Chapter 5
Cylindrical Cavities and Waveguides

We shall consider an electromagnetic field propagating inside a hollow (in the present case cylindrical) conductor. There are no sources inside the conductor, but we shall assume the material is isotropic with electric permittivity $\varepsilon$, and magnetic permeability, $\mu$. The speed of the propagating wave is $1/\sqrt{\varepsilon\mu}$. The direction of propagation will be along the cylindrical axis which is the $\hat{z}$ direction. We shall assume that $E(r, t) = E(r)e^{-i\omega t}$ and $B(r, t) = B(r)e^{-i\omega t}$. Maxwell’s equations give:

$$[\nabla^2 - \varepsilon\mu \frac{\partial^2}{\partial t^2}]E(r, t) = 0$$  

$$[\nabla^2 - \varepsilon\mu \frac{\partial^2}{\partial t^2}]B(r, t) = 0$$  

$$[\nabla^2 + \varepsilon\mu \omega^2]B(r) = 0$$  

$$[\nabla^2 + \varepsilon\mu \omega^2]E(r) = 0$$  

5.1  
5.2  
5.3  
5.4

Since the wave is propagating along the $\hat{z}$ direction we shall further assume that:

$$\nabla \times E = i\omega B(r);$$  

$$\nabla \times B = -i\omega \varepsilon\mu E(r).$$  

5.5  
5.6

Since the wave is propagating along the $\hat{z}$ direction we shall further assume that:

$$E(r) = E(x, y)e^{\pm jkz}$$  

$$B(r) = B(x, y)e^{\pm jkz}$$  

5.7a  
5.7b

Thus Eq. (5.3 and 5.4) become

$$[\nabla^2 + \varepsilon\mu \omega^2 - k^2]E(x, y) = 0$$  

$$[\nabla^2 + \varepsilon\mu \omega^2 - k^2]B(x, y) = 0$$  

5.8a  
5.8b

where
Finally, one can solve for

\[ \nabla^2_i = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]

\[ \nabla_i = \hat{x}\nabla_x + \hat{y}\nabla_y \]

The expressions in Eqs. 5.5 and 5.6 then become:

\[ \nabla \times \mathbf{E} = [\hat{z}\nabla_z + \nabla_i] \times [\hat{z}\mathbf{E}_z + \mathbf{E}_t] = i\omega \mathbf{B}(r) \] where
\[ \mathbf{E}_t = \mathbf{E} - \hat{z}\mathbf{E}_z = (\hat{z} \times \mathbf{E}) \times \hat{z} \]
\[ \mathbf{B}_t = \mathbf{B} - \hat{z}\mathbf{B}_z = (\hat{z} \times \mathbf{B}) \times \hat{z} \]

Then

\[ \nabla \times \mathbf{E} = \hat{z}\nabla_z \mathbf{E}_t - \hat{z} \times \nabla_i \mathbf{E}_z + \nabla_i \times \mathbf{E}_t = i\omega (\mathbf{B}_t + \hat{z}\mathbf{B}_z) \]
\[ \nabla \times \mathbf{B} = \hat{z}\nabla_z \mathbf{B}_t - \hat{z} \times \nabla_i \mathbf{B}_z + \nabla_i \times \mathbf{B}_t = -i\omega \epsilon \mu (\mathbf{E}_t + \hat{z}\mathbf{E}_z) \]

Thus,

\[ \hat{z} \times \nabla_z \mathbf{E}_t - \hat{z} \times \nabla_i \mathbf{E}_z = -i\omega \hat{z} \times (\hat{z} \times \mathbf{B}) \]
\[ \nabla_z \mathbf{E}_t + i\omega (\hat{z} \times \mathbf{B}_t) = \nabla_i \mathbf{E}_z \quad \text{(5.9)} \]
\[ \hat{z} \cdot (\nabla_i \times \mathbf{E}_t) = i\omega \mathbf{B}_z \quad \text{(5.10)} \]
\[ \nabla_z \mathbf{B}_t - i\omega \epsilon \mu (\hat{z} \times \mathbf{E}_t) = \nabla_i \mathbf{B}_z \quad \text{(5.11)} \]
\[ \hat{z} \cdot (\nabla_i \times \mathbf{B}_t) = -i\omega \epsilon \mu \mathbf{E}_z \quad \text{(5.12)} \]

