \rho_0 (v)] \ d^3 v, \tag{2.144}
\]

\[
\nabla \times E (r, t) = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B} (r, t), \tag{2.145}
\]

\[
\nabla \times B (r, t) = \varepsilon_{\infty} \frac{\partial}{\partial t} E (r, t) - \frac{4\pi e}{c} \int v \rho (r, v, t) \ d^3 v, \tag{2.146}
\]

\[
\nabla \cdot B (r, t) = 0. \tag{2.147}
\]

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3 Kerson Huang, Statistic Mechanics (Problem of Kinetic Theory, Chapt. 3, Wiley, 1963) See also Section 5.4

The quantum mechanical system is covered by Pines, The Many Body Problem (Benjamin, 1961) An interesting approach is provided by an article by Ehrenreich and Cohen (Phys, Rev, 115, 786-790 (1959)) which is reprinted on pp 255-259.
We have assumed that the background charges have a constant permittivity \( \varepsilon_i \).

This set of equations is non-linear in the electron density function. We will restrict ourselves to systems in which the electron charge density only deviates by a small amount from its equilibrium value. This permits us to linearize the equations by defining

\[
\delta \rho (r, v, t) = \rho (r, v, t) - \rho_0 (v)
\]  

and keeping only those terms involving the \( \delta \rho \) to first order or lower. In this approximation the transport equation becomes

\[
\frac{\partial}{\partial t} \delta \rho (r, v, t) + v \cdot \nabla_r \delta \rho (r, v, t) + a \cdot \nabla_v \rho_0 (v) = -\frac{\delta \rho (r, v, t)}{\tau}
\]  

The modified Maxwell’s equations are

\[
\nabla \cdot E (r, t) = 4\pi \varepsilon_i^{-1} (\varepsilon_e - \varepsilon_i) \int \delta \rho (r, v, t) \, dv,
\]

\[
\nabla \times B (r, t) = \varepsilon_i \frac{\partial}{\partial t} E (r, t) - \frac{4\pi e}{c} \int v \delta \rho (r, v, t) \, dv.
\]

We will assume that an electric field exists in the plasma and that this electric field is the real part of

\[
E (r, t) = E_0 \exp [i (k \cdot r - \omega t)]
\]

Neglecting the magnetic force in our transport equation we find the \( \delta \rho \) must satisfy

\[
\frac{\partial}{\partial t} \delta \rho (r, v, t) + v \cdot \nabla_r \delta \rho (r, v, t) - \frac{e}{m} E_0 \exp [i (k \cdot r - \omega t)] \cdot \nabla_v \rho_0 (v) = -\frac{\delta \rho (r, v, t)}{\tau}
\]

We look for a electron density variation of the form

\[
\delta \rho (r, v, t) = \Delta (v) \exp [i (k \cdot r - \omega t)]
\]

Using Eq. 2.153 we find that

\[
\Delta (v) = \frac{e\tau}{m} \left[ -i\omega \tau + iv \cdot k \tau + 1 \right]^{-1} E_0 \cdot \nabla_v \rho_0 (v)
\]

The spatial electron density is obtained by integrating \( \delta \rho (r, v, t) \) over the velocity

\[
n (r, t) = \exp (ik \cdot r - i\omega t) \int \int \frac{e\tau}{m} \frac{E_0 \cdot \nabla_v \rho_0 (v)}{v^2} \rho_0 (v) \, dv \, d^3 v
\]

In many cases \( \rho_0 (v) \) is only a function of the square of the electron’s velocity. For simplicity we will assume this property for \( \rho_0 (v) \). Then

\[
\nabla_v \rho_0 (v) = \frac{v}{\rho_0 (v)} \frac{\partial}{\partial v} \rho_0 (v)
\]

and \( \frac{\partial}{\partial v} \rho_0 (v) \) has no angular dependence. We can also let the \( \hat{z} \) for the \( v \) integration be along \( \hat{k} \), so that \( k \cdot v = kv \cos \theta \) and all the angular dependence in the integrand arises from \( \hat{v} \) and \( k \cdot \hat{v} \). We can also write

\[
n (r, t) = i \exp (ik \cdot r - i\omega t) \int \int \int \frac{e\tau}{mv} \left[ \frac{(E_0 \cdot \hat{k})\hat{k} + (E_0 \cdot \hat{x})\hat{x} + (E_0 \cdot \hat{y})\hat{y}}{\omega \tau - v \cdot k \tau + i} \right] \cdot \nabla_v \rho_0 (v) \, dv \, d^3 v
\]

\[
= \frac{1}{mv} \frac{e\tau}{\omega \tau - v \cdot k \tau + i} \int \int \int \left[ \frac{(E_0 \cdot \hat{k})\sin \theta \cos \phi + (E_0 \cdot \hat{y})\sin \theta \sin \phi}{\omega \tau - vk \cos \theta \tau + i} \right] \rho_0 (v) \, dv \, d^3 v
\]

\[
= \frac{1}{mv} \frac{e\tau}{\omega \tau - v \cdot k \tau + i} \int \int \int \left[ \frac{(E_0 \cdot \hat{k})\sin \theta \cos \phi + (E_0 \cdot \hat{y})\sin \theta \sin \phi}{\omega \tau - vk \cos \theta \tau + i} \right] \rho_0 (v) \, dv \, d^3 v
\]

\[
= \frac{1}{mv} \frac{e\tau}{\omega \tau - v \cdot k \tau + i} \int \int \int \left[ \frac{(E_0 \cdot \hat{k})\sin \theta \cos \phi + (E_0 \cdot \hat{y})\sin \theta \sin \phi}{\omega \tau - vk \cos \theta \tau + i} \right] \rho_0 (v) \, dv \, d^3 v
\]

\[
= \frac{1}{mv} \frac{e\tau}{\omega \tau - v \cdot k \tau + i} \int \int \int \left[ \frac{(E_0 \cdot \hat{k})\sin \theta \cos \phi + (E_0 \cdot \hat{y})\sin \theta \sin \phi}{\omega \tau - vk \cos \theta \tau + i} \right] \rho_0 (v) \, dv \, d^3 v
\]
The last term on the right is zero since both integrations over \( \phi \) vanish. We are then left with

\[
n(r, t) = i \frac{E_0 \cdot k}{k^2} \exp(i k \cdot r - i \omega t) \iint e^{\frac{e r}{m v \omega + i \tau - 1}} \frac{k \cdot v}{\omega - v \cdot k + i} \frac{d \rho_0(v)}{dv} d^3 v
\]  

(2.159)

It suffices for our analysis to carry out an expansion of the integrand in powers of \( (v \cdot k) / (\omega + i \tau^{-1}) \)

\[
n(r, t) = i \frac{E_0 \cdot k}{k^2} e^{i(k \cdot r - \omega t)} \iint \frac{e}{m v \omega + i \tau - 1} \frac{k \cdot v}{\omega + i \tau - 1} \left[ 1 + \frac{k \cdot v}{\omega + i \tau - 1} \right] \frac{d \rho_0(v)}{dv} d^3 v
\]  

(2.160)

The leading term in this expansion can be written as

\[
t E_0 \cdot ke^{i(k \cdot r - \omega t)} \frac{4 \pi e}{3m(\omega + \tau - 1)^2} \int_0^\infty \frac{d \rho_0(v)}{dv} v^3 dv
\]  

(2.161)

Using this first term as an approximation for \( n(r, t) \), one obtains

\[
n(r, t) = -i \frac{e \rho_0 E_0 \cdot k}{m(\omega + i \tau - 1)^2} e^{i(k \cdot r - \omega t)}
\]  

(2.162)

independent of the form taken for the equilibrium electron density. We should note that \( n(r, t) = 0 \) if \( E_0 \cdot k = 0 \).
Electron current density in plasma

In a similar way we can obtain the electron current density.

\[ \mathbf{j}_e(r, t) = \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) \int \frac{e^T}{m} \frac{\mathbf{E}_0 \cdot \mathbf{v}}{\mathbf{r} + i\omega \tau + i\mathbf{k} \cdot \mathbf{r} + 1} d^3 v \]  

We will restrict the analysis to the case in which \( \mathbf{E}_0 \) is perpendicular to \( \mathbf{k} \). To further simplify we will only seek the lowest order, non-zero term in \( (\mathbf{v} \cdot \mathbf{k} / [\omega + i\tau^{-1}] \). With these simplifications and letting \( \mathbf{E}_0 \) be along \( \hat{z} \) and

\[ \mathbf{j}_e(r, t) = i \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t) \int \int \int \frac{e^T}{mv} \mathbf{v} \cos \theta \left[ \sum_{n=0,\infty} \frac{\mathbf{k} \cdot \mathbf{v}}{\omega + i\tau^{-1}} \right]^n \frac{d\rho_0(v)}{dv} d^3 \nu' \]  

Consider the following integrations. Note that \( \mathbf{k} \cdot \mathbf{r} = 0 \) so that \( \mathbf{k} \cdot \mathbf{v} = k_x v_x + k_y v_y \).

\[ \int_0^{2\pi} \int_0^{\pi} \cos^m \theta \sin^j \theta d\theta d\phi = \int_0^{2\pi} \int_0^{\pi} \cos^n \theta \sin^{i+j} \theta \cos^i \phi \sin^j \phi \sin \theta d\theta d\phi = v^2 \int_0^{2\pi} \int_0^{\pi} \cos^n \theta \sin^{i+j} \theta \cos^i \phi \sin^j \phi \sin \theta d\theta d\phi \]  

\[ = v^2 \int_0^{2\pi} \cos^n \theta \sin^{i+j} \theta d(\cos \theta) \int_0^{2\pi} \cos^i \phi \sin^j \phi d\phi \]  

\[ = v^2 \int_{-1}^{1} u^n [1 - u^2]^{-1/2} du \int_0^{2\pi} \cos^i \phi \sin^j \phi d\phi \]  

\[ = 0 \] if \( n' \) is odd (This applies to all terms along \( \hat{x} \) and \( \hat{y} \))

\[ = v^2 \int_{-1}^{1} u^n [1 - u^2]^{-1/2} du \int_0^{2\pi} \cos^i \phi \sin^j \phi d\phi \]  

\[ = 0 \] if \( n' \) is odd (This applies to all terms along \( \hat{z} \))

So all the terms with along \( \hat{x} \) and \( \hat{y} \) vanish. The terms along \( \hat{z} \) vanish if \( i = j = 1 \) because

\[ \int_0^{2\pi} \cos \phi \sin \phi d\phi = 0 \]  

