Chapter II: General Coordinate Transformations

Before beginning this chapter, please note the Cartesian coordinate system below and the definitions of the angles $\theta$ and $\varphi$ in the spherical coordinate system. In the spherical coordinate system, $(r, \theta, \varphi)$ we shall use:

$$
\begin{align*}
x &= r \sin \theta \cos \varphi \\
y &= r \sin \theta \sin \varphi \\
z &= r \cos \Theta
\end{align*}
$$

and in the cylindrical coordinate system $(\rho, \varphi, z)$:

$$
\begin{align*}
x &= \rho \cos \varphi \\
y &= \rho \sin \varphi \\
\rho^2 &= x^2 + y^2.
\end{align*}
$$

![Figure 1](image.png)

**Figure 1.** The Cartesian coordinate system for 3-dimensional Euclidean space.
Chapter II: General Coordinate Transformations

Consider two coordinate systems in 3-dimensional Euclidian space:

1. a Cartesian system where a point is specified by \((x^1, x^2, x^3) = (x,y,z)\)

2. a general "q" system where a point is specified by \((q^1, q^2, q^3)\).

Each point \((x^1, x^2, x^3)\) corresponds to a unique set of real numbers \((q^1, q^2, q^3)\). Further, each \(x^i\) is a function of the \(q^i\), \(f^i(q^1,q^2,q^3)\), and each \(q^i = h^i(x^1,x^2,x^3)\) where all first partial derivatives of \(f^i\) and \(h^i\) exist. Using only the chain rule for differentiation, the following equations can be obtained:

\[
\begin{align*}
\text{dx}^i &= \frac{\partial f^i}{\partial q^1} dq^1 + \frac{\partial f^i}{\partial q^2} dq^2 + \frac{\partial f^i}{\partial q^3} dq^3 \\
&= \sum_j \frac{\partial x^i}{\partial q^j} dq^j \\
\text{also}
\end{align*}
\]

\[
\frac{\partial}{\partial x^i} = \sum_k \frac{\partial q^k}{\partial x^i} \frac{\partial}{\partial q^k}
\]

using chain rule

where we have used the notation \(x^i(q^1,q^2,q^3) = f^i(q^1,q^2,q^3)\) and \(q^i(x^1,x^2,x^3) = h^i(x^1,x^2,x^3)\).

Note that the differential, \(\text{dx}^i\), "transforms with" \([\partial x^i/\partial q^1]\) and the partial derivative "transforms with" \([\partial q^i/\partial x^i]\). So, using the summation notation (repeated indices \(\implies\) summation):

\[
\begin{align*}
\text{dx}^i &= \frac{\partial x^i}{\partial q^j} dq^j \equiv A^i_j dq^j \\
\frac{\partial}{\partial x^i} &= \frac{\partial q^j}{\partial x^i} \frac{\partial}{\partial q^j} \equiv B^i_j \frac{\partial}{\partial q^j}
\end{align*}
\]

Note the placement of the indices.
In general, \( \frac{\partial x^i}{\partial q^j} \neq \frac{\partial q^i}{\partial x^j} \). This means that the transformation

\[
A^i_j = \frac{\partial x^i}{\partial q^j} \equiv \text{contravariant transformation matrix}
\]

and

\[
B^j_i = \frac{\partial q^j}{\partial x^i} \equiv \text{covariant transformation matrix}
\]

transform differently under the coordinate transformation. The two transformations have been named \textbf{contravariant} and \textbf{covariant}, respectively.

Example: Transformation from Cartesian to spherical coordinates:

\[
\begin{align*}
\begin{array}{c}
x = r \sin \theta \cos \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \theta
\end{array}
\end{align*}
\]

\[
\begin{align*}
r & = \sqrt{x^2 + y^2 + z^2} = q^1 \\
\theta & = \cos^{-1}(z/r) = q^2 \\
\phi & = \tan^{-1}(y/x) = q^3
\end{align*}
\]

The differentials are given by:

\[
\begin{align*}
dx & = \sin \theta \cos \phi \, dr + r \cos \theta \cos \phi \, d\theta + -r \sin \theta \sin \phi \, d\phi \\
dy & = \sin \theta \sin \phi \, dr + r \cos \theta \sin \phi \, d\theta + r \sin \theta \cos \phi \, d\phi \\
dz & = \cos \theta \, dr + -\sin \theta \, d\theta + 0 \cdot d\phi
\end{align*}
\]