Also,

\[ \nabla_i \mathbf{E}_t + \nabla_z \mathbf{E}_z = 0 \quad \text{(5.13a)} \]
\[ \nabla_i \mathbf{B}_t + \nabla_z \mathbf{B}_z = 0 \quad \text{(5.13b)} \]

Finally, one can solve for \( \mathbf{E}_t \) and \( \mathbf{B}_t \) if \( \mathbf{E}_z \) and \( \mathbf{B}_z \) are known (and not both are zero).

\[ i\kappa \mathbf{E}_t = \nabla_i \mathbf{E}_z - i\omega (\hat{z} \times \mathbf{B}_t) \]
\[ i\kappa \mathbf{B}_t = \nabla_i \mathbf{B}_z + i\omega \epsilon \mu (\hat{z} \times \mathbf{E}_t) \]
\[ i\kappa (\hat{z} \times \mathbf{B}_t) = (\hat{z} \times \nabla_i \mathbf{B}_z) + i\omega \epsilon \mu (\hat{z} \times (\hat{z} \times \mathbf{E}_t)) \]

and

\[ i\kappa \mathbf{E}_t = \nabla_i \mathbf{E}_z - (i\omega \kappa) (\hat{z} \times \nabla_i \mathbf{B}_z) + (\omega^2 \epsilon \mu/k^2)(-\mathbf{E}_t) \]
\[ (\omega^2 \epsilon \mu - k^2) \mathbf{E}_t = i\kappa \nabla_i \mathbf{E}_z - i\omega (\hat{z} \times \nabla_i \mathbf{B}_z) \]
\[ \mathbf{E}_t = i(\omega^2 \epsilon \mu - k^2)^{-1}[k
abla_i \mathbf{E}_z - \omega (\hat{z} \times \nabla_i \mathbf{B}_z)] \] likewise
\[ (\omega^2 \epsilon \mu - k^2) \mathbf{B}_t = i(\omega^2 \epsilon \mu - k^2)^{-1}[k
abla_i \mathbf{B}_z + \omega \epsilon \mu (\hat{z} \times \nabla_i \mathbf{E}_z)] \]

For waves in the opposite direction change \( k \) to \( -k \).
Transverse electromagnetic wave (TEM):  $E_z$ and $B_z$ are zero everywhere inside cylinder.

For TEM waves $E_{TEM} = E_t$:

\[
\begin{align*}
\mathbf{\nabla} \times E_{TEM} &= 0 \quad 5.15a \\
\mathbf{\nabla} \cdot E_{TEM} &= 0 \quad 5.15b \\
k = k_0 = \omega \sqrt{\varepsilon \mu} \quad 5.15c \\
B_{TEM} &= \pm \omega \sqrt{\varepsilon \mu} (\hat{z} \times E_{TEM}) \quad 5.15d
\end{align*}
\]

Unfortunately, the TEM wave is not supported by a single hollow cylindrical conductor (with infinite conductivity). The surface must be an equipotential surface and inside such a conductor, the electric field vanishes. One needs two or more cylindrical surfaces (such as a coaxial cable) to support a TEM wave.

Boundary conditions at the surface

The existence of surface charge densities, $\sigma$, and surface current densities, $K$, at the interface provide the following boundary conditions:

\[
\begin{align*}
\hat{n} \cdot (D - D_c)|_S &= \sigma \\
\hat{n} \cdot (B - B_c)|_S &= 0 \\
\hat{n} \times (E - E_c)|_S &= 0 \\
\hat{n} \times (H - H_c)|_S &= K
\end{align*}
\]

In the conductor the electric field, $E_c$, (and for time varying electric fields $B_c$) is zero. Thus, inside the hollow cylinder the boundary conditions can only be satisfied at the interface when

\[
\begin{align*}
(\hat{n} \times E)|_S &= 0 \quad 516a \\
\hat{n} \cdot B|_S &= 0. \quad 5.16b
\end{align*}
\]

That is, the component of the electric field tangent to the interface ($E_z$) must equal zero at the surface:

\[
E_z|_S = 0 \quad 5.17
\]

The corresponding condition on $B_z$ is (see Eq. (5.11)):

\[
\hat{n} \cdot \nabla B_z|_S = \nabla \cdot B_z|_S = 0
\]

