Problem Set 1. (Fall 2006)

The charge density for a hydrogenic p state can be written as

\[ \rho(r, \theta) = \frac{2}{3} \frac{e}{a_0^3 64\pi} \left( \frac{r}{a_0} \right)^2 \exp \left( -\left( \frac{r}{a_0} \right)^2 \right) \left[ P_0(\cos \theta) - P_2(\cos \theta) \right] \]

\( e = 4.8 \times 10^{-10} \text{ statCoulomb}, \ a_0 = 0.529 \times 10^{-8} \text{ cm} \).

(a) Determine the multipole moments of this charge distribution and give the potential for large \( r \) in terms of these moments. (b) Determine the potential, near the origin, correct to order \( r^2 \). (c) Determine the interaction energy with a nuclear quadrupole moment, \( Q = 10^{-24} \text{ cm}^2 \), located at the origin.

Solution:

Part a:
The multipole expansion for the potential for \( r > r' \) is in Gaussian units (see Jackson, Eq. 4.2):

\[ \Phi(r, \theta, \varphi) = \sum_{l,m} \frac{4\pi}{2l + 1} \left[ \int \rho(r', \theta', \varphi') Y^*_{lm}(\theta', \varphi') r'^2 dr' d\Omega' \right] \frac{Y_{lm}(\theta, \varphi)}{r'^{l+1}} \]

where the multipole moment of the charge distribution, \( q_{lm} \), is given by:

\[ q_{lm} = \int \int \int \rho(r', \theta', \varphi') Y^*_{lm}(\theta', \varphi') r'^2 dr' d\Omega'. \]

We can write this in a general way by first noting that the charge distribution is

\[ \rho(r', \theta', \varphi') = \frac{2}{3} \frac{e}{a_0^3 64\pi} u^2 \exp(-u) \left[ \sqrt{4\pi} Y_{00}(\theta', \varphi') - \sqrt{\frac{4\pi}{5}} Y_{20}(\theta', \varphi') \right] \]

where \( u = \frac{r'}{a_0} \).

Thus the multipole moments are:

\[ q_{lm} = \frac{2}{3} \frac{a_0^3 e}{64\pi} \int u^{l+4} \exp(-u) du \int Y^*_{lm}(\theta', \varphi') \left[ \sqrt{4\pi} Y_{00}(\theta', \varphi') - \sqrt{\frac{4\pi}{5}} Y_{20}(\theta', \varphi') \right] d\Omega' \]

\[ = \frac{2}{3} \frac{a_0^3 e}{64\pi} (l + 4)! \left[ \sqrt{4\pi} \delta_{l0} \delta_{m0} - \sqrt{\frac{4\pi}{5}} \delta_{l2} \delta_{m0} \right] \]

Thus only \( m = 0 \) terms will be present, in particular only \( Y_{00}(\theta, \varphi) \) and \( Y_{20}(\theta, \varphi) \). The non-zero moments are:

\[ q_{00} = \int \int \rho(r, \theta) Y_{00}(\theta, \varphi) r^2 dr d\Omega = \frac{e}{\sqrt{4\pi}} \]

and

\[ q_{20} = \int \int \rho(r, \theta) Y_{20}(\theta, \varphi) r^2 dr d\Omega = -6 \frac{a_0^2 e}{\sqrt{4\pi}} \]

The potential for large \( r \), \( r >> a_0 \), is approximated by
\[ \Phi(r, \theta) = \sum_{l,m} \frac{4\pi}{2l+1} \frac{2}{3} \frac{a_0^l e}{64\pi} (l+4) \left[ \sqrt{4\pi} \delta_{l0} \delta_{m0} - \sqrt{\frac{4\pi}{5}} \delta_{l2} \delta_{m0} \right] Y_{lm}(\theta, \phi) \]
\[ = 4\pi \frac{2}{3} \frac{e}{64\pi} 4! \sqrt{4\pi} Y_{00}(\theta, \phi) - 4\pi \frac{2}{3} \frac{a_0^l e}{64\pi} 6! \sqrt{\frac{4\pi}{5}} \frac{Y_{20}(\theta, \phi)}{r^3} \]
\[ = \frac{e}{r} - 6 \frac{ea_0^2}{r^3} P_2(\cos \theta) \]

**Part b:** The Green’s function is
\[ G(r, \theta; r', \theta') = \sum_{l,m} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi) \]

