Appendix 1

Electric Field for $\ell = 0$ in the Radiation zone

Details of the derivation (page 78) for the electric field in the radiation zone, $kr \gg 1$, are given below.

\[ E_0(r) = \frac{ic}{\omega} \nabla \times B_0(r) = \]
\[ = k^2c^2 \frac{\omega}{c} \nabla \times \left[ \frac{r}{r} \times p_0 \right] f(r) \quad \text{where} \quad f(r) = \left[ \frac{\exp(ikr)}{r} \right] \left[ i - \frac{1}{kr} \right] \]
\[ = k \nabla \times \left[ r \times p_0 \right] \frac{1}{r} \frac{d}{dr} \left[ \frac{\exp(ikr)}{kr} \right] \]
\[ \nabla \times \left[ r \times p_0 \right] = \epsilon_{ijk} \hat{\epsilon}_j \epsilon_{kmn} x^m p_0^n \]
\[ = \hat{x}_i \nabla \left[ x^j p_0^j - x^i p_0^i \right] \]
\[ = p_0 - 3p_0 + r(p_0 \cdot \nabla) - p_0(r \cdot \nabla) \]

\[ E_0(r) = k[-2p_0 + r(p_0 \cdot \nabla) - p_0(r \cdot \nabla)] \frac{1}{r} \frac{d}{dr} h(r) \quad \text{where} \quad h(r) = \frac{\exp(ikr)}{kr} \]
\[ = k[r(p_0 \cdot \hat{r}) - p_0(r \cdot \hat{r})] \frac{1}{r} \frac{d}{dr} \left[ \frac{1}{r} h' \right] - 2k p_0 \frac{1}{r} h' \]
\[ = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] r \frac{d}{dr} \left[ \frac{1}{r} h' \right] - 2k p_0 \frac{1}{r} h' \]
\[ = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] [-\frac{1}{r} h' + h''] - 2k p_0 \frac{1}{r} h' \]
\[ = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] [-\frac{1}{r} h' + h''] \quad - 2k p_0 \frac{1}{r} h' \]
\[ = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] [-\frac{1}{r} h' + h'''] \quad - 2k p_0 \frac{1}{r} h' \]
\[ = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] [-k^2 h] + k p_0 \frac{1}{r} h' - k[\hat{r}(p_0 \cdot \hat{r})] \frac{1}{r} h' \]

Note that $h(r) = \frac{\exp(ikr)}{kr} = f_i(kr)$ is a solution to the spherical Bessell equation when $\ell = 0$ and $kr \neq 0$:

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} f_i(kr) \right) + (k^2 - \ell(\ell + 1)/r^2) f_i(kr) = 0 \]
\[ \frac{2}{r} f_i' + f_i'' + [k^2 - \ell(\ell + 1)/r^2] f_i = 0 \]

Finally,

\[ E_0(r) = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] [-k^2 h] - k[3\hat{r}(p_0 \cdot \hat{r}) - p_0] \frac{1}{r} h' \]
\[ = k[\hat{r}(p_0 \cdot \hat{r}) - p_0] [-k^2 h] - k[3\hat{r}(p_0 \cdot \hat{r}) - p_0] \frac{1}{r} h[i - \frac{1}{kr}] \]
\[ = -k^3[\hat{r}(p_0 \cdot \hat{r}) - p_0] \frac{\exp(ikr)}{kr} + k^3[3\hat{r}(p_0 \cdot \hat{r}) - p_0] \frac{1}{kr} \frac{\exp(ikr)}{kr} \left[ \frac{1}{kr} - i \right] \]
\[ = k^3[\hat{r} \times p_0] \frac{\exp(ikr)}{kr} + k^3[3\hat{r}(p_0 \cdot \hat{r}) - p_0] \frac{1}{kr} \left[ \frac{1}{kr} - i \right] \frac{\exp(ikr)}{kr} \]  

A1.1
The last equation, using \((\hat{r} \times p_0) \times \hat{r} = -\hat{r}(p_0 \cdot \hat{r}) + p_0\), converts the result to that given in Jackson, Eq. 9.18 on page 411. Whereas the magnetic induction, \(B\), is perpendicular to the "radius" vector, \(r\), at all distances, the electric field has components parallel to \(r\), and a component parallel to the dipole moment, \(p_0\). This derivation gives the electric dipole fields.

In the radiation zone \(B\) and \(E\) are given by:

\[
B(r, t) \rightarrow k^3 (\hat{r} \times p_0) \left[ \frac{\exp[i(kr - \omega t)]}{kr} \right], \quad \text{for } kr \gg 1
\]

\[
E(r, t) \rightarrow k^3 [\hat{r} \times (\hat{r} \times p_0)] \left[ \frac{\exp[i(kr - \omega t)]}{kr} \right] = B(r, t) \times \hat{r} \quad \text{for } kr \gg 1
\]

The units are given by \(k^3 p_0 \approx q/r^2\). The \(\frac{\exp[i(kr - \omega t)]}{kr}\) is an outgoing spherical wave, with \(k = \omega/c\).