Some of the terms for \( i = j = 2 \) do not vanish. A non zero term comes from the third term in the expansion.
Thus, keeping the two lowest order terms,

\[ j_e(r, t) \approx \frac{i(E_0 \cdot \hat{z})e^{i k r - i \omega t}}{2\pi} \int_{0}^{\infty} \frac{e}{m} \frac{v}{\omega + i \tau^{-1}} \frac{d\rho_0(v)}{dv} v^2 dv \cdot \int \cos^2 \theta d(\cos \theta) + \]

\[ i(E_0 \cdot \hat{z})e^{i k r - i \omega t} \int_{0}^{\infty} \frac{e}{m} \frac{v^2}{\omega + i \tau^{-1}} \left( \frac{k_x v_x + k_y v_y}{\omega + i \tau^{-1}} \right)^2 \frac{d\rho_0(v)}{dv} v^2 dv d\phi d(\cos \theta) \]

\[ = e^{i E_0 e^{i k r - i \omega t}} \int_{0}^{\infty} \frac{e}{m} \frac{v^3}{(\omega + i \tau^{-1})^3} \frac{d\rho_0(v)}{dv} dv \int_{0}^{\pi} \int_{0}^{2\pi} \cos^2 \theta |k v \sin \theta|^2 d\phi d(\cos \theta) \]

\[ = -i E_0 \exp(i k \cdot r - i \omega t) \int_{0}^{\infty} \frac{e}{m} \frac{k^2 v^5}{2m(\omega + i \tau^{-1})^3} \frac{d\rho_0(v)}{dv} dv \int_{0}^{\pi} \cos^2 \theta |1 - \cos^2 \theta| d(\cos \theta) \]

\[ = -i E_0 \exp(i k \cdot r - i \omega t) \int_{0}^{\infty} \frac{e}{m} \frac{k^2 v^5}{2m(\omega + i \tau^{-1})^3} \frac{d\rho_0(v)}{dv} dv \frac{2}{3} \]

\[ = -i E_0 \exp(i k \cdot r - i \omega t) \int_{0}^{\infty} \frac{e}{m} \frac{k^2 v^5}{2m(\omega + i \tau^{-1})^3} \frac{d\rho_0(v)}{dv} dv \frac{4}{15} \]

\[ = -i E_0 \exp(i k \cdot r - i \omega t) \frac{e}{m} \frac{1}{\omega + i \tau^{-1}} \frac{\rho_0}{d\phi} \frac{1}{\omega + i \tau^{-1}} \frac{\rho_0}{d\phi} \]

\[ \approx -i \frac{e}{m} E_0 \exp(i k \cdot r - i \omega t) \frac{\rho_0}{\omega + i \tau^{-1}} \] keeping only the first term

If \( \tau \) is very large (the relaxation to equilibrium takes a long time compared to the period of the field) then the electron current density will lead the electric field in time by \( 90^\circ \). That is, using \( -i = e^{-i \pi/2} \)

\[ j_e(r, t) \approx e^{-i \pi/2} \frac{e}{m} E_0 \exp(i k \cdot r - i \omega t) \frac{\rho_0}{\omega + i \tau^{-1}} \]

\[ = e \frac{E_0}{m} \frac{\rho_0}{\omega + i \tau^{-1}} e^{-i \pi/2 + i k \cdot r - i \omega t} \]

\[ = e \frac{E_0}{m} \frac{\rho_0}{\omega} \left[ 1 - \frac{i}{2 \pi} \tau_{field} e^{-i \pi/2 + i k \cdot r - i \omega t} \right] \]

\[ \approx e \frac{E_0}{m} \frac{\rho_0}{\omega} e^{i k \cdot r - \omega t - i \pi/2} = e \frac{E_0}{m} \frac{\rho_0}{\omega} e^{-i \pi/2} \]

This means that the electric field has the same phase as the current density at a later time \( t' = t + \frac{\pi}{2 \omega} \). Physically the current density lags the force by \( 90^\circ \) as expected.

\[ F = -eE_0 e^{i \pi} \]

\[ j_e(r, t) = -F \frac{\rho_0}{m \omega} e^{-i \pi/2} \]

\[ = F \frac{\rho_0}{m \omega} e^{i \pi/2} \]
That is, the force has the same phase as the current density at an earlier time, \( t' = t - \frac{\pi}{2\omega} \). Finally, in the limit that \( k \) and \( \omega \) go to zero we obtain the equation for the dc current

\[
j_c = -eJ_c = \frac{e^2 \rho_0 \tau}{m} E_0
\]  

(2.173)

This gives the standard equation for electrical conductivity

\[
\sigma_0 = \frac{e^2 \rho_0 \tau}{m}.
\]  

(2.174)

Assignment 7: For a metal the electron density would be approximated by the ‘fermi sea’,

\[
\rho_0(v) = 2 \left( \frac{m^*}{\hbar} \right)^3 \Theta(v_f - v), \quad v_f = (3\pi^2 \rho_0)^{1/3} \frac{\hbar}{m^*}
\]  

(2.175)

with \( m^* \) the effective mass of the electron. The equilibrium electron density for a charged plasma at temperature \( T \) is given by the Boltzmann distribution

\[
\rho_0(v) = \rho_0 \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{mv^2}{2k_B T} \right)
\]  

(2.176)

Using these equilibrium densities evaluate the correction terms for the electron density, \( n(r, t) \), and electron current density, \( j(r, t) \).
2.3.1 **Longitudinal plasma oscillation**  The electron density given by Eq. 2.162 will create a charge density which generates an electric field. We seek the condition for which the electric field generated by this charge density equals the electric field which creates the charge density variation. In this case the field and charge density are related by Gauss’s law ($\varepsilon_i$ is the permittivity of the background ions)

$$\nabla \cdot \mathbf{E}(r,t) = 4\pi \rho(r,t) / \varepsilon_i$$

$$i \mathbf{k} \cdot \mathbf{E}_0 e^{i(k \cdot \mathbf{r} - \omega t)} = -4\pi e n(r,t) / \varepsilon_i$$

$$= 4\pi i e^2 \rho_0 \mathbf{E}_0 \cdot \mathbf{k} / m\varepsilon_i (\omega + i\tau^{-1})^2 e^{i(k \cdot \mathbf{r} - \omega t)} \quad (2.177)$$

This equation can be satisfied for $\mathbf{E}_0 \cdot \mathbf{k} \neq 0$ only if

$$4\pi \frac{e^2 \rho_0}{m\varepsilon_i (\omega + i\tau^{-1})^2} = 1 \quad (2.178)$$

A plasma frequency is defined by

$$\omega_p^2 = \frac{4\pi e^2 \rho_0}{m\varepsilon_i} \quad (2.179)$$

and the undamped plasma will oscillate at this frequency. The plasma frequency is approximately $(6 \times 10^4 \text{rad} \cdot \text{cm}^{-3/2} / \text{s}) \cdot \rho_0^{1/2}$ and ranges from $10^9 \text{rad} / \text{s}$ for gaseous plasmas, through $10^{13} \text{rad} / \text{s}$ for doped semiconductors to $10^{16} \text{rad} / \text{s}$ for metals. For reference, the ‘observed’ relaxation time constant for currents in a metal is of order $10^{-14} \text{s}$ at room temperature. This would be the characteristic time between electron-phonon collisions.

2.3.2 **Transverse electromagnetic waves in a plasma**  A transverse electromagnetic wave would generate a current density in the plasma but not a charge density. The (transverse) electric field, $\mathbf{E}(r,t) = \mathbf{E}_0 \exp (i \mathbf{k} \cdot \mathbf{r} - i\omega t)$ must satisfy

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla \times \left[ -\frac{\partial}{\partial t} \mathbf{B} \right]$$

$$\nabla^2 \mathbf{E} + \nabla (\nabla \cdot \mathbf{E}) = -\frac{\partial}{\partial t} (\nabla \times \mathbf{B})$$

$$= -\frac{\partial}{\partial t} (\frac{\partial \mathbf{D}}{\partial t} - \frac{4\pi}{c} \mathbf{J}_o) \quad (2.181)$$

Note that for a transverse wave $\nabla \cdot \mathbf{E} = i\mathbf{k} \cdot \mathbf{E} = 0$ and (using the first order term for $\mathbf{J}_o = c \mathbf{j}_e$)

$$-\nabla^2 \mathbf{E} = \frac{-1}{c^2} \frac{\partial^2}{\partial t^2} \mathbf{D} + \frac{4\pi}{c} \frac{\partial}{\partial t} \left[ -i \frac{e^2}{m} \mathbf{E}_0 \exp (i \mathbf{k} \cdot \mathbf{r} - i\omega t) \frac{\rho_0}{\omega + i\tau^{-1}} \right]$$

$$k^2 \mathbf{E}_0 = \varepsilon \frac{\omega^2}{c^2} \mathbf{E}_0 - \frac{4\pi \omega e^2}{c^2} \frac{\rho_0}{\omega + i\tau^{-1}} \frac{\mathbf{E}_0}{m} \quad (2.183)$$

Hence, the dispersion relation (see Eq.127) emerges from:

$$\frac{(ek)^2}{\omega^2} \mathbf{E}_0 = (\varepsilon_i - \frac{4\pi e^2}{m} \frac{\rho_0}{\omega + i\tau^{-1}}) \mathbf{E}_0$$

$$= \varepsilon(\omega) \mathbf{E}_0 \quad (2.185)$$

$$= \varepsilon(\omega) \mathbf{E}_0 \quad (2.186)$$
We can identify the permittivity for the system as

\[ \varepsilon(\omega) = \varepsilon_i - \frac{4\pi e^2 \rho_0}{m \omega (\omega + i\tau^{-1})}. \]  

(2.187)

Using \( \omega_p^2 = \frac{4\pi e^2 \rho_0}{m \varepsilon_0} \) from Eq. 2.179

\[ \varepsilon(\omega) = \varepsilon_i \left[ 1 - \frac{\omega_p^2}{\omega (\omega + i\tau^{-1})} \right] = \varepsilon_i \left[ 1 - \frac{\omega_p^2 \tau^2 (\omega - i)}{\omega \tau (\omega^2 \tau^2 + 1)} \right] \]  

(2.188)

\[ = \varepsilon_i - \frac{4\pi \sigma_0}{\omega (\omega + i \tau)} = \varepsilon_i - \frac{4\pi \sigma_0 (\omega - i)}{\omega (\omega^2 \tau^2 + 1)} \]  

(2.189)

\[ = \varepsilon_i - \frac{4\pi \sigma_0 \tau}{(\omega^2 \tau^2 + 1)} + i \frac{\varepsilon_i \omega_p^2 \tau^2}{\omega \tau (\omega^2 \tau^2 + 1)} \]  

(2.190)

\[ = \varepsilon_i \left[ 1 - \frac{\omega_p^2 \tau^2}{(\omega^2 \tau^2 + 1)} \right] + i \frac{\varepsilon_i \omega_p^2 \tau^2}{\omega \tau (\omega^2 \tau^2 + 1)} \]  

(2.191)

showing the explicit relationships between the permittivity, the plasma frequency, and the conductivity. Figure 6 shows the real and imaginary parts of the permittivity for a metal with \( \omega_p \tau = 10 \). We note that the real part of the permittivity is negative below the plasma frequency and monotonically approaches permittivity of the background ions above the plasma frequency.

Equations 2.133 and 2.134 can be used to evaluate the real and imaginary parts of the index of refraction. The real and imaginary parts of the index of refraction are shown plotted versus \( \omega \tau \) in Fig. 7. In this plot the product, \( \omega_p \tau \), of the plasma frequency and the relaxation time, \( \tau \), has been taken to be 10 and the permittivity of the background ions has been taken to be 1.
We note that in the region near zero frequency the imaginary part of the permittivity is very large (approaching infinity at zero frequency). In this region the real and imaginary parts of the index of refraction are nearly equal. Indeed the real part of the index of refraction lies below the imaginary part for all frequencies below the plasma frequency. Note that the energy dissipated is proportional to the imaginary part of the permittivity (see Eq. 111 in Chapter 1). It will therefore not be meaningful to speak of a wave propagating in the medium at frequencies below the plasma frequency.

