Problem Set 3:

1. (Jackson 6.20).
   An example of the preservation of causality and finite speed of propagation in spite of the use of the Coulomb gauge is afforded by a unit strength dipole source that is flashed on and off at $t = 0$. The charge and current densities are
   \[
   \rho(r, t) = \delta(x)\delta(y)\delta'(z)\delta(t)
   \]
   \[
   J(r, t) = -e_3 \delta(x)\delta(y)\delta(z)\delta'(t)
   \]
   where a prime means differentiation with respect to the argument. This dipole is of unit strength and it points in the negative $z$ direction.
   
   (a) Show that the instantaneous Coulomb potential (6.23 in text) is
   \[
   \phi(r, t) = -\frac{1}{4\pi\epsilon_o} \frac{z}{r^3} \delta(t).
   \]
   
   (b) Show that the transverse current, $J_t$, is
   \[
   J_t(r, t) = -\left[ e_3 \frac{2}{3} \delta'''(r) + \frac{1}{4\pi} \frac{3n \cdot e_3}{r^3} - e_3 \right] \delta'(t)
   \]
   where $n = \hat{r}$, a unit vector along the $r$ direction and the $\frac{2}{3}$ factor multiplying the delta function comes from treating the gradient of $\frac{z}{r^3}$ according to (4.20 in text.)
   
   (c) Show that the electric and magnetic fields are causal and that the electric field components are given by:
   \[
   E(r, t) = e_3 \frac{c}{r} \left[ \delta''(ct - r) + \frac{1}{r^2} \delta(r - ct) - \frac{1}{r} \delta'(r - ct) \right]
   \]
   \[
   - r \frac{cz}{r^3} \left[ \delta''(r - ct) - \frac{3}{r} \delta'(r - ct) + \frac{3}{r^2} \delta(r - ct) \right]
   \]
   Hint: While the answer in part b displays the transverse current explicitly, the less explicit form,
   \[
   J_t(r, t) = -e_3 \delta(r)\delta'(t) - \delta'(t) \frac{1}{4\pi} \frac{\partial}{\partial z} \left( \nabla \frac{1}{r} \right)
   \]
   can be used with (6.47 in text) to calculate the vector potential and the fields for part c. An alternative method is to use the Fourier transforms in time of $J_t$ and $A$, the Green’s function (6.40) and its spherical wave expansion from Chapter 9.

2. Using the retarded Green’s function and the Lorentz gauge, find the electrostatic potential, $\Phi(r, t)$, generated by a point charge, $Q$, moving with velocity $v_o$. Assume $\Phi(r, t) \rightarrow 0$ and $\partial\Phi(r, t)/\partial t \rightarrow 0$ as $t \rightarrow -\infty$. Hint: Do the space integration first and recall the definition of the Dirac delta function whose argument is a function.
   
   You might also find it useful to use the following expressions to simplify the notation:
   \[
   \tau \equiv ct
   \]
   \[
   \tau' \equiv ct'
   \]
$\beta \equiv \frac{v_0}{c}$

$R \equiv r - \beta \tau = r - vot$

$W \equiv \tau - \tau'$

Note that $|R + \beta W| = |r - vot'|$