The time averaged Poynting vector is

\[
S(r)_{\text{avg}} = \frac{c}{8\pi} \text{Re}(E_0(r) \times B_0(r)^*)
\]

\[
|S(r)_{\text{avg}}| = \frac{ck^4}{8\pi} \frac{1}{r^2} \left[ (\hat{r} \cdot p_0) |\hat{r}| - (\hat{r} \cdot p_0) \right]
\]

\[
= \frac{ck^4}{8\pi} \frac{1}{r^2} \left[ (\hat{r} \cdot p_0)|\hat{r}|(\hat{r} \cdot p_0) - (\hat{r} \cdot p_0)(\hat{r} \cdot p_0)\right] - (\hat{r} \cdot p_0) + p_0(\hat{r} \cdot p_0)
\]

\[
= \frac{ck^4}{8\pi} \frac{1}{r^2} \left[ (\hat{r} \cdot p_0)^2 - (\hat{r} \cdot p_0)(\hat{r} \cdot p_0)\right] - (\hat{r} \cdot p_0) + p_0(\hat{r} \cdot p_0)
\]

\[
= \frac{ck^4}{8\pi} \frac{1}{r^2} \left[ (\hat{r} \cdot p_0)^2 - (\hat{r} \cdot p_0)(\hat{r} \cdot p_0)\right] - (\hat{r} \cdot p_0) + p_0(\hat{r} \cdot p_0)
\]

\[
= \frac{ck^4}{8\pi} \frac{1}{r^2} \left[ (\hat{r} \cdot p_0)^2 - (\hat{r} \cdot p_0)(\hat{r} \cdot p_0)\right] - (\hat{r} \cdot p_0) + p_0(\hat{r} \cdot p_0)
\]

\[
= \frac{ck^4}{8\pi} \frac{1}{r^2} \left[ (\hat{r} \cdot p_0)^2 - (\hat{r} \cdot p_0)(\hat{r} \cdot p_0)\right] - (\hat{r} \cdot p_0) + p_0(\hat{r} \cdot p_0)
\]

From Eq. (A1.4) one can see that the power is radiated only along the radial direction and the angular distribution is \(\sin^2 \theta\) where \(\theta\) is the angle between the radius vector and the dipole moment direction. \(S\) is the energy flux, or energy/area per unit time = power per unit area. So, \(dP_{\text{ave}}\) is given by
\[ dP_{\text{ave}} = |S(r)_{\text{avg}}| dA \]
\[ = |S(r)\text{avg}| r^2 d\Omega \]
\[ \frac{dP_{\text{ave}}}{d\Omega} = |S(r)\text{avg}| r^2 \]
\[ = \frac{ck^4}{8\pi} |p_0|^2 [1 - \cos^2 \theta] \]

The total time averaged power radiated by the dipole is proportional to the fourth power of the oscillation frequency:

\[ P_{\text{total}} = \int \int \frac{ck^4}{8\pi} |p_0|^2 [1 - \cos^2 \theta] d\Omega \]
\[ = \frac{ck^4}{8\pi} |p_0|^2 2\pi [1 - \cos^2 \theta](-\cos \theta) \]
\[ = \frac{ck^4}{8\pi} |p_0|^2 2\pi [2 - \frac{2}{3}] \]
\[ = \frac{ck^4}{3} |p_0|^2 \]
\[ = \frac{\omega^4}{3c^3} |p_0|^2 . \]

**Electric Field for \( l = 1 \) in the Radiation zone**

As found in the last section the \( l = 0 \) term in the expansion involves only the electric dipole characteristic of the source current. In this section we find that both the electric quadrupole and magnetic dipole contribute to the \( l = 1 \) term. In terms of the Cartesian coordinates

\[ rY^e_{1m}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} [z\delta_{m0} - \frac{1}{\sqrt{2}} (x + iy)\delta_{m1} + \frac{1}{\sqrt{2}} (x - iy)\delta_{m-1}] \]
\[ = \sqrt{\frac{3}{4\pi}} [z\delta_{m0} + \frac{1}{\sqrt{2}} (\delta_{m-1} - \delta_{m1})x - i\frac{1}{\sqrt{2}} (\delta_{m1} + \delta_{m-1})y] \]

Since the \( rY^e_{1m}(\theta, \phi) \) contain terms proportional to \( x, y, \) and \( z \) (or \( x_i \) with \( i = 1, 2, \) and 3) one looks for a way to convert an expression which converts \( J_0(r')r'Y^e_{1m} \) into a term containing \( \rho_0(r) \). The standard approach is to start with \( \nabla \cdot x_i x_j J(r) \) and use

\[ \nabla \cdot J(r) = -\frac{d}{dt}\rho(r) = i\omega \rho(r) \]

after some manipulations.