2.4 Propagation of plane waves in a medium

We now have two models for the index of refraction. Our next task is to investigate the propagation of a plane electromagnetic wave in an ionic crystal and in a metal. Reviewing the permittivities obtained for the models we note that the real part of the permittivity is an even function of $\omega$ and the imaginary part is an odd function of $\omega$. The properties of the real and imaginary parts of the indices of refraction are obtained from those of the permittivities and Eqs. 2.133 and 2.134. Like the permittivity we find that the real part of the index of refraction is an even function of $\omega$ and the imaginary part is an odd function of $\omega$. Using these properties a plane polarized wave travelling in the $+z$ direction can be described by the expression

$$E(z,t) = \int_{0}^{\infty} E_0(\omega) \exp(i[\text{Re}k(\omega) + i \text{Im}k(\omega)]z - \omega t) \, d\omega + c.c. = \int_{0}^{\infty} E_0(\omega) \exp(i\frac{\omega}{c}(n(\omega)z - ct)) \exp(-\frac{\omega}{c}\kappa(\omega)z) \, d\omega + c.c.$$  (2.192)

We will restrict our analysis to the case in which $|E_0(\omega)|$ has a single, ‘narrow’ peak of width $\Delta\omega$ at $\omega_0$. The peak will be considered narrow if

$$\frac{\Delta\omega}{n(\omega_0)} \frac{dn(\omega_0)}{d\omega_0} \ll 1.$$

2.4.1 Group velocity

For a wave composed of a narrow range of frequencies the wave vector $k(\omega)$ can be expanded about the central frequency to obtain an approximate expression for the propagation of the wave. In our example
the central frequency is $\omega_0$.

\[
\frac{\omega}{c} n(\omega) = \frac{\omega_0}{c} n(\omega_0) + \left[ n(\omega_0) + \omega_0 \frac{dn(\omega_0)}{d\omega_0} \right] \frac{\omega - \omega_0}{c} + \frac{1}{2} \left[ 2 \frac{dn(\omega_0)}{d\omega_0} + \omega_0 \frac{d^2n(\omega_0)}{d\omega_0^2} \right] \frac{(\omega - \omega_0)^2}{c} + \ldots
\] (2.193)

A similar expression holds for $\omega \kappa(\omega)/c$. Working to first order in $\omega - \omega_0$ we find

\[
E(z,t) = \exp\left(i\frac{\omega_0}{c}(n(\omega_0) z - ct)\right) \exp\left(-\frac{\omega_0}{c} \kappa(\omega_0) z\right)
\times \int_0^\infty E_0(\omega) \exp\left(i\frac{\omega}{c}\left\{ n(\omega_0) + \omega_0 \frac{dn(\omega_0)}{d\omega_0} \right\} z - ct\right) d\omega + c.c.
\] (2.194)

where we have neglected the variation in the imaginary part of the index of refraction. In this approximation the wave travels with the group velocity

\[
v_g(\omega_0) = c \left[ n(\omega_0) + \omega_0 \frac{dn(\omega_0)}{d\omega_0} \right]^{-1}
\] (2.195)

i.e.

\[
E(z,t) = \int_0^\infty E_0(\omega) \exp\left(i\frac{\omega - \omega_0}{v_g(\omega_0)} \{ z - v_g(\omega_0) t \} \right) d\omega
\times \exp(i\frac{\omega_0}{c}(n(\omega_0) z - ct)\right) \exp(-\frac{\omega_0}{c} \kappa(\omega_0) z) + c.c.
= F(z - v_g(\omega_0) t) \exp(i\frac{\omega_0}{c}(n(\omega_0) z - ct)\right) \exp(-\frac{\omega_0}{c} \kappa(\omega_0) z) + c.c.
\] (2.196)

This should describe a ‘broad’ wave packet with the envelope function travelling with speed $v_g(\omega_0)$ and the underlying wave travelling with speed $c/n(\omega_0)$. Figure 8 shows the group velocity for the index of refraction of an ionic crystal shown in Fig. 5. The normalized frequency is $\omega/\omega_T$ with the approximation (for the peaked $E_0(\omega)$) that $\omega \approx \omega_0$. The group velocity approaches 0 at $\omega/\omega_T = 1$ where $n(\omega_T)$ diverges.

![Figure 8. Group velocity for an ionic crystal below $\omega_T$.](image-url)
Fig. 9 shows the group velocity, above the resonance, in a region where \( n(\omega \approx \omega_0) \) is small and the phase velocity, \( c/n(\omega_0) \) is greater than the speed of light. The normalized frequency is \( \omega/\omega_T \).

![Graph showing group velocity above resonance](image)

Figure 9. Group velocity of an ionic crystal above \( \omega_T \).

The plasma presents a different aspect of the wave propagation problem. The group velocity can exceed the speed of light in the region where the index of refraction (see Fig. 7) is small and slowly varying. However, in this region there is no wave propagation as illustrated by Fig.10. This shows the phase change for the wave as the intensity falls by \( 1/e \). The normalized frequency is \( \omega\tau \), where \( \omega_p\tau = 10 \).

![Graph showing phase change in plasma](image)

Figure 10. Phase change in plasma
For reference the decay length, in units of $c\tau$, for the transverse wave in a plasma is shown in Fig. 11. For metallic plasmas $\omega_p \approx 10^{16}\text{rad/s}$ and the figure would correspond to a system with a damping constant of $\tau \approx 10^{-15}\text{s}$. In this case $c\tau \approx 100\text{nm}$.

![Figure 11. Plasma decay length / $c\tau$.](image)

### 2.4.2 Causality

Our concept of causality sequences events using the speed of light as the limiting factor. Let the time and location of two events be $(r, t)$ and $(r', t')$. The event at $(r', t')$ is in the past of the event at $(r, t)$ if $|r - r'| < c(t - t')$. This means that $t$ is so much larger than $t'$ that the signal (emitted at $(r', t')$) has long since reached the observation point $r$. If $t$ is not quite large enough (but $t > t'$) the event is in the present of $(r, t)$ and $|r - r'| > c|t - t'|$. That is, the signal has not yet reached the observation point, $r$. The event is in the future of $(r, t)$ if $t' > t$ and $|r - r'| < c(t' - t)$. This means that the signal has not yet been generated.

Causality requires that a given event can only be influenced by events occurring in its past. Therefore no signal may propagate faster than the speed of light. In particular, a wave in a dispersive medium will propagate with a speed which does not exceed the vacuum speed of light. This places constraints on the index of refraction and the electric permittivity. Let $u(x, t)$ be a wave propagating in a medium with a permittivity $\varepsilon(\omega)$ and let $n(\omega) = \varepsilon(\omega)^{1/2}$. For example

$$u(x, t) = \int e^{-i\omega t} \left[ A(\omega) \exp \left( i \frac{\omega n(\omega) x}{c} \right) + B(\omega) \exp \left( -i \frac{\omega n(\omega) x}{c} \right) \right] d\omega$$  \(2.197\)

Let $u(0, t)$ and $[\partial u(x, t)/\partial x]_{x=0}$ be given. Then

$$u(0, t) = \int_{-\infty}^{\infty} e^{-i\omega t} [A(\omega) + B(\omega)] d\omega \quad \text{and}$$  \(2.198\)

$$\int_{-\infty}^{\infty} e^{i\omega t} u(0, t) dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(\omega - \omega')t} [A(\omega) + B(\omega)] d\omega dt$$

$$= 2\pi \int_{-\infty}^{\infty} [A(\omega) + B(\omega)] \delta(\omega - \omega') d\omega$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} u(0, t) dt = [A(\omega') + B(\omega')]$$
Finally, in the integrals for \( A(\omega) \) and \( \omega \), we subtract off this asymptotic dependence. But \( \omega \) and similarly

\[
\text{\( A(\omega) = \frac{1}{2} \sum_{-\infty}^{\infty} e^{i\omega(t-t')} \left\{ u(0,t) - i\frac{c}{2\pi \omega n(\omega)} \frac{\partial}{\partial x'} u(x,t) \bigg|_{x'=0} \right\} dt \)}
\]

\[
\text{\( B(\omega) = \frac{1}{2} \sum_{-\infty}^{\infty} e^{i\omega(t-t')} \left\{ u(0,t) + i\frac{c}{2\pi \omega n(\omega)} \frac{\partial}{\partial x'} u(x,t) \bigg|_{x'=0} \right\} dt \)}
\]

In the integrals for \( A(\omega) \) and \( B(\omega) \) let \( \omega' \) be replaced by \( \omega \). And let \( t \) be replaced by \( t' \) to distinguish it from the value of time on the left hand side in the following expression: 

\[
 u(x,t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')} \exp \left( i\frac{\omega n(\omega)}{c} x \right) \left\{ u(0,t') - i\frac{c}{2\pi n(\omega)} \left[ \frac{\partial u(x',t')}{\partial x'} \right]_{x'=0} \right\} d\omega dt'
\]

Next, the integration over \( \omega \) can be converted into a contour integration in the complex \( \omega \) plane.

\[
 u(x,t) = \frac{1}{4\pi} \int_{C} e^{-i\omega(t-t')} \exp \left( i\frac{\omega n(\omega)}{c} x \right) \left\{ u(0,t') - i\frac{c}{2\pi n(\omega)} \left[ \frac{\partial u(x',t')}{\partial x'} \right]_{x'=0} \right\} d\omega dt'
\]

with \( C \) denoting the contour which passes above any poles or cuts on the real \( \omega \) axis. To observe the constraints imposed by causality it suffices to consider the following expressions for \( x > 0 \),

\[
g(x,t-t') = -\frac{1}{4\pi} \int_{C} e^{-i\omega(t-t')} \exp \left( i\frac{\omega n(\omega)}{c} x \right) \frac{i\omega n(\omega)}{\omega n(\omega)} d\omega
\]

and

\[
\frac{\partial}{\partial x} g(x,t-t') = f(x,t-t') = \frac{1}{4\pi} \int_{C} e^{-i\omega(t-t')} \exp \left( i\frac{\omega n(\omega)}{c} x \right) d\omega
\]

A general property of \( \epsilon(\omega) \) and \( n(\omega) \) is that for large \( |\omega| \) they approach one. To obtain well behaved integrands for large \( |\omega| \) we will subtract off this asymptotic dependence. But first we consider the functions with \( n(\omega) = 1 \), call them \( g_0(x,t-t') \) and \( f_0(x,t-t') \). The following are evaluated using a contour in the lower half plane and the Cauchy residue
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Theorem. The residue is that of a simple pole at \( \omega = 0 \).

\[
\begin{align*}
g_0(x,t-t') &= -\frac{1}{4\pi} \int_C e^{-i\omega(t'-t)} \exp \left( i\frac{\omega x}{c} \right) \frac{ic}{\omega} d\omega = -\frac{2\pi i}{4\pi} [-ic\Theta(t-t' - \frac{x}{c})] \quad (2.206) \\
&= -\frac{c}{2} \Theta(t-t' - \frac{x}{c}) \quad \text{Note: countour direction } C \text{ is negative} \quad (2.207)
\end{align*}
\]

\[
\begin{align*}
f_0(x,t-t') &= \frac{\partial}{\partial x} g(x,t-t') = \frac{\partial}{\partial x} \frac{c}{2} \Theta(t-t' - \frac{x}{c}) = \frac{1}{2} \delta(t-t' - \frac{x}{c}) \quad (2.208)
\end{align*}
\]

These expressions are consistent with the relationship between \( g(x,t) \) and \( f(x,t) \). As expected \( g_0(x,t-t') \) and \( f_0(x,t-t') \) are ‘causal’ in that a disturbance at the origin at time \( t_0 \) does not effect the wave at \( x \) until the time \( t \) satisfying \( c(t-t') = x \). We now consider \( g(x,t-t') - g_0(x,t-t') \)

\[
\begin{align*}
g(x,t-t') - g_0(x,t-t') &= -\frac{ic}{4\pi} \int_C \frac{e^{-i\omega(t'-t-x/c)\omega}}{\omega} \left\{ \frac{\exp \left( i\omega [n(\omega) - 1] x/c \right)}{n(\omega)} - 1 \right\} d\omega \quad (2.210)
\end{align*}
\]

This is the fourier transform of a function which vanishes as \( \omega \to \pm \infty \). This transform is zero for \( x > c(t-t') \) if

\[
\gamma(\omega) = \frac{1}{\omega} \left\{ \frac{\exp \left( i\omega [n(\omega) - 1] x/c \right)}{n(\omega)} - 1 \right\} \quad (2.211)
\]

has no singularities or cuts in the upper half complex \( \omega \) plane.