Thus,

\[
\begin{bmatrix}
\frac{dx}{dr} \\
\frac{dy}{d\theta} \\
\frac{dz}{d\phi}
\end{bmatrix} = 
\begin{bmatrix}
\sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
\sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
\cos \theta & -r \sin \theta & 0
\end{bmatrix}
\begin{bmatrix}
dr \\
d\theta \\
d\phi
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{dr}{dq^1} \\
\frac{d\theta}{dq^2} \\
\frac{d\phi}{dq^3}
\end{bmatrix}
\]
In the above equation $A_{1}^{1} = \sin\theta \cos\phi$ and $A_{2}^{3} = -\sin\theta$, etc. The elements of the contravariant transformation matrix are obtained from the expression for the differentials of $dx$, $dy$ and $dz$.

One can also show that:

$$
\begin{bmatrix}
\frac{dr}{d\theta} \\
\frac{d\theta}{d\phi} \\
\frac{d\phi}{dq}
\end{bmatrix}
= \begin{bmatrix}
\sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
(\cos \theta \cos \phi) / r & (\cos \theta \sin \phi) / r & -\sin \theta / r \\
-\sin \phi / (r \sin \theta) & \cos \phi / (r \sin \theta) & 0
\end{bmatrix}
\begin{bmatrix}
dx \\
dy \\
dz
\end{bmatrix}
$$

And using the chain rule one finds:

$$
dq^k = \frac{\partial q^k}{\partial x^j} \ dx^j = [A^{-1}]^k_j \ dx^j
$$

Note that $\partial y / \partial \theta = r \cos \theta \sin \phi$ is not simply related to $\partial \theta / \partial y = (\cos \theta \sin \phi) / r$. One can see also from the above equations that $A^{-1} = B^T$ and $B^{-1} = A^T$. In the above expressions different summation indices were used for each of the sums. This is a good habit to adopt. Try to do this in all your derivations.

**Sometimes you will find the notation:** $A_i^j$ for $B_i^j$. We won't use this notation since it is confusing. Keep in mind that $A_i^j \neq A_i^j$. If you don't understand this comment look again carefully at the indices on the above equations.

**Exercise:** Evaluate the $A$ (contravariant) matrix for the transformation (from Cartesian to spherical coordinates) if $r = 3$, $\theta = \pi/4$ and $\phi = \pi/2$.

**Exercise:** Find the $A^{-1}$ matrix for the same values of $r$, $\theta$, and $\phi$ in the above exercise.
Rules for keeping track of indices:

1. Always use different index symbols for different summations;
2. Indices on the left side of the equation match with the "left-most" index on the right hand side;
3. "Superscripts" found in the "denominator" (such as the k in $\partial/\partial x^k$) function as subscripts in the expression;
4. $A^i_j$ and $B^m_k$ have four "positions", two "upper" and two "lower" (see below);
5. A "left" position on $A^i_j$ or $B^m_k$ denotes a row index; the "right" position denotes column index;
6. Indices are always "matched" in an equation (look for the pairs);

Contravariant and covariant vectors:

A triplet of quantities in the Cartesian system, $(V^1, V^2, V^3)$, [such as $(dx^1, dx^2, dx^3)$] which transforms into its counterpart, $(V'^1, V'^2, V'^3)$, in the q system [such as $(dq^1, dq^2, dq^3)$] via the $A^i_j$ matrix is said to be a set of contravariant vector components. Contravariant vector components are denoted by superscripts.

$$V^i = A^i_j V'^j$$
contravariant vector transformation

A triplet of quantities in the Cartesian system, $(V_1, V_2, V_3)$, [such as $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$] which transforms into its counterpart, $(V'_1, V'_2, V'_3)$, in the q system [such as $(\partial/\partial q^1, \partial/\partial q^2, \partial/\partial q^3)$] via the $B^m_k$ matrix is said to be a set of covariant vector components. Covariant vector components are denoted by subscripts.

$$V_k = B^m_k V'_n$$
covariant vector transformation
The differentials, $dx^i$, transform like contravariant vector components under a generalized (linear) coordinate transformation. The "linear" means that the transformation is linear in the differentials. The partial derivatives, $\partial/\partial x^k$, transform like covariant vector components under a generalized (linear) coordinate transformation. In fact, any triplet which transforms "like the differentials" under a generalized linear coordinate transformation is said to be a contravariant vector. A triplet which transforms "like the partial derivatives" is said to be a covariant vector under the generalized linear coordinate transformation.

Later we shall find out how to find the proper contravariant and covariant components of a physical vector. Two "physical" vectors are always simple to deal with: $dr$ and $\nabla f(x,y,z)$.