\[ \nabla \cdot x_i x_j J(r) = \sum_{n=1}^{3} \frac{\partial}{\partial r_n} [J_n(r)x_i x_j] = x_i x_j \nabla \cdot J(r) + x_j J_i(r) + x_i J_j(r) \]
\[ x_j J_i(r) = -\frac{1}{2} x_i x_j \nabla \cdot J(r) - \frac{1}{2} x_j J_i(r) + \frac{1}{2} x_i J_j(r) + \frac{1}{2} \sum_{n=1}^{3} \frac{\partial}{\partial r_n} [J_n(r)r_i r_j] \]

It follows that the integrand for \( l = 1 \) the terms in the integrand have the form
\[ x_i J_i(r) = -\frac{1}{2} x_i x_j \nabla \cdot J(r) + \frac{1}{2} (x_i J_j(r) - x_j J_i(r)) + \frac{1}{2} \sum_{n=1}^{3} \frac{\partial}{\partial r_n} [J_n(r)x_i x_j] \]
\[ = \frac{1}{2} x_i x_j \frac{\partial}{\partial t} \rho(r) + \frac{1}{2} (x_i J_j(r) - x_j J_i(r)) + \frac{1}{2} \sum_{n=1}^{3} \frac{\partial}{\partial r_n} [J_n(r)x_i x_j] \]
\[ x_i J_{0i}(r) = -\frac{1}{2} i \omega x_i x_j \rho_0(r) + \frac{1}{2} (x_i J_{0j}(r) - x_j J_{0i}(r)) + \frac{1}{2} \sum_{n=1}^{3} \frac{\partial}{\partial r_n} [J_{0n}(r)x_i x_j] \]
\[ x_i J_0(r) = -\frac{1}{2} i \omega x_i r \rho_0(r) + \frac{1}{2} (x_i J_0(r) - r J_0(r)) + \frac{1}{2} \nabla \cdot [J_0(r)x_i r] \]

The first term gives the quadrupole moment of the charge distribution:
\[ \left[ \iiint J_0(r') r' Y_{1,m}(\theta', \phi')^* d^3 r' \right]_{\text{quad}} = -\frac{i \omega}{2} \iiint \rho_0(r') r' r' Y_{1,m}(\theta', \phi') d^3 r' \]

The second term gives the magnetic moment of the current distribution, and the third term gives zero (one converts the divergence to a surface integral outside of and enclosing the source where there is no current density). As one can see from this analysis the Cartesian form of the expansion provides reasonably simple expressions for the low lying terms in the multipole expansion. It is when higher moments of the current and charge distributions are important that the power of the multipole expansion is useful. In this case the rotational properties of the current distribution can be used to select the appropriate multipole in the expansion. This can be rewritten using the unit vectors
\[ e_\pm = \frac{1}{\sqrt{2}} [\pm e_1 + i e_2] \]
\[ e_0 = e_3 \]
\[ e_1 = \frac{1}{\sqrt{2}} [e_+ - e_-] \]
\[ e_2 = \frac{1}{\sqrt{2}} [e_+ + e_-] \]

In this basis
\[ r' = r' \sqrt{\frac{4\pi}{3}} [Y_{1,-1}(\theta', \phi') e_+ + Y_{1,1}(\theta', \phi') e_- + Y_{1,0}(\theta', \phi') e_0] \]
and
\[ \left[ \iiint J_0(r') r' Y_{1,m}(\theta', \phi')^* d^3 r' \right]_{\text{quad}} = -\frac{i \omega}{2} \iiint \rho_0(r') r' \sqrt{\frac{4\pi}{3}} [Y_{1,-1}(\theta', \phi') e_+ + Y_{1,1}(\theta', \phi') e_- + Y_{1,0}(\theta', \phi') e_0] r' Y_{1,m}(\theta', \phi') d^3 r' \]
\[ \left[ \iiint J_0(r') r' Y_{1,m}(\theta', \phi')^* d^3 r' \right]_{\text{quad}} = -\frac{i \omega}{2} \sqrt{\frac{4\pi}{3}} \sum_{m=-1}^{+1} e_m' \iiint \rho_0(r') r'^2 Y_{1,m}(\theta', \phi') Y_{1,m}(\theta', \phi') d^3 r' \]