The problem with a cut can be illustrated by the integral

\[
C(t) = \int_{-\infty-i\delta}^{\infty-i\delta} \omega^{-1/2} \exp \left( -i\omega t \right) d\omega \quad (2.212)
\]

where the path of integration is taken to be just below the real \( \omega \) axis. To use Cauchy’s theorem to evaluate the integral for \( t < 0 \) we would attempt to close the path in the upper half plane. The difficulty is that \( \omega^{-1/2} \) ‘cuts the plane’. Let \( \omega = |\omega| e^{i\phi} \) with \(-\pi < \phi \leq \pi\) then as \( \omega \) approaches the negative real axis from below \( \omega^{1/2} \to -i |\omega|^{1/2} \) while as \( \omega \) approaches the
negative real axis form above $\omega^{1/2} \to +i|\omega|^{1/2}$. This discontinuity occurs because the function $z^2$ maps the complex plane with $-\pi < \phi \leq \pi$ into the ‘two sheets’ $-2\pi < \phi' \leq 2\pi$.

To avoid the ‘second sheet’ when the path is closed, the path must not cross the cut. When the path is closed in the upper half plane the segments of the path are, in the limit $R \to \infty$, (1) $-R - i\delta \to +R - i\delta$, (2) $z = R \cdot e^{i\phi}$ from $+R - i\delta$ to $-R + i\delta$, (3) $-R + i\delta \to i\delta$, (4) $z = \delta \cdot e^{i\phi}$ from $+i\delta$ to $-i\delta$, and (5) $-i\delta \to -R - i\delta$. This path is shown in Fig. 1. We will now evaluate $C(t)$. For $t > 0$ $C(t) = 0$ and for $t < 0$ we obtain the integral of the discontinuity if the integrand across the cut.

\[
C(t) = -\int_{-\infty}^{0} \frac{\exp(-i\omega t)}{(\omega)^{1/2}} (-2i) d\omega = \frac{2i}{\sqrt{-t}} \int_{0}^{\infty} u^{-1/2} \exp(-iu) \, du = (1 + i) \sqrt{\frac{2\pi}{-t}}, \quad t < 0
\]  

(2.213)

We note that a cut in the upper half plane would lead to a non-zero value for the integral in the case that the path was to be closed in the upper half plane.

Omit: Assignment 8: The claim is that the cut in the plane defining $\omega^{1/2}$ can be taken along any line going from zero to infinity in the upper half plane. (Note the physics of a problem determines whether the path of integration is taken above or below the cut. There’s no physics in this problem.) Check whether this claim is correct by taking $\omega = |\omega| \exp(i\phi)$ with $-3\pi/2 < \phi \leq \pi/2$.

The requirement that $\gamma(\omega)$ have no cuts or poles in the upper half complex plane is satisfied if $n(\omega) = \varepsilon(\omega)^{1/2}$ has no poles or cuts in the upper half complex plane. In this case $\varepsilon(\omega)$ can neither have a pole nor a zero in the upper half plane. This condition is satisfied by the model permittivities for the ionic crystals and for the charged plasmas.

A similar argument applies to $f(x, t - t') - f_0(x, t - t')$. This function is the fourier transform

\[
f(x, t - t') - f_0(x, t - t') = \frac{1}{4\pi} \int_{C} e^{-i\omega (t-t'-x/c)} \left[ \exp\left( i \frac{\omega |n(\omega) - 1| x/c}{c} \right) - 1 \right] d\omega
\]  

(2.214)

Again the function being transformed vanishes as $\omega \to \pm\infty$ and the integral will be zero if $t > 0$ and $\varepsilon(\omega)$ has no poles or zeros in the upper half plane.
2.4.3 Wave dispersion in a medium

In order to examine the dispersion character for a wave in a medium we take \( u(0,t) = u_0 \delta(t) \) and \( \left[ \frac{\partial u(x,t)}{\partial t} \right]_{x=0} = 0 \). These boundary conditions will provide identical waves travelling in the \(+x\) and \(-x\) directions. Since the Fourier transform of a delta function is a constant this acts as a ‘white light’ source. From Eq. 200,

\[
\begin{align*}
\mathbf{u}(x,t) &= \frac{1}{4\pi} \int_C e^{-i\omega(t-t')} \left[ \exp \left( i \frac{\omega n(\omega)x}{c} \right) + \exp \left( -i \frac{\omega n(\omega)x}{c} \right) \right] u(0,t') \, d\omega \, dt' \\
\end{align*}
\]

\[u(x,t) = \frac{u_0}{2\pi} \int_C e^{-i\omega t} \cos \left( \frac{\omega n(\omega)x}{c} \right) u_0 \delta(t') \, d\omega, \quad t > 0
\]

\[u(x,t) = u(0,t) = u_0 \text{ for } t > 0
\]

Since \( u(x,t) = u(-x,t) \) it suffices to consider the wave travelling in either the \(+x\) or \(-x\) direction.

\[
\begin{align*}
\mathbf{u}(x,t) &= \frac{u_0}{4\pi} \int_C \exp \left( -i\omega \left( t - n(\omega)x \right) \right) \, d\omega + \frac{u_0}{4\pi} \int_C \exp \left( -i\omega \left( t + n(\omega)x \right) \right) \, d\omega, \quad t > 0
\\
&= u_+(x,t) + u_-(x,t)
\end{align*}
\]

We will chose the wave travelling in the \(+x\) direction for which the transform is

\[
\begin{align*}
\mathbf{u}_+(x,t) &= \frac{u_0}{4\pi} \int_C \exp \left( -i\omega \left( t - n(\omega)x \right) \right) \, d\omega, \quad t > 0
\\
&= \frac{u_0}{4\pi} \int_{-\infty}^{\infty} \exp \left( -i\omega \left( t - n(\omega)x \right) \right) \, d\omega
\\
&= \frac{u_0}{4\pi} \int_{-\infty}^{\infty} \cos \left( -\omega \left( t - n(\omega)x \right) \right) + i \sin \left( -\omega \left( t - n(\omega)x \right) \right) \, d\omega
\end{align*}
\]

Since \( n(\omega) = n(-\omega) \) only the real part of the last expression survives to give

\[
\begin{align*}
\mathbf{u}_+(x,t) &= \frac{u_0}{2\pi} \text{Re} \int_{0}^{\infty} \exp \left( -i\omega \left( t - n(\omega)x \right) \right) \, d\omega, \quad t > 0
\end{align*}
\]

In the following we will assume, incorrectly, that the imaginary part of the permittivity vanishes. That is,

\[
k = \left[ \text{Re} \varepsilon(\omega) + i\omega \right] \frac{1}{c} = [n(\omega) + ikap(\omega)] \frac{\omega}{c}
\]

and \( kap(\omega) = 0 \). Thus, using Eq. 2.189,

\[
\begin{align*}
n(\omega) &= \varepsilon_i^{1/2} \left[ 1 - \frac{\omega_p^2 + \tau^2}{\omega^2 + \tau^2 + 1} \right]^{1/2}
\\
&= \varepsilon_i^{1/2} \left[ 1 - \frac{\omega^2 + \tau^2}{\omega^2 + \tau^2} \right]^{1/2} \quad \text{since } \tau \rightarrow \infty
\end{align*}
\]

so that

\[
\omega n(\omega) = \varepsilon_i^{1/2} \left[ \omega^2 - \omega_p^2 \right]^{1/2}
\]
This will allow us to use a special case of the ‘method of steepest descent’ which is less tedious. The method is known as the ‘stationary phase approximation’. The approximation will be illustrated using the electric permittivity of an undamped plasma, $\tau \to \infty$.

For an undamped plasma with a plasma frequency $\omega_p$ we have that $\omega_n (\omega) = \sqrt{\omega^2 - \omega_p^2}$ for $\omega > \omega_p$. For $-\omega_p \leq \omega \leq \omega_p$ we obtain $\omega_n (\omega) = i \sqrt{\omega_p^2 - \omega^2}$. It is convenient to define a time $t_0 = x/c$ which is the time it takes for a light signal travelling in a vacuum to go from the origin to the observation point $x$. We also let $\omega = \xi \omega_p$ and $t = \nu t_0$. We have determined that this system satisfies causality so that $\nu \geq 1$. With these definitions the function can be written as

$$u_+ (x, t) = \frac{u_0 \omega_p}{2\pi} \text{Re} \int_1^\infty \exp \left(-i\omega_p t_0 \left[ \nu \xi - \sqrt{\xi^2 - 1} \right] \right) d\xi + \frac{u_0 \omega_p}{2\pi} \text{Re} \int_0^1 \exp \left(-\omega_p t_0 \left[ i\nu \xi + \sqrt{1 - \xi^2} \right] \right) d\xi$$

(2.223)

Our problem is to obtain an estimate for the value of this function if $\omega_p t_0 \gg 1$. The second integral is exponentially decaying with distance and so will be assumed to be negligible.

The phase, $\phi(\omega)$, occurring in the first integral is $\omega_p t_0 f(\nu, \xi)$ with

$$f(\nu, \xi) = \nu \xi - \sqrt{\xi^2 - 1}$$

(2.224)

Fig. 13 gives $f(\nu, \xi) = \phi(\omega)/\omega_p t_0$ for three observation times, $1.1 t_0, 1.2 t_0, \text{and} 1.3 t_0$.

Regions in which the phase varies ‘rapidly’ will contribute little to the integral. For large $\omega_p t_0$ the major contribution to

---

4 This is a technique for obtaining an approximation to an integral of the form

$$\int_C \exp \left[ a u (t) \right] dt$$

for large $a$. The saddle points of $u (z)$, $z$ complex, are located and the path of integration is distorted so as to pass through these saddle points. In going through a saddle point the path is along the line for which the curvature is negative. The integral is approximated by using only the neighborhoods of the saddle points and approximating $u (z)$ by a quadratic in the distance from the saddle point. The classic reference for this is Courant & Hilbert, Methods of Mathematical Physics, Vol. 1 (Interscience Publishers, pp526-532) Another reference is Approximation methods in quantum mechanics by A. B. Migdal and V. Kraitov (pp 6-8, W. A. Benjamin, 1969)
the integral will come from the region where the magnitude of the phase is minimum or has zero slope. This occurs at

$$\frac{\partial f}{\partial \xi} = 0 = v - \frac{\xi}{\sqrt{\xi^2 - 1}} \quad \text{at} \quad \xi = \xi_o$$ \hspace{1cm} (2.225)

$$\xi_o \equiv \frac{\omega}{\omega_p} = \frac{v}{\sqrt{v^2 - 1}} = \frac{1}{\sqrt{1 - \left(\frac{t_0}{t}\right)^2}}$$ \hspace{1cm} (2.226)

where we note that

$$\xi_o^2 - 1 = \frac{1}{v^2 - 1}$$ \hspace{1cm} (2.227)

The frequency for which the slope vanishes is shown in Fig. 14

For $t \rightarrow t_0$ the frequency of the minimum of the magnitude of the phase approaches infinity. On the other extreme as $t \rightarrow \infty$ the frequency of the minimum approaches the plasma frequency. We deduce then that the frequency observed at the observation point starts high and decreases with time. The range of frequencies which contribute to the observed signal also decreases with time. This can be obtained by considering the reciprocal of the curvature of the phase as a function of $\omega$.