The general potential due the charge distribution is
\[ \Phi(r, \theta, \phi) = \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) \left[ \int_0^\infty \int_{r_{<}}^{r_{>}} \frac{r'^l}{r_{>}} \rho(r', \theta', \phi') Y_{lm}^*(\theta', \phi') r'^2 dr' d\Omega' \right] 
\]
\[ = \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) \int_0^\infty \frac{r'^l}{r_{>}} \frac{2}{5} \frac{e}{a_0^3 64\pi} u^2 \exp(-u) dr' \cdot 
\]
\[ \int Y_{lm}^*(\theta', \phi') \left[ \sqrt{4\pi} Y_{00}(\theta', \phi') - \sqrt{\frac{4\pi}{5}} Y_{20}(\theta', \phi') \right] d\Omega' \]
\[ = \frac{2}{3} \frac{e}{64\pi} \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm}(\theta, \phi) \left[ \sqrt{4\pi} \delta_{l0} \delta_{m0} - \sqrt{\frac{4\pi}{5}} \delta_{l2} \delta_{m0} \right] \cdot 
\]
\[ \left[ \frac{a_0^l}{r_{>}} \int_0^{r/a_0} u^{l+4} \exp(-u) du + \frac{r^l}{a_0^{l+1}} \int_0^{\infty} u^{l-1} \exp(-u) du \right] \]

For \((r/a_0) \ll 1\) the first integral (in which one can expand the exponential) will give terms proportional to \((r/a_0)^n\) with \(n > 2\).

And since \((r/a_0) \ll 1\) the integral from \(r/a_0\) to infinity can be approximated by the integral from zero to infinity. This approximation gives the expression correct to \((r/a_0)^2\).

\[ = \frac{e}{24} \left[ \frac{1}{r} \int_0^{r/a_0} u^4 \exp(-u) du + \frac{1}{a_0} \int_0^{\infty} u^3 \exp(-u) du \right] 
\]
\[ - \frac{e}{24} \frac{1}{5} P_2(\cos \theta) \left[ \frac{a_0^2}{r^3} \int_0^{r/a_0} u^6 \exp(-u) du + \left[ \frac{r}{a_0} \right]^2 \int_0^{\infty} u \exp(-u) du \right] \]
\[ \approx \left[ \frac{1}{24} \cdot 3! - \frac{1}{120} P_2(\cos \theta) \cdot 1! \cdot \left( \frac{r}{a_0} \right)^2 \right] \frac{e}{a_0} \]

**Part c:** To determine the interaction energy of the charge distribution above with the nuclear quadrupole moment, \(Q\), located at the origin, we need to calculate

\[ W = \iiint \rho_{\text{nuclear}}(r) \Phi(r, \theta, \phi) d^3x \]
\[ \approx -\frac{1}{6} \sum_{ij} Q_{ij} \frac{\partial}{\partial x_i} \Phi(x) \] at \(r \ll a_0\)
See footnote about nuclear quadrupoles at the bottom of Jackson, page 151:

\[ Q_{11} = Q_{22} = -\frac{1}{2} Q_{33} \]
\[ Q_{33} = eQ \]
\[ Q_{ij} = 0 \text{ for } i \neq j \]

\[ W = -\frac{1}{6} \left[ Q_{11} \frac{\partial^2 \Phi}{\partial x^2} + Q_{22} \frac{\partial^2 \Phi}{\partial y^2} + Q_{33} \frac{\partial^2 \Phi}{\partial z^2} \right] \]
\[ = -\frac{1}{6} \frac{1}{2} Q_{33} \left[ -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2 \frac{\partial^2}{\partial z^2} \right] \left[ \frac{1}{24} \cdot 3! - \frac{1}{120} P_2(\cos \theta) \cdot 1! \cdot \left( \frac{r}{a_0} \right)^2 \right] \frac{e}{a_0} \]
\[ = \frac{e}{a_0} \frac{1}{6} \frac{Q_{33}}{240} \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 2 \frac{\partial^2}{\partial z^2} \right] \frac{1}{2} \left[ 3(r \cos \theta)^2 - r^2 \right] \]
\[ = \frac{e}{a_0} \frac{1}{6} \frac{Q_{33}}{240} \left[ -1 - 1 - 4 \right] \]
\[ = -\frac{e}{a_0} \frac{Q_{33}}{240} \]

The interaction energy, \( U = W \), is

\[ U = -\frac{e^2}{a_0} \frac{Q}{240a_0^2} \]
\[ = -26 \text{ eV} \frac{Q}{240a_0^2} = -26 \text{ eV} \times 1.49 \times 10^{-10} \]
\[ = -4 \times 10^{-9} \text{ eV} \]

Planck’s constant is \( 4.1 \times 10^{-15} \text{ eV} \cdot \text{s} \). The frequency is approximately \( 1 \text{ MHz} \).
1.2: Obtain the transverse and longitudinal components of the current densities:

\[ a) \quad j(r,t) = \text{Re}\left\{ e^{-i\omega t} i_0 \left[ \Theta\left( z + \frac{a}{2} \right) - \Theta\left( z - \frac{a}{2} \right) \right] \delta(x)\delta(y) \right\} \hat{k} \]

\[ b) \quad j(r,t) = \text{Re}\left\{ e^{-i\omega t} i_0 \delta(z)\delta(\rho - b) \right\} \hat{\phi}, \quad \text{cylindrical coordinates} \]

(c) For each current density obtain the related charge density (charge conservation) and the potential which vanishes at points infinitely far from the charges.