---

**Exercise:** Find the contravariant components of the vector $dr$ in the Cartesian coordinate system.

---

**Exercise:** Find the contravariant components of the vector $dr$ in the spherical coordinate system.

---

**Exercise:** Find the covariant components of the vector $\nabla = \hat{x}^i (\partial/\partial x^i)$ in the spherical coordinate system.

---

**Exercise:** Find the covariant components of the vector $\nabla = \hat{r}^i (\partial/\partial r^i)$ in the cylindrical coordinate system.
Kronecker Delta Symbol and Other Notation

The Kronecker delta symbol is $\delta_{ij} = \{ 0 \text{ if } i \neq j \text{ and } 1 \text{ if } i=j \}$.

Generalized Kronecker delta:

$$\delta_{ijk...m}^{i'j'k'...m'} = \begin{cases} 
0 & \text{if any two superscripts (or subscripts) are equal} \\
& \text{or if } \{i,j,k,...,m\} \neq \{i',j',k',...,m'\} \\
+1 & \text{if subscripts are an even permutation of superscripts} \\
-1 & \text{if subscripts are an odd permutation of superscripts}
\end{cases}$$

Examples:

$$\delta_{123}^{132} = -1 \quad \delta_{123}^{132} = +1 \quad \delta_{123}^{132} = 0$$

Numerically, $\delta_{ij}^i = \delta_{ij} = \delta_{ij}$. But in an expression, these can carry different information.

PERMUTATION SYMBOLS (COMPLETELY ANTISYMMETRIC "TENSORS")

$$\epsilon_{ijk...m} = \delta_{ijk...m}^{123...n} = \begin{cases} 
0 & \text{if any two superscripts are equal} \\
& \text{or if superscripts are not same set as } 1,2,3...n \\
+1 & \text{if the superscripts are an even permutation of } 1,2,3,...n \\
-1 & \text{if the superscripts are an odd permutation of } 1,2,3,...n
\end{cases}$$

$$\epsilon_{ijk...m} = \delta_{ijk...m}^{123...n} = \begin{cases} 
0 & \text{if any two subscripts are equal} \\
& \text{or if subscripts are not same set as } 1,2,3...n \\
+1 & \text{if the subscripts are an even permutation of } 1,2,3,...n \\
-1 & \text{if the subscripts are an odd permutation of } 1,2,3,...n
\end{cases}$$

The above symbols are special cases of the generalized Kronecker delta and are often used with expressions involving permutations. The above symbols are "almost" tensors as we shall see in the chapter on tensors.

Examples:

$$\epsilon_{123} = \epsilon_{132} = \epsilon_{231} = +1$$
$$\epsilon_{123} = \epsilon_{132} = \epsilon_{231} = -1$$
$$\epsilon_{112} = \epsilon_{223} = \epsilon_{323} = 0, \text{ etc.}$$

$$\epsilon_{123} = \epsilon_{312} = \epsilon_{231} = +1$$
$$\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$$
$$\epsilon_{313} = \epsilon_{322} = \epsilon_{211} = 0, \text{ etc.}$$

Example:

$$\epsilon_{1234} = +1 \quad \epsilon_{0123} = 0$$
The Metric Tensor

The metric tensor, $g_{jk}$, for the $q^i$ system is defined as follows:

\[ ds^2 = g_{jk} dq^j dq^k \]

\[ j,k \text{ summation implied} \]

$g =$ metric tensor

where \( ds^2 = (dx)^2 + (dy)^2 + (dz)^2 = dx_i dx_i \) (summation over $i$ implied)

What is $g_{jk}$?

\[
\begin{align*}
\text{ds}^2 &= \mathbf{dr} \cdot \mathbf{dr} = dx_i dx_i \\
&= \left[ \frac{\partial x^j}{\partial q^i} \right] dq^j \cdot \left[ \frac{\partial x^k}{\partial q^i} \right] dq^k \quad \text{Summation over} \ i \\
&= \left[ \frac{\partial x^j}{\partial q^i} \right] \left[ \frac{\partial x^k}{\partial q^i} \right] dq^j dq^k \quad \text{Summation over} \ i \\
&= g_{jk} dq^j dq^k
\end{align*}
\]

Thus

\[
\boxed{g_{jk} = \left[ \frac{\partial x^j}{\partial q^i} \right] \left[ \frac{\partial x^k}{\partial q^i} \right] \quad (\text{Summation over} \ i)}
\]