Since we can not have both $E_z$ and $B_z$ equal to zero everywhere inside the cylinder, there are two simple cases which satisfy the boundary conditions:
Transverse Magnetic (TM) wave: \( B_z = 0 \) everywhere and \( E_z|_S = 0 \)

\[ E_t = i(\omega^2 \varepsilon \mu - k^2)^{-1}k \nabla_t E_z \]  
\[ B_t = i(\omega^2 \varepsilon \mu - k^2)^{-1}\omega \varepsilon \mu (\hat{z} \times \nabla_t E_z) \]  
\[ 0 = [\nabla_t^2 + \varepsilon \mu \omega^2 - k^2]E_z(x, y) \]

Transverse Electric (TE) wave: \( E_z = 0 \) everywhere and \( \frac{\partial B_z}{\partial n}|_S = 0 \)

\[ B_t = i(\omega^2 \varepsilon \mu - k^2)^{-1}k \nabla_t B_z \]  
\[ E_t = i(\omega^2 \varepsilon \mu - k^2)^{-1}[-\omega (\hat{z} \times \nabla_t B_z)] \]  
\[ 0 = [\nabla_t^2 + \varepsilon \mu \omega^2 - k^2]B_z(x, y) \]

The differential equations (5.18d) for \( E_z \) and (5.19d) for \( B_z \) and the boundary conditions (5.18) and (5.19) give rise to eigenvalues of \( k \) (dependent on \( \omega \)) for which the propagation is allowed. Since the boundary conditions for \( E_z \) and \( B_z \) are different, the eigenvalues are also different. The allowed TE and TM waves (and the TEM wave, if it exists) provide a complete set of waves from which one can construct an arbitrary electromagnetic disturbance in the waveguide or cavity.

Modes in a rectangular waveguide:

We shall determine the TE modes in a rectangular waveguide with dimensions \( a \) in \( x \) and \( b \) in \( y \) (with \( a > b \)) as shown in Fig. 8.5.

Note that this means \( E_z = 0 \) everywhere (\( E \) is transverse). \( E_t \) will be found from \( B_z \). So, first one must solve Eq. 5.8 for \( B_z \):  
\[
\left[ \nabla_t^2 + \varepsilon \mu \omega^2 - k^2 \right]B_z(x, y) = 0
\]
\[
\left[ -\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mathbf{k}' \cdot \mathbf{k}' \right]B_z(x, y) = 0
\]

The general solution is:

\[ B_z(x, y) = C_1 e^{+i\mathbf{k}' \cdot \mathbf{r}} + C_2 e^{-i\mathbf{k}' \cdot \mathbf{r}}, \text{ where} \]
\[ \mathbf{r} = x\hat{x} + y\hat{y} \]
The form for $B_z(x,y)$ which is non-zero when $x = y = 0$ is:

$$B_z(x,y) = B_o \cos(k'_x x) \cos(k'_y y)$$

5.20

The boundary conditions are

$$\frac{\partial}{\partial n} B_z|_S = 0$$

$$\frac{\partial}{\partial x} B_o \cos(k'_x x) \cos(k'_y y)|_{x=a} = 0$$

$$\frac{\partial}{\partial y} B_o \cos(k'_x x) \cos(k'_y y)|_{y=b} = 0$$

These give:

$$\sin(k'_x a) = \sin(k'_x 0) = 0 \text{ or } k'_x = m\pi/a; \ m = 0, 1, 2, \ldots$$

$$\sin(k'_y b) = \sin(k'_y 0) = 0 \text{ or } k'_y = n\pi/b; \ n = 0, 1, 2, \ldots$$

$$B_z(x,y) = B_o \cos(m\pi x/a) \cos(n\pi y/b)$$

5.22a

$$(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 = \epsilon \mu \omega^2 - k^2; \quad m, n = 0, 1, 2, 3, \ldots$$

$$k^2 = \epsilon \mu \omega^2 - \left[ (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \right] > 0$$

5.22d

$$\omega > \frac{1}{\sqrt{\epsilon \mu}} \left[ (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \right]^{1/2}$$

5.22e

$$\omega_{\min} = \frac{\pi}{a \sqrt{\epsilon \mu}}; \quad m = 1, n = 0$$

5.22f

$$\omega_{mn} = \frac{\pi}{\sqrt{\epsilon \mu}} \left[ (\frac{m}{a})^2 + (\frac{n}{b})^2 \right]^{1/2}$$