(d) Verify that Eq. 40 is correct.

Solution (a): From Eq. 44a the longitudinal current density is

\[ J_l(r,t) = -\frac{1}{4\pi} \nabla \int \int \int \frac{\nabla' \cdot j(r',t')}{|r - r'|} \, d^3r' \]

\[ J_l(r,t) = -\text{Re} \frac{i_0 e^{-i\omega t}}{4\pi} \nabla \int \int \int \frac{\left[ \delta(z' + \frac{a}{2}) - \delta(z' - \frac{a}{2}) \right] \delta(x')\delta(y')}{|r - r'|} \, d^3r' \]

\[ = -\text{Re} \left( e^{-i\omega t} \right) \frac{i_0}{4\pi} \nabla \left[ \left( x^2 + y^2 + (z + a/2)^2 \right)^{-1/2} - \left( x^2 + y^2 + (z - a/2)^2 \right)^{-1/2} \right] \]

\[ = -\cos(\omega t) \frac{i_0}{4\pi} \left[ \frac{xi + yj + (z - a/2)k}{\left( x^2 + y^2 + (z + a/2)^2 \right)^{3/2}} - \frac{xi + yj + (z + a/2)k}{\left( x^2 + y^2 + (z - a/2)^2 \right)^{3/2}} \right] \]

and the transverse current density (Eq. 44b) is
\[ J_t(r, t) = \frac{1}{4\pi} \nabla \times \iiint \frac{\nabla' \times j(r', t)}{|r - r'|} d^3r' \]

\[ = \cos(\omega t) \frac{i_0}{4\pi} \nabla \times \iiint \frac{[\Theta(z' + \frac{a}{2}) - \Theta(z' - \frac{a}{2})][\delta(x')\delta(y')i - \delta'(x')\delta(y')j]}{|r - r'|} d^3r' \]

\[ = \cos(\omega t) \frac{i_0}{4\pi} \nabla \times \int_{-a/2}^{a/2} dz' \left[ \int \frac{\delta'(y')i}{|r - r'|_{y'=0}} dy' - \frac{\delta'(x')j}{|r - r'|_{y'=0}} dx' \right] \text{ integrate by parts} \]

\[ = \cos(\omega t) \frac{i_0}{4\pi} \nabla \times \int_{-a/2}^{a/2} dz' \left[ \delta(y')i - \int \frac{\partial}{\partial y'} \left( \frac{1}{|r - r'|_{y'=0}} \right) dy' + \right. \]

\[ - \left. \frac{\delta(x')j}{|r - r'|_{y'=0}} + \int \delta(x')j \frac{\partial}{\partial x'} \left( \frac{1}{|r - r'|_{y'=0}} \right) dx' \right] \]

\[ = \cos(\omega t) \frac{i_0}{4\pi} \nabla \times \left[ \rho \hat{\Phi} \int_{-a/2}^{z+a/2} \frac{du}{(\rho^2 + u^2)^{3/2}} \right] \]

\[ = \cos(\omega t) \frac{i_0}{4\pi} \left[ \frac{xi + yj}{(x^2 + y^2 + (z - a/2)^2)^{3/2}} - \frac{xi + yj}{(x^2 + y^2 + (z + a/2)^2)^{3/2}} \right] \]

\[ + \cos(\omega t) \frac{i_0}{4\pi} k \int_{-a/2}^{z+a/2} \frac{\partial}{\partial \rho} \left[ \frac{\rho}{(\rho^2 + u^2)^{3/2}} \right] du \]

The integrand of the last term can be rewritten by noting that, with \( r = \sqrt{\rho^2 + z^2} \), the divergence of the electric field due to a unit charge at the origin is

\[ \nabla \cdot \left( \frac{r}{r^3} \right) = \frac{\partial}{\partial \rho} \left( \frac{\rho}{(\rho^2 + z^2)^{3/2}} \right) + \frac{\partial}{\partial z} \left( \frac{z}{(\rho^2 + z^2)^{3/2}} \right) = 4\pi \delta(x)\delta(y)\delta(z) \]

or

\[ \frac{\partial}{\partial \rho} \left( \frac{\rho}{(\rho^2 + z^2)^{3/2}} \right) = -\frac{\partial}{\partial z} \left( \frac{z}{(\rho^2 + z^2)^{3/2}} \right) - 4\pi \delta(x)\delta(y)\delta(z) \]