The metric tensor is symmetric since

\[
\left[ \frac{\partial x^j}{\partial q^i} \right] \left[ \frac{\partial x^k}{\partial q^i} \right] = \left[ \frac{\partial x^i}{\partial q^j} \right] \left[ \frac{\partial x^k}{\partial q^j} \right] = g_{kj} = g_{jk}. \quad \text{Summation over} \ i
\]

THE METRIC TENSOR IN CARTESIAN SYSTEMS:

\[
dx^i = [R^{-1}]^j_i \ dx^j = \left[ \frac{\partial x^i}{\partial x^k} \right] dx^k
\]

Thus

\[
g_{jk} = \sum [R^{-1}]^j_i [R^{-1}]^i_k
\]

\[
= \sum [R^{-1}]^T_j \ [R^{-1}]_k
\]

\[
= \sum R_j^i \ [R^{-1}]^i_k
\]

\[
= [R R^{-1}]_k = \delta_{jk}
\]

and

\[
g_{\text{cartesian}} = 1
\]
The metric tensor:

\[ g_{ij} = A_i A_j \]

\[ g_{ij} = \text{det} \left[ \frac{\partial q^j}{\partial x^i} \right] = \text{det} \left[ B_i^j \right] = \text{det}(B) \]

where \( \frac{\partial q^j}{\partial x^i} \) implies a matrix whose i,j component is \( \frac{\partial q^j}{\partial x^i} \).

\[ \text{Jacobian} = \text{det} \left[ \frac{\partial q^j}{\partial x^i} \right] = \text{det} \left[ B_i^j \right] = \text{det}(B) \]

\[ \text{proof:} \]

\[ g_{jk} = \sum \left[ \frac{\partial x^j}{\partial q^i} \right] \left[ \frac{\partial x^k}{\partial q^i} \right] \]

\[ = \sum A_i^j A_i^k \]

\[ = \sum [A_i^T]^i_A^j \]

\[ = [A_i^T A_i^j] \]

\[ \text{Exercise:} \quad \text{Find } g \text{ for the spherical coordinate system.} \]
**Exercise:** Find the metric tensor for a $q^i$ system if

$$ds^2 = 4dq^1dq^1 - q^2 dq^1dq^2 + \pi dq^2dq^3 + 7q^1 dq^2dq^2 - dq^3dq^3.$$ 

**Exercise:** Find the metric tensor for the cylindrical coordinate system:

\[
\begin{align*}
x &= \rho \cos \phi \\
y &= \rho \sin \phi \\
z &= z
\end{align*}
\]

\[
\rho = [x^2+y^2]^{1/2} \quad \phi = \tan^{-1}[y/x].
\]
The unit vectors, $\mathbf{q}_i$, in the $\mathbf{q}_i$ system

The equations $q^i = q(x^1, x^2, x^3) = c^i$ where $i = 1, 2, 3$ each represent a surface of constant $q^i$ in the three dimensional space. Below is a schematic of three such surfaces which intersect at a point. This point can be labelled by its $(x, y, z)$ value, or by $(q^1, q^2, q^3) = (c^1, c^2, c^3)$.

Unit vectors perpendicular to a surface of constant $q^i$ can be obtained from the $\nabla q^i$:

$$\mathbf{q}_i = \frac{\nabla q^i}{|\nabla q^i|} = \sum_k \frac{\hat{x}^k}{\nabla x^k} \frac{1}{|\nabla q^i|}$$

No sum on $i$

$$|\nabla q^i| = \sqrt{\sum_j \left( \frac{\partial q^i}{\partial x^j} \right)^2}^{1/2}$$

Note that $q^i = B_{ij} \hat{x}^j / [\beta^i]$ (with no sum on $i$!) where $\beta^i = |\nabla q^i|$.

Later we shall put the $\beta^i$ into another more convenient form.

Exercise: Find $\mathbf{r}$, $\mathbf{\theta}$, and $\mathbf{\phi}$ in the spherical coordinate system.
The Inverse Metric Tensor

Claim: \( g^{-1} = B^T B \).

proof:

1. We note from page II-2 that expression a) below holds; from page II-3 we find expression b):
   
   a) \[ \frac{\partial q^i}{\partial x^j} = B^k_j. \]
   
   b) \[ \frac{\partial q^i}{\partial x^j} = (A^{-1})^k_j. \]

   Thus \( B^T k_j = (A^{-1})^k_j \) and \( B^T = A^{-1} \).