$$\frac{\partial^2 \phi}{\partial \omega^2} (\omega; x, t) = \frac{t_0}{\omega_p} (\xi^2 - 1)^{-1.5}$$
$$\quad = \frac{t_0}{\omega_p} (\nu^2 - 1)^{3/2} \quad \text{at} \quad \xi = \xi_o$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = \omega_p t_0 (\nu^2 - 1)^{3/2} \quad \text{at} \quad \xi = \xi_o$$ \hspace{1cm} (2.228)
Note also that

\[ \phi(\xi_0) = \omega_p t_0 f(\upsilon, \xi) |_{\xi_0} = \omega_p t_0 [\upsilon \xi_0 - \sqrt{\xi_0^2 - 1}] \]

\[ = \omega_p t_0 [\upsilon \frac{\upsilon}{\sqrt{\upsilon^2 - 1}} - \frac{1}{\sqrt{\upsilon^2 - 1}}] \]

\[ = \omega_p t_0 \upsilon \sqrt{\upsilon^2 - 1} \]  

\[
\frac{d^2 \phi}{d\omega^2}(\omega_p t_0) 
\]

Reciprocal phase curvature at minimum

\[ \frac{d^2 \phi}{d\omega^2}(\omega_p t_0) \approx \frac{1}{2} \omega_p t_0 \upsilon \sqrt{\upsilon^2 - 1} \]

\[ + \frac{1}{2} \omega_p t_0 [\upsilon^2 - 1]^{3/2} (\xi - \xi_0)^2 \]  

(2.229)

This is shown in Fig. 15. The approximation to \( u_+(x, t) \) is obtained by expanding the phase about the minimum, keeping only the quadratic terms \(^5\)

\[
\phi(\xi) = \phi(\xi_0) + \phi'(\xi_0)(\xi - \xi_0) + \frac{1}{2} \phi''(\xi_0)(\xi - \xi_0)^2 + \ldots
\]

\[ \approx \phi(\xi_0) + \frac{1}{2} \phi''(\xi_0)(\xi - \xi_0)^2 \]

\[ = \omega_p t_0 \upsilon \sqrt{\upsilon^2 - 1} + \frac{1}{2} \omega_p t_0 [\upsilon^2 - 1]^{3/2} (\xi - \xi_0)^2 \]  

(2.230)

Putting the above expression into the integrand we have,

\[ u_+(x, t) \approx \frac{\omega_p}{2\pi} \text{Re} \left[ \int_1^\infty \exp \left[ -i[\phi(\xi_0) + \frac{1}{2} \phi''(\xi_0)(\xi - \xi_0)^2] \right] d\xi \right] \]  

(2.231)

\(^5\) Integrals of the form

\[ C(z) = \int_0^z \cos \left( \frac{\pi t^2}{2} \right) dt \]

\[ S(z) = \int_0^z \sin \left( \frac{\pi t^2}{2} \right) dt \]

are Fresnel integrals. (Abramowitz and Stegun, Handbook of Mathematical Functions, p.300, Dover) Both integrals go to 0.5 as \( z \to \infty \).
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\[ u_+ (x, t) \approx \frac{u_0 \omega_p}{2\pi} \Re \left[ \exp \left( -i \phi(\xi_0) \right) \int_1^\infty \exp \left[ -i \frac{1}{2} \phi''(\xi_0) (\xi - \xi_0)^2 \right] d\xi \right] \]

\[ = \frac{u_0 \omega_p}{2\pi} \Re \left[ \exp \left( -i \phi(\xi_0) \right) \left( \frac{1}{\pi} \phi''(\xi_0) \right)^{-1/2} \int_{W(\xi = 1)}^\infty \exp \left[ -i \frac{\pi}{2} W^2 \right] dW \right] \]

\[ = \frac{u_0 \omega_p}{2\pi} \Re \left[ \exp \left( -i \phi(\xi_0) \right) \left( \frac{1}{\pi} \phi''(\xi_0) \right)^{-1/2} \int_{W(\xi = 1)}^\infty \{ \cos \left( \frac{\pi}{2} W^2 \right) - i \sin \left( \frac{\pi}{2} W^2 \right) \} dW \right] \quad (2.232) \]

where we used

\[ W^2 = \frac{1}{\pi} \phi''(\xi_0) (\xi - \xi_0)^2 \quad \text{and} \quad (2.233) \]

\[ W(\xi = 1) = \left( \frac{1}{\pi \omega_p t_0} \right)^{1/2} \left[ \frac{t_0^2}{t - t_0} - 1 \right]^{3/4} (1 - \xi_0)^2 \]

\[ \approx 0 \quad \text{for large} \quad \frac{t}{t_0} \quad \text{where} \quad \xi_0 \approx 1 \quad (2.234) \]

One can do the integrals (see the footnote) to obtain

\[ u_+ (x, t) \approx \frac{u_0 \omega_p}{2\pi} \Re \left[ \exp \left( -i \phi(\xi_0) \right) \left( \frac{1}{\pi} \phi''(\xi_0) \right)^{-1/2} \left( 1 - i \right) / 2 \right] \quad (2.235) \]

\[ u_+ (x, t) \approx \frac{u_0}{4 \sqrt{\pi} t_0} \Re \left[ \frac{\exp \left( -i \omega_p t_0 \sqrt{u^2 - 1} \right)}{(u^2 - 1)^{3/4}} (1 - i) \right] \]

\[ = \frac{\sqrt{2} u_0}{4 \sqrt{\pi} t_0} \Re \left[ \frac{\exp \left( -i \omega_p t_0 \sqrt{u^2 - 1} \right)}{(u^2 - 1)^{3/4}} \exp(i \pi / 4) \right] \]

\[ = \frac{u_0}{2 \sqrt{2} \pi (\omega_p t_0)^{1/4}} \cos \left( \frac{\omega_p \sqrt{t^2 - (x/c)^2} + \pi / 4}{(t^2 - (x/c)^2)^{3/4}} \right) \quad (2.236) \]

This approximation is good for \( \omega_p t_0 \gg 1 \). Considering the range of plasma frequencies, from \( 10^9 \text{rad/s} \) for gaseous plasmas through \( 10^{16} \text{rad/s} \) for metallic plasmas, the distance of the observation point from the source should be much greater than \( 10 \text{cm} \) for the gaseous plasmas and \( 1 \mu \text{m} \) for the metals. The plot in Fig. 16 shows \( u_+ (x, t) \) for a metallic plasma with \( \omega_p = 10^{16} \text{rad/s} \) and \( \omega_p t_0 = 1000 \) at three times, \( t \). Note that the frequency decreases as time increases.
2.5 Analyticity and the Kramers-Kronig relationship

We have determined that causality requires $\varepsilon(\omega)$ to be analytic and have no zeros in the upper half complex $\omega$ plane. In the case that the local approximation holds, this can be seen by considering the causal expression

$$D(r,t) = E(r,t) + \int_0^\infty G(\tau) E(r,t-\tau) d\tau$$ (2.238)

In this expression $G(\tau)$ is a property of the system and is generally expected to vanish as $\tau \to \infty$. (The exception to this behavior is furnished by the electrical conductors for which $G(\tau) \to 4\pi \sigma_0$, the DC conductivity$^6$.) The Fourier transform, in time, of this expression along with the relationship between the frequency components of $D$ and $E$ yields

$$\int_{-\infty}^{\infty} D(r,\omega) e^{i\omega t} dt = \int_{-\infty}^{\infty} E(r,\omega) e^{i\omega t} dt + \int_0^\infty G(\tau) E(r,t-\tau) e^{i\omega t} d\tau$$

$$D(r,\omega) = E(r,\omega) + \int_{-\infty}^{\infty} G(\tau) e^{i\omega \tau} E(r,t-\tau) e^{i\omega (t-\tau)} d(t-\tau)d\tau$$

$$\varepsilon(\omega) E(r,\omega) = E(r,\omega) + \int_0^\infty G(\tau) e^{i\omega \tau} E(r,\omega) d\tau$$

$$\varepsilon(\omega) - 1 = \int_0^\infty G(\tau) \exp(i\omega \tau) d\tau$$ (2.240)

$^6$ In the case of a charged plasma

$$G(\tau) = \frac{1}{2\pi} \int_C \left[ \frac{4\pi \sigma_0}{\omega(\omega t + i)} \right] e^{-i\omega t} d\omega$$

$$= 4\pi \sigma_0 \left[ 1 - e^{-t/\tau} \right]$$

For $\tau \to 0$ $G(\tau) \to 0$ which can be interpreted as demonstrating the finite time it takes for a system to respond to a stimulus. While for $\tau \to \infty$ $G(\tau) \to 4\pi \sigma_0$. 

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Section 2.5 Analyticity and the Kramers-Kronig relationship

Since $D$ and $E$ are real $G(\tau)$ will also be real. It follows, as previously noted, that

$$\varepsilon(\omega) = \varepsilon(-\omega)^*.$$  \hfill (2.241)

It is also possible to obtain a relationship between $G$ and its derivatives at $\tau = 0$ and the high frequency behavior of $\varepsilon(\omega)$. To arrive at the relationship we note that

$$\int_0^\infty G(\tau) \exp(i\omega \tau) d\tau = \frac{1}{i\omega} \left( G(\tau) \exp(i\omega \tau) \right|_0^\infty - \int_0^\infty \frac{dG(\tau)}{d\tau} \exp(i\omega \tau) d\tau \right) \quad \text{(2.242)}$$

and

$$\int_0^\infty \frac{d^n G(\tau)}{d\tau^n} \exp(i\omega \tau) d\tau = \frac{1}{i\omega} \left( \frac{d^n G(\tau)}{d\tau^n} \exp(i\omega \tau) \right|_0^\infty - \int_0^\infty \frac{d^{n+1} G(\tau)}{d\tau^{n+1}} \exp(i\omega \tau) d\tau \right) \quad \text{(2.243)}$$

For all materials which are not DC conductors (and for which $\lim_{n \to \infty} \frac{d^n G(\tau)}{d\tau^n} = 0$)

$$\varepsilon(\omega) - 1 = \sum_{n=1}^\infty \left( \frac{-1}{i\omega} \right)^n \frac{d^{n-1} G(\tau')}{d\tau^{n-1}} \bigg|_{\tau = 0} \quad \text{(2.244)}$$

The asymptotic behavior of $\varepsilon(\omega)$ for large $\omega$ can be deduced from this expression. It is reasonable to expect that the response of the system takes a finite time, i.e., the system does not instantaneously change. In the case that this is true we have that $G(0) = 0$ and the first two terms in the expansion are

$$\varepsilon(\omega) - 1 \approx -\omega^{-2} G'(0) - i\omega^{-3} G''(0)$$ \hfill (2.245)

$${\text{Re}}(\varepsilon(\omega) - 1) \approx \frac{-1}{\omega^2} G'(0)$$ \hfill (2.246)

$${\text{Im}}(\varepsilon(\omega) - 1) \approx \frac{-1}{\omega^3} G''(0)$$ \hfill (2.247)

That is, the real part of $\varepsilon(\omega) - 1$ vanishes as $\omega^{-2}$ and the imaginary part of $\varepsilon(\omega)$ vanishes as $\omega^{-3}$.

Finally, for complex $\omega$ in the upper half (Im $\omega > 0$)

$$\varepsilon(\omega) - 1 = \int_0^\infty G(\tau) e^{i\omega \tau} d\tau \leq \int_0^\infty G(\tau) |e^{i\tau \text{Re} \omega} e^{-\tau \text{Im} \omega} d\tau \leq \int_0^\infty G(\tau) d\tau \quad \text{(2.248)}$$

If $\int_0^\infty G(\tau) d\tau$ is finite then $\varepsilon(\omega) - 1$ will be finite and an analytic function of complex $\omega$ in the upper half (Im $\omega > 0$) complex plane. This has some important consequences. The $\varepsilon(\omega) - 1$ for $\omega$ real exists and is the limit of a function which is analytic in the upper half plane.