5.22g

For a non-trivial solution, $m$ and $n$ can not both be zero. Equation (5.22d) provides a cutoff on the wave vector, $k$, since for $k^2 < 0$ the factor $e^{ikz}$ becomes $e^{\pm ikz}$ and the wave would not propagate. The full solution for each TE$_{mn}$ mode is:

$$\mathbf{B}_t = i(\omega^2 \epsilon \mu - k^2)^{-1}k\nabla B_z e^{i(kz - \omega t)};$$

5.23a

$$\mathbf{B}_t(m,n) = -ik\left[ (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \right]^{-1} B_o, m, n \{ \delta(m\pi x/a) \sin(m\pi x/a) \cos(n\pi y/b) \} e^{i(kz - \omega t)}$$

$$+ \delta(n\pi/b) \cos(m\pi x/a) \sin(n\pi y/b) \} e^{i(kz - \omega t)}$$

$$\mathbf{E}_t = -\frac{\omega}{k}(\hat{\mathbf{z}} \times \mathbf{B}_t)$$

$$k^2 = \epsilon \mu \omega^2 - \left[ (\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2 \right]$$

The solution for $m = 1, n = 0$ is:
\[
\begin{align*}
\mathbf{B}_t &= -i \frac{k\alpha}{\pi} B_o \hat{s} \sin(\pi x/a) \exp i(kz - \omega t) \\
\hat{z} B_z &= \hat{z} B_o \cos(\pi x/a) \exp i(kz - \omega t) \\
\mathbf{E}_t &= i \frac{oa}{\pi} B_o \hat{s} \sin(\pi x/a) \exp i(kz - \omega t)
\end{align*}
\]

There is no propagation for
\[
\omega < \frac{1}{\epsilon\mu} \left( \frac{\pi}{a} \right)^2
\]

Note the 90 degree phase difference between the $B_x$ and $B_z$ arising from the $-i = e^{-i\pi/2}$ factor. The $\mathbf{B}_t$ and $\mathbf{E}_t$ are 180 degrees out of phase.

Usually one designs the wave guide so that the $m = 1, n = 0$ mode is the dominant TE mode. One can define the general $k_{mn}$ as follows:

\[
\begin{align*}
k_{mn} &= \sqrt{\epsilon\mu} \sqrt{\omega^2 - \omega_{mn}^2} \\
\omega_{mn} &= \frac{\pi}{\sqrt{\epsilon\mu}} \left[ \left( \frac{m}{a} \right)^2 + \left( \frac{n}{b} \right)^2 \right]^{1/2}
\end{align*}
\]

For each mode, the $k_{mn}$ varies with frequency $\omega > \omega_{mn}$. The $\omega_{mn}$ is the cutoff frequency for the mode. In Fig. 8.4 from Jackson is a plot of $k_{mn}/(\omega \sqrt{\epsilon\mu})$ as a function of $\omega$, where $k_\lambda = k_{mn}$.

It is often convenient to choose the dimensions of the waveguide so that at the operating frequency only the lowest mode can occur. Since the wave number, $k_{mn}$, is always less than the "free space" value, $\sqrt{\epsilon\mu} \omega$, the wavelength in the waveguide is always larger than the free space wavelength.

For the TM modes:

![Figure 8.4](image)  
Figure 8.4 Wave number $k_\lambda$ versus frequency $\omega$ for various modes $\lambda$. $\omega_{\lambda}$ is the cutoff frequency.
\( E_z = E_0 \sin(m\pi x/a) \sin(n\pi y/b) e^{i(kz - \omega t)} \)  

\( E_z |_{S} = 0 \) at \( x = 0, a, \) and \( y = 0, b \)

\[
E_t = i(\omega^2 \epsilon \mu - k^2)^{-1} k \nabla E_z
\]

\[
= i(\omega^2 \epsilon \mu - k^2)^{-1} k E_0 [\hat{x}(m\pi/a) \cos(m\pi x/a) \sin(n\pi y/b) \\
+ \hat{y}(n\pi/b) \sin(m\pi x/a) \cos(n\pi y/b)] e^{i(kz - \omega t)}
\]

\[
B_t = \frac{\omega \epsilon \mu}{k} (\hat{\mathbf{z}} \times E_t)
\]

\[
k^2 = \epsilon \mu \omega^2 - \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right]
\]

In the TM modes if \( n = 0 \) or if \( m = 0, E_z = 0. \) Hence \( E_t \) and \( B_t \) are also zero. The next possible mode is \( n = m = 1 \) with

\[
\omega_{11} = \frac{\pi \sqrt{\epsilon \mu}}{a} \left[ \left( \frac{1}{a} \right)^2 + \left( \frac{1}{b} \right)^2 \right]^{1/2}
\]

\[
\omega_{\text{min TM}} = \omega_{11} > \omega_{\text{min TE}} = \frac{\pi \sqrt{\epsilon \mu}}{a}
\]

Thus \( \text{TE}_{10} \) mode has the smallest cutoff frequency.