2. But \( g^{-1} = [A^T A]^{-1} = A^{-1} [A^T]^{-1} = B^T B \)

   where we have used \( [A^T]^{-1} = [A^{-1}]^T = [B^T]^T = B \). (to be shown later).

Note that \( [g^{-1}]_{jk} = \sum_n [B^T]^j_n B_n^k. \)
The components of the inverse metric tensor, \( \mathbf{g}^{-1} \), are denoted by \( g^{ij} \) -- with superscripts! Later we shall see that the superscripts on \( \mathbf{g}^{-1} \) and the subscripts on \( \mathbf{g} \) have definite implications. Normally, one does not use a special symbol for the inverse metric tensor -- when written with components.

\[
\begin{align*}
g_{ij} &\equiv \text{metric tensor components} \\
g^{ij} &\equiv \text{inverse metric tensor components}
\end{align*}
\]

**Claim:** \( \beta^i = |\nabla q^i| = \sqrt{g^{ii}} \) (no sum on \( i \)).

**proof:**

1. \( g^{ii} \) (no sum on \( i \)) is the diagonal (ii) component of the inverse metric tensor.

\[
g^{ii} = [B^TB]^{ii} = [A^{-1}]^i_k \quad B_k^i \quad \text{sum on } k, \text{ no sum on } i.
\]

\[
= B_k^i \quad B_k^i \quad \text{sum on } k \text{ only.}
\]

\[
= [\partial q^i/\partial x^k] \quad [\partial q^i/\partial x^k] \quad \text{sum on } k \text{ only.}
\]

\[
= [ (\partial q^i/\partial x^k) \quad \mathbf{\hat{x}}^k ] : [ (\partial q^i/\partial x^k) \quad \mathbf{\hat{x}}^k ] = \nabla q^i \cdot \nabla q^i \quad \text{(no sum on } i).}
\]

\[
= (\partial q^i/\partial x^k) (\partial q^i/\partial x^k) \quad \mathbf{\hat{x}}^k \cdot \mathbf{\hat{x}}^n = (\partial q^i/\partial x^k) (\partial q^i/\partial x^k) \quad \delta^{kn}
\]

2. Thus \( g^{ii} = \nabla q^i \cdot \nabla q^i = |\nabla q^i|^2 \) no sum on \( i \).

3. So, \( \beta^i = |\nabla q^i| = \sqrt{g^{ii}} \) (no sum on \( i \)).
Use of $g$ to convert from contravariant to covariant components

Claim:

$$dx_i = [g_x]_{ij} dx^j \quad \text{and} \quad dq_k = g_{km} dq^m$$

proof:

1. Let $dx^j = A^j_n \, dq^n$; multiply on the left by $[g_x]_{ij} = [g_{\text{cartesian}}]_{ij} = 1_{ij}$ and sum over $j$:

$$[g_x]_{ij} dx^j = [g_x]_{ij} A^j_n \, dq^n = [g_x]_{ij} A^j_n \, \delta^n_{n'} \, dq^{n'} \quad \text{where 1 is replaced by} \; g^{-1}g \; \text{and} \; g \; \text{is for} \; q^i \; \text{system.}$$

$$= [g_x]_{ij} A^j_n \, g^{nm} \, g_{mn'} \, dq^{n'}$$

(Recall that superscripts $\Rightarrow g^{-1}$)

2. Now use $g^{-1} = B^T B$:

$$[g_x]_{ij} dx^j = [g_x]_{ij} A^j_n \, [B^T B]^{nm} \, g_{mn'} \, dq^{n'}$$

$$= \delta_{ij} \, [AB^T B]^m_{n'} \, g_{mn'} \, dq^{n'}$$

$$= [AB^T B]^m_{n'} \, g_{mn'} \, dq^{n'} = [AA^{-1} B]^m_{n'} \, g_{mn'} \, dq^{n'}$$

$$= B_i^m \, g_{mn'} \, dq^{n'}$$

$$= B_i^m \, [g_{mn'} \, dq^{n'}]$$

3. Thus $[g_x]_{ij} dx^j = dx_i$ and $[g_{mn'} \, dq^{n'}] = dq_m$ transform like covariant vectors and

$$dx_i = B_i^m \, dq_m.$$

QED

Note that we have defined the $dx^j$ and $dq^{n'}$ in terms of $dx^x$ and $dq^x$.
The previous proof can be used to show that for any \(v^i\) in the Cartesian system, the proper covariant vector components can be formed from \(v_i = \left[g_x\right]_{ij} v^j\) -- and in the \(q^i\) system,

\[
v'_m = \left[g_{mn'}\right] v'^{n'}
\]