### 2.5.1 The Kramers-Kronig relations

Since $\varepsilon(\omega) - 1$ is an analytic function in the upper half complex plane then, by Cauchy’s residue theorem,

$$\varepsilon(z) = 1 + \frac{1}{2\pi i} \oint_C \frac{\varepsilon(z') - 1}{z' - z} \, dz'$$ \hfill (2.249)

with the imaginary part of all points on the curve $C$ restricted to be greater than or equal to zero. Let the path lie along the real axis and let it be closed by the semi-circle of infinite radius. ($\varepsilon(\omega) - 1$ will vanish on this semi-circle.) In addition let $z = \omega + i\delta$ with $1 \gg \delta > 0$ then

$$\varepsilon(\omega) = 1 + \lim_{\delta \to 0} \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{\varepsilon(\omega') - 1}{\omega' - \omega - i\delta} d\omega'.$$ \hfill (2.250)
Section 2.5 Analyticity and the Kramers-Kronig relationship

This can be rewritten using the relationships between path integrals, principal value integrals, and delta functions. The principal value of the integral, where \( f(z') \) is analytic in upper half plane, is:

\[
P \int \frac{f(z')}{z'-z} dz' = \int_C \frac{f(z')}{z'-z} dz' - \int_{C_\delta} \frac{f(z')}{z'-z} dz'
\]  

(2.251)

where \( C_\delta \) is a circular curve of radius \( \delta' \) which "surrounds" the point, \( z \). If \( z \) lies on the real axis then \( C_\delta \) is a semi-circle of radius \( \delta' \) taken below the point \( z \) and

\[
P \int \frac{f(z')}{z'-z} dz' = \int_C \frac{f(z')}{z'-z} dz' - \int_{C_\delta} \frac{f(z')}{z'-z} dz' = 2\pi i f(z + i\delta') - \int_{C_\delta} \frac{f(z')}{z'-z} d(z' - z)
\]  

(2.252)

\[
= 2\pi i f(z + i\delta) - \int_{C_\delta} \frac{f(z')}{z'-z} d(z' - z)
\]  

(2.253)

\[
= 2\pi i f(z + i\delta) - \int_{\pi}^{2\pi} \frac{f(z + \delta' e^{i\phi})}{\delta' e^{i\phi}} d(\delta' e^{i\phi}) \text{ with } z' - z = \delta' e^{i\phi}
\]  

(2.254)

\[
= 2\pi i f(z + i\delta) - \int_{\pi}^{2\pi} \frac{f(z) + f'(z) \frac{\delta' e^{i\phi}}{2}}{i} d\phi
\]  

(2.255)

\[
= 2\pi i f(z + i\delta) - i f(z)\pi - f'(z) \frac{\delta' e^{i2\pi - e^{i\pi}}}{} + ...
\]  

(2.256)

Now let \( \delta' \) go to 0. Finally, let \( \delta \rightarrow 0 \). Then,

\[
P \int \frac{f(z')}{z'-z} dz' = i f(z)\pi
\]  

(2.257)

Applying this to our problem, the integral taken below the pole equals the principal part of the integral plus \( \pi i \) times the integral with a delta function

\[
\varepsilon(\omega) = 1 + \frac{1}{2\pi i} |P| \int \frac{\varepsilon(\omega') - 1}{\omega' - \omega} d\omega' + i\pi(\varepsilon(\omega) - 1)
\]  

(2.258)

or

\[
\varepsilon(\omega) = 1 + \frac{1}{\pi i} P \int \frac{\varepsilon(\omega') - 1}{\omega' - \omega} d\omega'
\]  

(2.259)

Using the fact that the integrand vanishes on the upper half circle at \( \omega' = \infty \), this yields relationships between the real and imaginary parts of \( \varepsilon(\omega) \)

\[
\text{Re} \varepsilon(\omega) = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Im} \varepsilon(\omega')}{\omega' - \omega} d\omega'
\]  

(2.260)

\[
\text{Im} \varepsilon(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Re} \varepsilon(\omega') - 1}{\omega' - \omega} d\omega'
\]  

(2.261)
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which, using \( \text{Re} \varepsilon (\omega) = \text{Re} \varepsilon (-\omega) \) and \( \text{Im} \varepsilon (\omega) = - \text{Im} \varepsilon (-\omega) \), can be written as

\[
\text{Re} \varepsilon (\omega) = 1 + \frac{1}{\pi} P \left[ \int_{-\infty}^{0} \frac{\text{Im} \varepsilon (\omega') d\omega'}{\omega' - \omega} + \int_{0}^{\infty} \frac{\text{Im} \varepsilon (\omega') d\omega'}{\omega' - \omega} \right] = 1 + \frac{1}{\pi} P \left[ \int_{0}^{\infty} \frac{\text{Im} \varepsilon (-\omega'') d\omega''}{-\omega'' - \omega} + \int_{0}^{\infty} \frac{\text{Im} \varepsilon (\omega') d\omega'}{\omega' - \omega} \right] = 1 + \frac{1}{\pi} P \left[ \int_{0}^{\infty} \frac{\text{Im} \varepsilon (\omega'') d\omega''}{\omega'' + \omega} + \int_{0}^{\infty} \frac{\text{Im} \varepsilon (\omega') d\omega'}{\omega' - \omega} \right] = 1 + \frac{2}{\pi} \int_{0}^{\infty} \frac{\text{Im} \varepsilon (\omega') d\omega'}{\omega^2 - \omega'^2}
\]

\[
\text{Im} \varepsilon (\omega) = -\frac{1}{\pi} P \left[ \int_{-\infty}^{0} \frac{\text{Re} \varepsilon (\omega') d\omega'}{\omega' - \omega} + \int_{0}^{\infty} \frac{\text{Re} \varepsilon (\omega') d\omega'}{\omega' - \omega} \right] = -\frac{1}{\pi} P \left[ \int_{0}^{\infty} \frac{\text{Re} \varepsilon (-\omega'') d\omega''}{-\omega'' - \omega} + \int_{0}^{\infty} \frac{\text{Re} \varepsilon (\omega') d\omega'}{\omega' - \omega} \right] = -\frac{2}{\pi} \int_{0}^{\infty} \frac{\text{Re} \varepsilon (\omega') d\omega'}{\omega^2 - \omega'^2}
\]

These are the classic Kramers-Kronig relations between the real and imaginary parts of the electric permittivity. Similar relations hold for the frequency response function for any causal system. Similar relationships are very useful in quantum mechanics (particularly in scattering theory), circuit analysis in electrical engineering, etc.

The significance of these relations is that it provides a way to measure or calculate one characteristic of a system and then determine a completely different characteristic of the system. For example the existence of an absorption peak can be seen to lead to anomalous dispersion, i.e., \( dn (\omega) / d\omega < 0 \). This is easily demonstrated. Assume that the system has an absorption peak at \( \omega = \omega_0 \). For simplicity we take

\[
\text{Im} \varepsilon (\omega) = \frac{\pi}{2} \alpha \omega_0 \delta (\omega - \omega_0) + ...
\]

where the additional terms vary slowly in the region of \( \omega_0 \) and \( \alpha \) is a dimensionless constant. Equation 2.261 gives the real part of \( \varepsilon (\omega) \) in the region of \( \omega_0 \) (but not at \( \omega_0 \)),

\[
\text{Re} \varepsilon (\omega) = 1 + \frac{2}{\pi} \int_{0}^{\infty} \frac{\pi \alpha \omega_0 \delta (\omega' - \omega_0) \omega'}{\omega'^2 - \omega^2} d\omega' + \Delta \varepsilon
\]

\[
= 1 + \frac{\alpha \omega_0^2}{\omega_0^2 - \omega^2} + \Delta \varepsilon
\]

with \( \Delta \varepsilon (\omega) \) slowly varying in the neighborhood of \( \omega_0 \). The contribution of the absorption peak to the real part of \( \varepsilon (\omega) \) is seen as a rapid variation in \( \text{Re} \varepsilon (\omega) \) from positive below \( \omega_0 \) to negative above \( \omega_0 \).

-----------------------------------------------

**Assignment 9a:** Jackson, 7.22, (2nd edition 7.14) Use the Kramers-Kronig relations to calculate the real part of \( \varepsilon (\omega) \) given the imaginary part of \( \varepsilon (\omega) \) for positive \( \omega \) as

(a) \( \text{Im} \varepsilon (\omega) = \lambda (\theta (\omega - \omega_1) - \theta (\omega - \omega_2)) \), \( \omega_2 > \omega_1 > 0 \)

(b) \( \text{Im} \varepsilon (\omega) = \frac{\lambda \gamma \omega}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega^2} \)

In each case sketch the behavior of \( \text{Im} \varepsilon (\omega) \) and \( \text{Re} \varepsilon (\omega) \) as functions of \( \omega \). Comment on the similarities or differences of the results as compared with curves in Fig. 7.8 of Jackson.

**Assignment 9b:** Jackson, 7.23, (2nd edition, 7.15) Let \( \varepsilon (\omega) \) be the permittivity of a system containing free electrons. In this case, as seen in the lectures covering the permittivity of a charged plasma, there is a singularity at \( \omega = 0 \). The singularity
is removed in the function
\[ f(\omega) = \varepsilon(\omega) - \frac{4\pi\sigma_0 i}{\omega} \]
with \( \sigma_0 \) the DC conductivity of the electrons. Since the function \( f(\omega) \) is analytic for \( \text{Im} \omega \geq 0 \) it satisfies Kramer-Kronig type relations. Obtain the modified Kramer-Kronig relations for \( \varepsilon(\omega) \):

\[ \text{Re} \varepsilon(\omega_1) = 1 + \frac{2}{\pi} P \int_0^\infty \frac{\omega' \text{Im} \varepsilon(\omega')}{\omega'^2 - \omega_1^2} d\omega' \quad (2.266) \]

It follows that

\[ \text{Re} \varepsilon(\omega) - \text{Re} \varepsilon(\omega_1) = \frac{2}{\pi} P \int_0^\infty \omega' \text{Im} \varepsilon(\omega') \frac{1}{\omega'^2 - \omega_1^2} \frac{1}{\omega'^2 - \omega^2} d\omega' \]

\[ = \frac{2}{\pi} (\omega^2 - \omega_1^2) \cdot P \int_0^\infty \frac{\omega' \text{Im} \varepsilon(\omega')}{(\omega'^2 - \omega^2)(\omega'^2 - \omega_1^2)} d\omega' \quad (2.267) \]

This form for \( \varepsilon(\omega) \) reduces the integral’s dependence on the values of \( \text{Im} \varepsilon(\omega) \) at large \( \omega \). This is known as a ‘subtracted dispersion relation’. This process of subtraction can be repeated for each known value of \( \varepsilon(\omega) \). A similar relationship can be generated, starting with Eq. 2.262, for the imaginary part \( \varepsilon(\omega) \).