In this form we can use the coupling of the spherical harmonics (‘vector coupling’ for angular momentum). For this particular case we need the coupling of two \( l = 1 \) ‘states’. In Jackson’s notation
In this case

\[ Y_{11}(\theta, \phi)^2 = \sqrt{\frac{3}{10\pi}} Y_{22}(\theta, \phi) \]

\[ Y_{10}(\theta, \phi)Y_{11}(\theta, \phi) = \sqrt{\frac{3}{20\pi}} Y_{21}(\theta, \phi) \]

\[ Y_{1-1}(\theta, \phi)Y_{11}(\theta, \phi) + Y_{00}(\theta, \phi)^2 = \sqrt{\frac{3}{10\pi}} Y_{20}(\theta, \phi) \]

\[ -2Y_{1-1}(\theta, \phi)Y_{11}(\theta, \phi) + Y_{00}(\theta, \phi)^2 = \frac{3}{\sqrt{4\pi}} Y_{00}(\theta, \phi) \]

with

\[ Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}(\theta, \phi)^*. \]

This approach is particularly useful if the charge distribution has a simple symmetry, say it is

\[ \rho_0(\mathbf{r}) = \sqrt{\frac{10\pi}{3}} Q \exp(-ar) Y_{22}(\theta, \phi). \]

In this case

\[
\left[ \iiint J_0(\mathbf{r}') r' Y_{1m}(\theta', \phi')^* d^3r' \right]_{quad}
= -\frac{i\omega}{2} \sqrt{\frac{4\pi}{3}} \sum_{m' = -1}^{+1} \mathbf{e}_{m'} \left[ \iiint \rho_0(\mathbf{r}') r'^2 Y_{1m}^*(\theta', \phi') Y_{1m}(\theta', \phi') d^3r' \right]
= -\frac{i\omega}{2} \sqrt{\frac{4\pi}{3}} \sum_{m' = -1}^{+1} \mathbf{e}_{m'} \int_0^\infty Q \exp(-ar') r'^4 dr' \left[ \int \sqrt{\frac{10\pi}{3}} Y_{22}(\theta', \phi') (-1)^m Y_{1m}^*(\theta', \phi') Y_{1m}(\theta', \phi') d\Omega' \right]
= \frac{i\omega}{2} \sqrt{\frac{4\pi}{3}} \sum_{m' = -1}^{+1} \mathbf{e}_{m'} \int_0^\infty Q \exp(-ar') r'^4 dr' \delta_{m1} \int Y_{22}(\theta', \phi') Y_{22}(\theta', \phi') d\Omega'
= \frac{i\omega}{2} \sqrt{\frac{4\pi}{3}} \mathbf{e}_+ \int_0^\infty Q \exp(-ar') r'^4 dr' \delta_{m1} = i12 \omega \sqrt{\frac{4\pi}{3}} Q 2a^{-5} \delta_{m1} \mathbf{e}_+
\]

The vector potential for this charge density is

\[ \mathbf{A}_0(\mathbf{r}) \bigg|_{l=1,quad} = \frac{4\pi k}{c} \sum_{m=-1}^{+1} \frac{k}{3} \exp \left[ i \frac{kr - \pi}{2} \right] Y_{1m}(\theta, \phi) i12 \omega \sqrt{\frac{4\pi}{3}} Q 2a^{-5} \delta_{m1} \mathbf{e}_+. \]

\[ \mathbf{A}_0(\mathbf{r}) = 16\pi i \frac{1}{3} \frac{k^2}{\alpha^2} Q 2 \exp(ikr)(-i) \sqrt{\frac{4\pi}{3}} Y_{1,1}(\theta, \phi) \mathbf{e}_+ \]

\[ = -8\sqrt{2} \pi \frac{k^2}{\alpha^2} Q 2 \exp[ikr] \sin(\theta) e^{i\phi} \mathbf{e}_+ \]

In the radiation region the magnetic induction is given by

\[ \mathbf{B}_0(\mathbf{r}) \rightarrow 8\sqrt{2} \pi \frac{k^3}{\alpha^2} Q 2 \exp[ikr] \sin(\theta) e^{i\phi} [\cos \theta \mathbf{e}_+ - 2^{-1/2} \sin \theta \ e^{i\phi} \mathbf{e}_3] \]

and the electric field is

\[ \mathbf{E}_0(\mathbf{r}) = -i \frac{\mathbf{E}_0}{\alpha} \times \mathbf{B}_0(\mathbf{r}) \]

The Poynting vector for the radiation fields is
The system does not radiate along the z axis.

Note that for any value of $\ell$ there is a contribution from the charge density $\ell + 1$ moment:

$$
\left[ \iiint J_0(r') r'^\ell Y_{1m}(\theta', \phi')^* d^3 r' \right]_{\ell + 1} = -\frac{i\omega}{2} \sqrt{\frac{4\pi}{3}} \sum_{m' = -1}^{+1} e^{*}_{m'} \iiint \rho_0(r') r'^\ell Y_{1m'}(\theta', \phi') Y_{1m}(\theta', \phi')^* d^3 r'
$$