**Exercise:** Show that \(v'^n = g^{nk} v'_k\).

---

**Claim:** \(g\) is diagonal iff \(q^i \cdot q^j = \delta^{ij}\)

**(diagonal \(g\) \(\iff\) orthogonal coordinate system)**

**Proof:**

1. **Case I:** show that \(q^i \cdot q^j = \delta^{ij} \implies g\) is diagonal.
   
   a) Assume \(q^i \cdot q^j = \left[\nabla q^i / \nabla g_{ij}\right] \cdot \left[\nabla q^j / \nabla g_{ij}\right]\) (no sum on \(i\) or \(j\)) = \(\delta^{ij}\)
   
   b) Thus,
   
   \[
   \sqrt{g^{ii}} \sqrt{g^{jj}} \delta^{ij} \text{ no sum on } i \text{ or } j = \delta^{ij} g^{ii} = \nabla q^i \cdot \nabla q^j \text{ (no sum on } i) = [\partial q^i / \partial x^k] x^k \cdot [\partial q^j / \partial x^k] x^k
   \]
   
   \[
   = B^i_k B^j_n \delta^{kn}
   \]
   
   \[
   = [B^T B]_{ij}
   \]
   
   \[
   = [g^{-1}]_{ij}
   \]
   
   c) So \(\delta^{ij} g^{ii} \text{ no sum on } i = [g^{-1}]_{ij} \text{ and } g^{-1}\) is diagonal.
   
   d) If \(g^{-1}\) is diagonal, \(g\) is also.

2. **Case II:** assume \(g\) is diagonal and show \(q^i \cdot q^j = \delta^{ij}\).
   
   a) \(g^{-1}\) is diagonal if \(g\) is.
   
   b) But,
   
   \[
   [g^{-1}]_{ij} = [B^T B]_{ij} = B^i_k B^j_n \delta^{kn} = \nabla q^i \cdot \nabla q^j \text{ (all the steps in case I are reversible)}
   \]
   
   c) Therefore (since steps in case I are reversible):
   
   \[
   [g^{-1}]_{ij} \delta^{ij} g^{ii} \text{ no sum on } i \implies \nabla q^i \cdot \nabla q^j = \delta^{ij} C, \text{ where } C = \text{constant.}
   \]
Claim: If \( \mathbf{A} \) is orthogonal \( \mathbf{g} = 1 \).

proof:

1. \( \mathbf{A}^T = \mathbf{A}^{-1} \) since \( \mathbf{A} \) is orthogonal.
2. thus, \( \mathbf{A}^T \mathbf{A} = \mathbf{g} = \mathbf{A}^{-1} \mathbf{A} = 1 \).

The \( \mathbf{u}^i \) and \( \mathbf{u}_i \) basis vectors for the \( q^k \) system

Claim: \( \sqrt{g^{ii}} \mathbf{u}_i \) transforms into \( \mathbf{x}^k \) via \( \mathbf{A} \).

proof:

1. \( \sqrt{g^{ii}} \mathbf{u}_i \) transforms into \( \mathbf{x}^k \) via \( \mathbf{A} \), (see box on page II-10 and \( \beta^i \) on page II-12).

2. so, \( \mathbf{u}_i = \mathbf{B}_k^i \mathbf{x}^k \) multiply both sides by \( \mathbf{A}^n_i \) and sum over \( i \):

\[
\mathbf{A}^n_i \mathbf{u}_i = \mathbf{A}^n_i [\mathbf{A}^{-1}]^i_k \mathbf{x}^k = [\mathbf{AA}^{-1}]^n_k \mathbf{x}^k = \delta^n_k \mathbf{x}^k = \mathbf{x}^n.
\]

Since the \( \mathbf{u}_i \) and \( \mathbf{x}^n \) transform "according to" \( \mathbf{A} \) we can (using same steps as on page II-13) show that:

\[
\mathbf{u}_j = g_{ji} \mathbf{u}^i
\]

gives the corresponding basis "vectors" which transform according to \( \mathbf{B} \):

\[
\mathbf{x}_k = \mathbf{B}_k^j \mathbf{u}_j.
\]

The \( \mathbf{u}_i \) and \( u^i \) vectors serve as basis vectors -- playing the "role" of \( \mathbf{x}^k \) and \( \mathbf{x}_i \) in the "general" coordinate system. We shall show on the next page that these generalized basis vectors obey an orthogonality condition and their dot products can be determined from the metric tensor, \( \mathbf{g} \). The geometrical interpretation of the \( \mathbf{u}^k \) and \( \mathbf{u}_n \) vectors can be seen from a simple example, after we derive the special conditions on the next page.
Orthogonality and dot products for "basis vectors".