**2.5.2 Sum rules** The Kramer-Kronig relations can be used to obtain some general properties of the integrals of the real and imaginary parts of the permittivity. We have noted that, at high frequency, the permittivity varies as \( \omega^{-2} + O(\omega^{-3}) \). Physically at high frequencies the binding forces for the electrons becomes negligible and the system responds as a plasma. With this interpretation we define the plasma frequency for the system

\[ \omega_p^2 \equiv \lim_{\omega \to \infty} \left[ \omega^2 (1 - \varepsilon(\omega)) \right] \quad (2.268) \]

If \( \text{Im} \varepsilon(\omega) \) varies as \( \omega^{-3} + O(\omega^{-4}) \) for high frequencies then Eq. 2.261 gives

\[ \omega_p^2 = \left( \lim_{\omega \to \infty} \omega^2 (1 - \text{Re} \varepsilon(\omega)) \right) - \omega^2 i \text{Im} \varepsilon(\omega) \]

\[ = \left( \lim_{\omega \to \infty} \omega^2 (1 - \text{Re} \varepsilon(\omega)) - \omega^2 i(\omega^{-3} + O(\omega^{-4})) \right) \]

\[ = \lim_{\omega \to \infty} \left[ -\omega^2 \frac{2}{\pi} P \int_0^\infty \frac{\omega' \text{Im} \varepsilon(\omega')}{\omega'^2 - \omega^2} d\omega' - i(\omega^{-1} + O(\omega^{-2})) \right] \]

\[ = \lim_{\omega \to \infty} \left[ -\frac{2}{\pi} P \int_0^\infty \frac{\omega' \text{Im} \varepsilon(\omega')}{\omega'^2 - 1} d\omega' - i(\omega^{-1} + O(\omega^{-2})) \right] \]

\[ \omega_p^2 = \frac{2}{\pi} \int_0^\infty \omega \text{Im} \varepsilon(\omega) d\omega \quad (2.270) \]

This is the sum rule for oscillator strengths.

It is interesting to consider the relationship between the coefficients in Sellmeier’s equation, Eq. 2.122, for the
permittivity and the plasma frequency defined in Eq. 2.270. The sum rule requires that

\[
\omega_p^2 = \frac{2}{\pi} \sum_m \int_0^\infty \omega \text{Im} \frac{A_m}{\omega_m^2 - i \gamma_m \omega - \omega^2} d\omega
\]

(2.271)

This integrand has simple poles at \( \omega_{\pm} \) and \( \omega_+^* \), with

\[
\omega_{\pm} = i \frac{\gamma}{2} \pm \sqrt{\omega_m^2 - \frac{\gamma^2}{4}}
\]

In this case we rewrite the integral as

\[
\omega_p^2 = \frac{1}{\pi} \sum_m \int_{-\infty}^{\infty} \frac{A_m \gamma_m \omega_m^2}{(\omega - \omega_{\pm})(\omega - \omega_{\pm}^*)} d\omega
\]

(2.273)

and evaluate it using Cauchy’s residue theorem. Closing the path in the upper half complex \( \omega \) plane we obtain the residues of the poles in the upper half plane (at \( \omega_+ \) and \( \omega_- \))

\[
\omega_p^2 = 2i \sum_m \frac{A_m \gamma_m \omega_m^2}{(\omega_+ - \omega_-)(\omega_- - \omega_+^*)}
\]

(2.274)

The definition of the plasma frequency given in Eq. 2.268 is quite general and would reflect the contributions of the responses of all charged particles in the system. The coefficients in Sellmeier’s equation, which approximates the permittivity as a sum of resonances, gives the ‘strength’ of each resonance. In principle the sources of the resonances include optically active phonons, plasmons, electronic excitations, electronic ionizations, etc.

A second sum rule is obtained if \( \text{Re} \varepsilon (\omega) - 1 \sim -\omega_p^2/\omega^2 + O (\omega^{-4}) \) and \( \text{Im} \varepsilon (\omega) \sim O (\omega^{-3}) \) for large \( \omega \). The asymptotic relationship between the real and imaginary parts of \( \varepsilon (\omega) \) can be obtained using Equation 2.262. We write this expression as

\[
\text{Im} \varepsilon (\omega) = \frac{2}{\pi \omega} \left[ \int_0^{\omega_m} \text{Re} [\varepsilon (\omega') - 1] d\omega' + P \int_{\omega_m}^{\infty} \text{Re} [\varepsilon (\omega') - 1] d\omega' \right]
\]

(2.276)
with $\omega \gg \omega_m$ and $\omega_m$ sufficiently large that the asymptotic form for the real part of $\varepsilon(\omega)$ provides a valid approximation. The approximated equation is

$$\text{Im} \varepsilon(\omega) \approx \frac{2}{\pi \omega} \int_{0}^{\omega_m} \text{Re} [\varepsilon(\omega') - 1] \left[ 1 + (\omega'/\omega)^2 \right] d\omega' + \frac{2}{\pi \omega} P \int_{\omega_m}^{\infty} \frac{1}{1 - (\omega'/\omega)^2} \left[ -\omega_p^2/\omega'^2 + O(\omega'^{-4}) \right] d\omega'$$

$$\text{Im} \varepsilon(\omega) \approx \frac{2}{\pi \omega} \left[ \int_{0}^{\omega_m} \text{Re} [\varepsilon(\omega') - 1] d\omega' - [\omega_p^2/\omega_m + O(\omega^{-2})] \right] (2.277)$$

Note that for an integral over finite limits with $f(x) = f(-x)$

$$P \int_{-a}^{a} \frac{f(x)}{x} dx = \lim_{\delta \to 0} \left[ \int_{-a}^{-\delta} \frac{f(x)}{x} dx + \int_{\delta}^{a} \frac{f(x)}{x} dx \right]$$

$$= \lim_{\delta \to 0} \left[ -\int_{-\delta}^{0} \frac{f(-x)}{x} dx + \int_{\delta}^{a} \frac{f(x)}{x} dx \right] = 0 (2.278)$$

Since $\text{Im} \varepsilon(\omega) \sim O(\omega^{-3})$ for large $\omega$ it follows that

$$\int_{0}^{\omega_m} \text{Re} [\varepsilon(\omega') - 1] d\omega' - \omega_p^2/\omega_m = 0 \quad (2.279)$$

or

$$\omega_m^{-1} \int_{0}^{\omega_m} \text{Re} \varepsilon(\omega') d\omega' = 1 + \omega_p^2/\omega_m^2 \quad (2.280)$$

This is sometimes called a superconvergence relation. This latter relation, Eq. 2.280, does not hold for conductors while the sum rule for oscillator strengths, Eq. 2.270, will hold for conductors.

### 2.5.3 Some useful relations between path integrals

The Kramers-Kronig relations involve principal value integrals. For these integrals the integrand will have simple poles on the real axis and the integral is taken along the real axis excluding the location of the pole.
Section 2.5 Analyticity and the Kramers-Kronig relationship

The principal value integral

equals one half of

plus

For example let \( f(x) \) have no singularities on the real axis then

\[
P \int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx = \lim_{\delta \to 0} \left[ \int_{-\infty}^{-\delta} \frac{f(x)}{x} \, dx + \int_{\delta}^{\infty} \frac{f(x)}{x} \, dx \right]
\]

(2.281)

Often it is easier to evaluate the integral using Cauchy’s residue theorem. This requires a closed path and the principal value integration leaves a gap in the path. This problem is surmounted using the identity illustrated in Fig. 16. The integrand is assumed to have a simple pole located at the ‘x’ on the real axis. The identity holds because one can show that the integrals along the half circles in the two contours cancel. In this case

\[
P \int_{-\infty}^{\infty} \frac{f(x)}{x} \, dx = \frac{1}{2} \left[ \int_{-\infty, C_+}^{\infty} \frac{f(x)}{x} \, dx + \int_{-\infty, C_-}^{\infty} \frac{f(x)}{x} \, dx \right]
\]

(2.282)

The principal value integral has been replaced by two integrals but Cauchy’s residue theorem can often be used to evaluate each integral.

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Another relationship between the path integrals is often useful. This identity is illustrated in Fig. 17 where again ‘x’ marks the location of the pole. The direction of integration along the circular path around the pole has been selected so as to cancel the half circle for $C_+$ and add the half circle for $C_-$. If both identities are combined the principal value integral is seen to equal (1) the integral along path $C_+$ plus one half the integral around the pole or (2) the integral along the path $C_-$ minus one half the integral around the pole.

2.6 Vector characteristic of electromagnetic wave

The final property of electromagnetic waves we shall consider is their vector characteristic. In an isotropic medium the electric field is perpendicular to the direction of propagation of the wave. A plane wave with the electric field given by $E(r,t) = E_1(r,t)\varepsilon_1$ is said to be linearly polarized with polarization vector $\varepsilon_1$. Similarly $E(r,t) = E_2(r,t)\varepsilon_2$ (with $\varepsilon_1 \cdot \varepsilon_2 = 0$) describes a linearly polarized wave with polarization vector $\varepsilon_2$. A general plane electromagnetic wave with wave vector $k$ ($k \cdot \varepsilon_i = 0$, $k \times \varepsilon_1 = k\varepsilon_2$) is given by

$$E = (E_1\varepsilon_1 + E_2\varepsilon_2)\exp(i(k \cdot r - \omega t)) \quad (2.283)$$
$$B = \frac{c}{\omega} k \times E \quad (2.284)$$

In general the amplitudes $E_1$ and $E_2$ are complex numbers.

If the phases of $E_1$ and $E_2$ differ by an $n\pi$ the wave is linearly polarized along the line at an angle of $\theta = \tan^{-1}((-1)^n E_2/E_1)$ with $\varepsilon_1$. In this case the amplitude of the wave is $\sqrt{E_1^2 + E_2^2}$.

An alternate representation of the wave can be given in terms of the complex basis vectors $\varepsilon_{\pm}$

$$\varepsilon_{\pm} = 2^{-1/2} (\varepsilon_1 \pm i\varepsilon_2) \quad (2.285)$$

$$\varepsilon_{\pm} \cdot \varepsilon_{\mp}^* = 1$$
$$\varepsilon_{\pm} \cdot \varepsilon_{\pm}^* = 0$$
$$\varepsilon_{\mp}^* = \varepsilon_{\pm}$$

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Using these the electric field for a general plane wave is

\[ \mathbf{E} = 2^{-1/2} [(E_1 - iE_2) \varepsilon_+ + (E_1 + iE_2) \varepsilon_-] \exp [i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \]  

(2.286)

A wave described by \( \mathbf{E}(\mathbf{r}, t) = E_0 \varepsilon_\pm \exp (i[\mathbf{k} \cdot \mathbf{r} - \omega t]) \) is a circularly polarized wave. The components of this wave are

\[ E_1(\mathbf{r}, t) = 2^{-1/2} |E_0| \cos (\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_0) \]  

(2.287)

\[ E_2(\mathbf{r}, t) = \mp 2^{-1/2} |E_0| \sin (\mathbf{k} \cdot \mathbf{r} - \omega t + \phi_0) \]  

(2.288)

With the upper sign of \( \varepsilon_\pm \) the wave is said to be left hand circularly polarized and with the lower sign it is right hand circularly polarized. This is determined by the direction of rotation of the electric field in the advancing wave (at a fixed time not at a point). An observer (when facing the oncoming wave) sees the \( \mathbf{E} \) rotating counterclockwise (clockwise) at a fixed space point for the \( \varepsilon_+ \) (\( \varepsilon_- \)) polarization. In modern terminology the left hand circularly polarized wave has positive helicity while the right hand circularly polarized wave has negative helicity. The sign is the sign of the projection of the angular momentum of the wave on the direction of wave propagation.