With this identity the tranverse current is

\[ J_t(r, t) = \cos(\omega t) \frac{i_0}{4\pi} \left[ \frac{xi + yj}{(x^2 + y^2 + (z - a/2)^2)^{3/2}} - \frac{xi + yj}{(x^2 + y^2 + (z + a/2)^2)^{3/2}} \right] \]

\[ - \cos(\omega t) \frac{i_0}{4\pi} k \int_{-a/2}^{z+a/2} \left[ \frac{\partial}{\partial z} \left( \frac{u}{(\rho^2 + u^2)^{3/2}} \right) - 4\pi \delta(x)\delta(y)\delta(u) \right] du \]

\[ = \cos(\omega t) \frac{i_0}{4\pi} \left[ \frac{xi + yj + (z - a/2)k}{(x^2 + y^2 + (z - a/2)^2)^{3/2}} - \frac{xi + yj + (z + a/2)k}{(x^2 + y^2 + (z + a/2)^2)^{3/2}} \right] \]

\[ + \cos(\omega ti_0)[\Theta(z + a/2) - \Theta(z - a/2)]\delta(x)\delta(y)k \]
Clearly $\mathbf{J}_l + \mathbf{J}_t = \mathbf{J}$ and the result should be correct since each was calculated independently.

Solution (b): Using cylindrical coordinates we see that the longitudinal current density is zero. That is,

$$
\mathbf{J}_l(r,t) = -\frac{1}{4\pi} \nabla \int \int \int \frac{\nabla' \cdot \mathbf{j}(r',t')}{|\mathbf{r} - \mathbf{r}'|} \, d^3r' \\
= -\frac{1}{4\pi} \nabla \int \int \int \frac{\nabla' \cdot \text{Re} \{ e^{-i\omega t} i_0 \delta(z') \delta(\rho' - b) \} \hat{\phi}}{|\mathbf{r} - \mathbf{r}'|} \, d^3r' \\
= -\frac{1}{4\pi} \nabla \int \int \int \frac{1}{\rho'} \frac{\partial}{\partial \phi'} \text{Re} \{ e^{-i\omega t} i_0 \delta(z) \delta(\rho - b) \} \, d^3r' \\
= 0
$$

This current density will therefore not cause the charge density to vary with time. (A possible physical system would be a uniformly charged circular ring. The current would be generated when the circular ring rotates about an axis which passes through its center and lies perpendicular to the plane of the ring.) In this example $\mathbf{J}_t = \mathbf{J}.$

Solution (c): To find the charge density we use conservation of charge:

$$
\nabla \cdot \mathbf{j} + \frac{\partial}{\partial t} \rho = 0
$$

For the current density of part (a) the charge density is

$$
\rho(r,t) = \text{Re} \left[ \frac{i}{\omega} e^{-i\omega t} i_0 \frac{\partial}{\partial z} \left\{ \left[ \Theta \left( z + \frac{a}{2} \right) - \Theta \left( z - \frac{a}{2} \right) \right] \delta(x) \delta(y) \right\} \right] \\
= \frac{i_0}{\omega} \sin(\omega t) \left[ \delta \left( z + \frac{a}{2} \right) - \delta \left( z - \frac{a}{2} \right) \right] \delta(x) \delta(y)
$$

This is just two point charges $-i_0 \sin(\omega t) / \omega$ at $z = a/2$ and $+i_0 \sin(\omega t) / \omega$ at $z = -a/2$. The potential obtained using Gauss’s law is

$$
\phi(r,t) = \left( \frac{i_0}{\omega} \sin(\omega t) \right) \left[ \frac{1}{\sqrt{x^2 + y^2 + (z + a/2)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z - a/2)^2}} \right].
$$

For the current density of part (b) $\nabla \cdot \mathbf{j} = 0$ gives $\rho(r,t) = 0$ and a potential which is zero.

Solution (d): We only need to check the relationship for the current density given in part (a).

$$
\mathbf{J}_l = \frac{1}{4\pi} \nabla \left[ \frac{\partial}{\partial t} \phi(r,t) \right] \\
= \frac{(i_0 \cos(\omega t))}{4\pi} \nabla \left[ \frac{1}{\sqrt{x^2 + y^2 + (z + a/2)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z - a/2)^2}} \right].
$$
It follows that Eq. 40 is correct, that is

\[ \nabla \times \mathbf{J}_f = 0 \]

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