\[
\begin{align*}
\vec{u}_i \cdot \vec{u}_j &= g_{ij} \\
\vec{u}^i \cdot \vec{u}^j &= g^{ij} \\
\vec{u}^i \cdot \vec{u}^j &= \delta^i_j
\end{align*}
\]

derivations:

1. \( \vec{u}_i \cdot \vec{u}_j = [B^{-1}]_i^k x^k \cdot [B^{-1}]_j^n x^n \)
   
   \[= [A^T]^k_i [A^T]^n_j x^k \cdot x^n \]
   
   \[= [A^T]^k_i A^n_j \delta_{kn} \]
   
   \[= [A^T]^k_i A^k_j \text{ sum over } k \]
   
   \[= [A^T A]_{ij} = g_{ij}. \]

2. \( \vec{u}^i \cdot \vec{u}^i = [A^{-1}]^i_k x^k \cdot [A^{-1}]^i_n x^n \)
   
   \[= [B^T]^k_i B^j_n x^k \cdot x^n \]
   
   \[= [B^T]^k_i B^k_j \delta^{kn} \]
   
   \[= [B^T B]^{ij} \]
   
   \[= g^{ij}. \]

3. \( \vec{u}^i \cdot \vec{u}_j = [A^{-1}]^i_k x^k \cdot [B^{-1}]_j^n x^n \)
   
   \[= [A^{-1}]^i_k [A^T]^n_j x^k \cdot x^n \]
   
   \[= [A^{-1}]^i_k [A^T]^n_j \delta^k_n \]
   
   \[= [A^{-1}]^i_k A^k_j \]
   
   \[= [A^{-1} A]^i_j \]
   
   \[= \delta^i_j. \]
Ordinary \((=f_i)\), contra- and co-variant components of \(F\):

\[
F = f_1 q^1 + f_2 q^2 + f_3 q^3
\]

\[
F = F' \bar{u}^i
\]

\[
F = F'^j \bar{u}_j
\]

\[
F'_i = F \cdot \bar{u}_i \quad \text{and} \quad F'^k = F \cdot \bar{u}^k
\]

[When \textbf{bold face} is not available, use \(^\wedge\) and \(\bar{}\) (bars) over symbols to denote vectors.]

Derivations:

1. \(F = f_1 q^1 + f_2 q^2 + f_3 q^3 = \sum_i f_i q^i\) where \(f_i = \text{ordinary components of } F\)

2. Substitute \(u^i = \sqrt{g^{ii}} q^i\) \(\text{no sum on } i\) in above.

\[
F = \sum_i f_i q^i = \sum_i f_i \left[ u^i \sqrt{g^{ii}} \right]
\]

\[
= \sum_i \left[ f_i \sqrt{g^{ii}} \right] u^i
\]

3. \(F \cdot u_j = \sum_i \left[ f_i \sqrt{g^{ii}} \right] u^i u_j\)

\[
= \sum_i \left[ f_i \sqrt{g^{ii}} \right] \delta^i_j
\]

\[
= \left[ f_j \sqrt{g^{jj}} \right] \text{ no sum on } j
\]

On the following page we shall show that \(F \cdot u_j = F'_j\), and transforms like the covariant component of \(F\). (Note that in the \(q^i\) system, the components are distinguished from those in the Cartesian system by primes.) Once this is accomplished, we can find the contravariant component of \(F\), \(F'^k\), from \(F'^k = g^{kn} F_n\).
**Claim:**

\[ \overline{F} \cdot \overline{u}_j = F'_j \]

* = covariant component of \( F \) in \( q^i \) system

**proof:**

1. \( F = F \cdot \bar{x}^i = f_n \bar{u}^i = (f_n/\sqrt{g^{nn}}) u^i \); dot with \( \bar{x}_j \).

2. \( \overline{F}_i \cdot \overline{x}_j = (f_n/\sqrt{g^{nn}}) \bar{u}^\nu \overline{\bar{x}}^\nu \cdot \overline{\bar{x}}_j \) transform \( u^\nu \) (see page II-15)

   \[ \overline{F}_j = (f_n/\sqrt{g^{nn}}) \left[ A^{-1} \right]_{k}^{n} \bar{x}^\nu \cdot \bar{x}_j \]

   \[ \overline{F}_j = \overline{F} \cdot \overline{u}_n \cdot [B^T]_{k}^{n} \delta_k^j \] (see p. II-17)

   \[ \overline{F} \cdot \overline{x}_j = B_k^n \overline{F} \cdot \overline{u}_n \]

4. So \( \overline{F} \cdot \overline{u}_n \) and \( \overline{F} \cdot \overline{x}_j \) are covariant components of \( F \) (they transform via \( B \)).