The general polarized wave has elliptical polarization. It is most easily seen using the circularly polarized waves. The semimajor axis is reached when the counter-circulating electric fields are aligned. Its length is therefore

\[ a = \sqrt{2} (|E_+| + |E_-|) \]

. The semiminor axis is reached when the fields are anti-aligned resulting in the length

\[ b = \sqrt{2} (||E_+| - |E_-||) \]

. If the waves are in phase the major axis is along \( \varepsilon_1 \) and the minor axis is along \( \varepsilon_2 \). Let \( \phi_+ \) be the phase of the positive helicity wave and \( \phi_- \) the phase of the negative helicity wave. If \( \theta_\pm \) are the angles between the electric fields and \( \varepsilon_1 \) then

\[ \theta_+ = \mathbf{k} \cdot \mathbf{r} - \omega t + \phi_+ \quad \text{and} \]

\[ \theta_- = -\mathbf{k} \cdot \mathbf{r} + \omega t + \phi_- \]

The alignment of these two vectors resulting in the semimajor axis occurs for \( \theta_+ - \theta_- = 0, 2N\pi \). This has solutions

\[ \mathbf{k} \cdot \mathbf{r} - \omega t = -0.5 (\phi_+ + \phi_-), -0.5 (\phi_+ + \phi_-) + N\pi \]

. The resultant angles with \( \varepsilon_1 \) are

\[ \theta_{\text{major}} = 0.5 (\phi_+ - \phi_-), 0.5 (\phi_+ - \phi_-) + N\pi. \]

### 2.6.1 Stoke’s parameters

The polarization of an electromagnetic wave can be determined by intensity measurements. These measurements yield the magnitude of the amplitudes of \( \varepsilon_\pm \) and \( \varepsilon_{1,2} \) and the relative phases of these amplitudes. In terms of the field expression the amplitudes are

\[ a_\pm e^{i\delta_\pm} = \varepsilon_\pm \cdot \mathbf{E}(\mathbf{r}, t) \]  

(2.289)

\[ a_j e^{i\delta_j} = \varepsilon_j \cdot \mathbf{E}(\mathbf{r}, t) \]  

(2.290)

The measurements are the following:

1. The total intensity

\[ s_0 = a_1^2 + a_2^2 = a_+^2 + a_-^2 = \]  

(2.291)

2. The difference in linear polarization intensities along \( \varepsilon_1 \) and \( \varepsilon_2 \)

\[ s_1 = a_1^2 - a_2^2 = 2a_+ a_- \cos (\delta_- - \delta_+) \]  

(2.292)
Consider a plane electromagnetic wave (wave vector $k$ and angular frequency $\omega$) travelling in a medium with a permittivity $\varepsilon_i(\omega)$ and permeability $\mu_i(\omega)$. We assume that this medium fills the space $z<0$ and that the region $z>0$ is filled with a medium with a permittivity $\varepsilon_t(\omega)$ and permeability $\mu_t(\omega)$. From our investigations of Maxwell’s equations we have learned that the tangential components of $\mathbf{E}$ and $\mathbf{H}$ and the normal components of $\mathbf{D}$ and $\mathbf{B}$ are continuous at the interface between the materials. Without any loss of generality we can assume that the electric field part of the wave is described by the real part of

$$E_i(r, t) = E_i \exp \left[ i (k_1 x + k_3 z - \omega t) \right],$$

i.e., the plane of incidence is perpendicular to the $y$ axis. The continuity conditions at the interface require that the transmitted and reflected electric fields have the same spatial variation as the incident field along the planar interface,

$$E_t(r, t) = E_t \exp \left[ i (k_1 x + k'_3 z - \omega t) \right]$$

$$E_r(r, t) = E_r \exp \left[ i (k_1 x - k'_3 z - \omega t) \right]$$

with

$$k_1^2 + k_3^2 = \varepsilon_i(\omega) \mu_i(\omega) \frac{\omega^2}{c^2}$$

$$k_1^2 + k'_3^2 = \varepsilon_t(\omega) \mu_t(\omega) \frac{\omega^2}{c^2}$$

This requirement is seen to yield Snell’s law of refraction along with the ‘law of reflection’ at interfaces between media.
Section 2.7 Reflection and refraction at a plane interface between dielectrics

To relate the various fields to the electric fields we consider the spatial and temporal Fourier transform of the source free Maxwell’s equations along with the constituent equations for an infinite medium.

\[
\begin{align*}
B_0 &= \frac{c}{\omega} k \times E_0 \quad (2.303) \\
H_0 &= \mu(\omega)^{-1} B_0 \quad (2.304) \\
D_0 &= \varepsilon(\omega) E_0 \quad (2.305)
\end{align*}
\]

It is convenient to consider two cases, the first with the incident electric field perpendicular to the plane of incidence

\[
E_i = E_i e_2 \quad (2.306)
\]

and the second with the incident electric field parallel to the plane of incidence

\[
E_i = E_i k_3 e_1 - k_1 e_3 \quad (2.307)
\]

In each case the electric fields of the transmitted and reflected waves will lie in the same planes as the incident wave.

2.7.1 Electric field perpendicular to plane of incidence

With the incident electric field perpendicular to the plane of incidence the electric fields of the transmitted and reflected waves will have the amplitudes

\[
\begin{align*}
E_t &= E_t e_2 \\
E_r &= E_r e_2
\end{align*}
\]

The continuity requirement for the component of the electric field which is tangential to the interface gives

\[
E_i + E_r = E_t \quad (2.310)
\]

The tangential components of the magnetic field vector, \(H\), for the incident, transmitted\(^8\), and reflected waves are

\[
\begin{align*}
[H_i]_1 &= -c\mu_i(\omega)^{-1} k_3 E_i / \omega \\
[H_t]_1 &= -c\mu_t(\omega)^{-1} k_3' E_t / \omega \\
[H_r]_1 &= c\mu_i(\omega)^{-1} k_3 E_i / \omega
\end{align*}
\]

The continuity requirement for the component of the magnetic field which is tangential to the interface gives

\[
\mu_i(\omega)^{-1} k_3 [E_i - E_r] = \mu_t(\omega)^{-1} k_3' E_t \quad (2.314)
\]

The solution of Equations 2.310 and 2.314 is

\[
\begin{align*}
E_r &= \frac{\mu_t(\omega) k_3 - \mu_i(\omega) k_3'}{\mu_i(\omega) k_3 + \mu_i(\omega) k_3'} E_i \\
E_t &= \frac{2\mu_t(\omega) k_3}{\mu_i(\omega) k_3 + \mu_i(\omega) k_3'} E_i
\end{align*}
\]

The average component of the incident Poynting vector which is perpendicular to the interface is

\[
[S_i]_{avg} = \text{Re} (k_3 / \mu_i(\omega)) |E_i|^2 \quad (2.317)
\]

The corresponding average component of the reflected Poynting vector is

\[
[S_r]_{avg} = -\text{Re} (k_3 / \mu_i(\omega)) |E_i|^2 \left| \frac{\mu_t(\omega) k_3 - \mu_i(\omega) k_3'}{\mu_i(\omega) k_3 + \mu_i(\omega) k_3'} \right|^2 \quad (2.318)
\]

\(^8\) It is assumed here that \(k_3'\) is real. If it is complex then the imaginary part is positive, i.e., the wave is damped out going into the second medium.
Section 2.7 Reflection and refraction at a plane interface between dielectrics

Generally the ratio of this reflected component to the incident component gives the fraction of the incident intensity which is reflected

\[
R_\perp = \frac{\mu_t(\omega) k_3 - \mu_i(\omega) k_3'}{\mu_t(\omega) k_3 + \mu_i(\omega) k_3'}^2
\]

(2.319)

The component of the average transmitted Poynting vector which is perpendicular to the interface is

\[
[S_t]_{\text{avg}} = \text{Re} \left( \frac{k_3'}{\mu_t(\omega)} \right) \frac{2 \mu_t(\omega) k_3}{\mu_t(\omega) k_3 + \mu_i(\omega) k_3'} |E_i|^2
\]

(2.320)

and the fraction of the incident intensity which is transmitted is

\[
T_\perp = \frac{k_3'/\mu_t(\omega) + \text{c.c.}}{k_3/\mu_t(\omega) + \text{c.c.}} \left( \frac{2 \mu_t(\omega) k_3}{\mu_t(\omega) k_3 + \mu_i(\omega) k_3'} \right)^2
\]

(2.321)

In the case that \( k_3 \) and the permeabilities are real we have that

\[
1 - R_\perp = \frac{2 \mu_t(\omega) \mu_i(\omega) k_3 (k_3' + \text{c.c.})}{\mu_t(\omega) k_3 + \mu_i(\omega) k_3'} = T_\perp
\]

(2.322)

If \( k_3 \) is complex (as is generally true) an interference between incident and reflected waves occurs.

Assignment 11: The material of a thin film, in air, has a permittivity given by \( \varepsilon = 10 + 0.5i \). (Assume the permeabilities are one.) Light with a wave number \( k_0 \) (real) is directed at the film at normal incidence. (a) By explicit calculation of each part evaluate the fraction of the intensity which is (1) reflected, (2) transmitted, (3) absorbed for film thickness \( d \) satisfying \( 0 < k_0d < 2\pi \). (b) Repeat part (a) with \( \varepsilon = 10 - 0.5i \). (−absorption is creation) (c) Compare and discuss the structures seen in plots of your results.

2.7.2 Electric field parallel to the plane of incidence With the electric field parallel to the plane of incidence the magnetic fields will be perpendicular to this plane. The magnetic field of the incident wave has an amplitude

\[
H_i = H_1 e_2
\]

(2.323)

and the magnetic fields of the reflected and transmitted waves have amplitudes

\[
H_r = H_r e_2
\]

(2.324)

\[
H_t = H_t e_2.
\]

(2.325)

The fourier transform of the source free Ampere-Maxwell law relates the electric and magnetic fields

\[
k \times H = -\varepsilon(\omega) \frac{\omega}{c} E
\]

(2.326)

This gives the components of the electric fields which are parallel to \( e_1 \),

\[
(E_i)_1 = \frac{ck_3}{\omega \varepsilon(\omega)} H_i
\]

(2.327)

\[
(E_r)_1 = -\frac{ck_3}{\omega \varepsilon(\omega)} H_r
\]

(2.328)

\[
(E_t)_1 = \frac{ck_3'}{\omega \varepsilon(\omega)} H_t
\]

(2.329)
Continuity of the tangential electric fields yields

\[ (E_i)_1 + (E_r)_1 = (E_t)_1 \]  
(2.330)

while continuity of the tangential magnetic fields gives

\[ \frac{\varepsilon_i(\omega)}{k_3} [(E_i)_1 - (E_r)_1] = \frac{\varepsilon_t(\omega)}{k_3} (E_t)_1 \]  
(2.331)

The solution of these equations is

\[ (E_r)_1 = \frac{\varepsilon_i(\omega) k'_3 - \varepsilon_t(\omega) k_3}{\varepsilon_i(\omega) k'_3 + \varepsilon_t(\omega) k_3} (E_i)_1 \]  
(2.332)

\[ (E_t)_1 = \frac{2\varepsilon_i(\omega) k'_3}{\varepsilon_i(\omega) k'_3 + \varepsilon_t(\omega) k_3} (E_i)_1 \]  
(2.333)

At normal incidence Equations 2.315 and 2.316 and Equations 2.332 and 2.333 give identical results

\[ E_r = \frac{[\mu_t(\omega) \varepsilon_i(\omega)]^{1/2} - [\mu_i(\omega) \varepsilon_t(\omega)]^{1/2}}{[\mu_t(\omega) \varepsilon_i(\omega)]^{1/2} + [\mu_i(\omega) \varepsilon_t(\omega)]^{1/2}} E_i \]  
(2.334)

\[ E_t = \frac{2 [\mu_t(\omega) \varepsilon_i(\omega)]^{1/2}}{[\mu_t(\omega) \varepsilon_i(\omega)]^{1/2} + [\mu_i(\omega) \varepsilon_t(\omega)]^{1/2}} E_i \]  
(2.335)

### 2.7.3 Systems with multiple surfaces

In the case that the electromagnetic wave passes through multiple surfaces there will generally be Both incident and reflected waves on each side of the interfaces. The most common example of these systems involve the thin film coating of optical elements. The problem is to design and manufacture a coating which will provide a given reflectance or transmittance between elements. The importance of the problem is reflected in the large number of formulations of the problem and the multitude of computer codes designed to solve the problem.