Higher order modes:

The following shows the \( E_t \) for some TE modes \( (E_z = 0) \). (Taken from N. Stoyanov, Department of Chemistry, MIT, Ph.D. thesis, 2003)
Summary of TE and TM

TE modes:

\[ E_z = 0; \quad \frac{\partial B_z}{\partial n} \mid_S = 0 \]

\[ B_z(x,y) = B_0 \cos(m \pi x/a) \cos(n \pi y/b) e^{i(kz-\omega t)} \]

\[ B_t = ik \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right]^{-1} \nabla B_z \]

\[ E_t = -\frac{\omega}{k} (\hat{z} \times B_t) \]

\[ k^2 = \varepsilon \mu \omega^2 - \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right] \]

TM modes:

\[ B_z = 0; \quad E_z \mid_S = 0 \]

\[ E_z(x,y) = E_0 \sin(m \pi x/a) \sin(n \pi y/b) e^{i(kz-\omega t)} \]

\[ E_t = ik \left[ \left( \frac{m \pi}{a} \right)^2 + \left( \frac{n \pi}{b} \right)^2 \right]^{-1} \nabla E_z \]

\[ B_t = \frac{\omega \varepsilon \mu}{k} (\hat{z} \times E_t) \]

Energy Flow in the Waveguide for TE Modes

The time averaged flux of energy is given by the real part of the following expression
\[
S = \frac{1}{2} (\mathbf{E} \times \mathbf{H}^*)
= \frac{1}{2\mu} (\mathbf{E}_t \times (\mathbf{B}_t + \mathbf{z} \mathbf{B}_z)^*)
= -\frac{\omega}{2\mu k} (\mathbf{z} \times \mathbf{B}_t) \times (\mathbf{B}_t + \mathbf{z} \mathbf{B}_z)^*
= \frac{\omega}{2\mu k} [\mathbf{z} \mathbf{B}_t \cdot \mathbf{B}_t^* - \mathbf{B}_t \mathbf{B}_z^*]
\]

5.30

\[
\hat{z} \cdot S = \frac{\omega}{2\mu k} \left[ -k^2 \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] \nabla_l B_z \cdot \nabla_l B_z^* - \nabla_l B_z \cdot \nabla_l B_z^*
= C(\omega, m, n) \nabla_l B_z \cdot \nabla_l B_z^*
\]

Now we can integrate this over the cross section of the waveguide to find the power:

\[
P_{TE} = \int \int \hat{z} \cdot S dA
= C(\omega, m, n) \int \int [\nabla_l \cdot \nabla_l B_z^+] dA
= C(\omega, m, n) \int [\nabla_l \cdot (B_z \nabla_l B_z^*) - B_z \nabla_l^2 B_z^+] dA
= C(\omega, m, n) [\oint B_z \nabla_l B_z^* \cdot d\mathbf{r} - \int \int B_z \nabla_l^2 B_z^* dA]
\]

boundary

\[
= C(\omega, m, n) [\oint B_z \nabla_l B_z^* \cdot d\mathbf{r} - \int \int B_z \nabla_l^2 B_z^* dA]
= C(\omega, m, n) [\oint B_z (\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y}) B_z^* \cdot d\mathbf{r} - \int \int B_z \nabla_l^2 B_z^* dA]
= C(\omega, m, n) [\oint B_z (dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y}) B_z^* \cdot d\mathbf{r} - \int \int B_z \nabla_l^2 B_z^* dA]
= 0 + C(\omega, m, n) \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] \int \int B_z B_z^* dA
= \frac{\omega}{2\mu} \left[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right] \int \int B_z B_z^* dA
\]

5.31

The corresponding value for TM modes is

\[
P_{TM} = \frac{\omega^2 \mu}{2\varepsilon \mu} \left[ \sqrt{1 - \frac{\omega_m^2}{\omega^2}} \right] \int \int E_z E_z^* dA
\]

5.32

Thus at fixed frequency, the power is inversely proportional to the \(\omega_m^2\) and the smallest mode numbers correspond to maximal power.