.....

**Claim:**

\[ \overline{F} \cdot \overline{u}^k = F'^k = \text{contravariant component} \]

**proof:**

1. First note that \( \overline{F} \cdot \overline{u}^i = \left[ F'_n u^n \right] \cdot \overline{u}^i = \overline{F} \cdot g^{nk} \) (see page II-16)

2. Consider \( F_k = B_k^n F_n' \) and multiply by \( [g^1]_{kj} \) and sum over \( k \)

3. \[ F_k [g^1]_{kj} = [g^1]_{kj} B_k^n \delta_n^{n'} F_{n'} \]

   \[ = \delta_{kj} B_k^n [g g^{-1}]_n^{n'} F_{n'} \]

   \[ = \delta_{kj} B_k^n g_{nm} g^{mn'} F_{n'} \]

   \[ = \delta_{kj} B_k^n \sum_i [A^T]_{n}^{i} A_{m}^{i} g^{mn'} F_{n'} \] use \( A^T = B^{-1} \)

   \[ = \sum_i [B B^{-1}]_{ji} A_{m}^{i} \left[ g^{mn'} F_{n'} \right] = A_{m}^{i} \left[ g^{mn'} F_{n'} \right] \]

so \( g^{mn'} F_{n'} \) and \( F_k [g^1]_{kj} \) transform via \( A_{m}^{i} \) and are contravariant components of \( F \).
\[ \bar{F} = F'_k \bar{u}^k = F'^j \bar{u}_j \]

**Derivation:**

1. \[ F = f_i q^i = (f_i \sqrt{g^{nn}}) u^a = (F \cdot u_b) u^a = F'_b u^a \text{----as shown on p. II-18.} \]

2. \[ g_{nm} F^{mn} u^n = F^m [g_{mn} u^n] = F^{mn} u_n. \]

\[ \bar{F} \cdot \bar{F} = F'_j F^j \]

**Derivation:**

1. \[ F \cdot F = F'_i u^i \cdot F^k u_k = F'_i F^k \delta^i_k = F'_i F^i. \]

\[ F_i F^i = F'_k F^{ik} \]

**Derivation:**

1. \[ F \cdot F = f_i f^i = B^m_i F_m A^i_k F^k \]
   
   \[ = [A^{-1}]^m_i A^i_k F^m F^k \]
   
   \[ = [A^{-1}A]^m_k F^m F^k \]
   
   \[ = \delta^m_k F^m F^k \]
   
   \[ = F_k F^k \]
Exercises:

1. Assume that in the s,t,w ( = q¹,q²,q³) a physical quantity is given by $\mathbf{F} = 6u_1 + 4u_3$. Find the contravariant components of $\mathbf{F}$ in the s,t,w system (leave answer in terms of s,t,w).

2. Given that for the s,t,w system $\mathbf{g}$ is as shown,
find the covariant components of $\mathbf{F}$ in the stw system.
Leave answer in terms of s,t,w.

$$
\mathbf{g} = \begin{bmatrix}
6 & 2 & -5 \\
2 & 3 & 2 \\
-5 & 2 & 10
\end{bmatrix}
$$

3. Find $F_jF^j$ (note unprimed components $\Rightarrow$ x system).

4. Find $u_1·u_3$. 
**Covariant Form**

When expressions are written totally in terms of covariant and contravariant vector components with proper summation over all indices (such as $F_i F'$) they can be easily evaluated in either $(x,y,z \text{ or } q')$ system. Further, one can set the expression for the quantity in, say, the $xyz$ system equal to the expression in the expression in terms of the $q'$. This is a powerful tool. It means you don't have to use $A$ (or $B$). This is used extensively in special relativity and in general relativity. It can also be of use in any problem where one wants the answer, say, in the $xyz$ coordinate system, but finds it easier to use a coordinate system which follows fluid flows, or isobars, etc.

**A Physical Picture of the Coordinate Systems**

1. assume that 
   
   $s = q^1 = 3x \cdot \cos \alpha + 3y \cdot \sin \alpha$  
   $t = q^2 = 3x \cdot \cos \beta + 3y \cdot \sin \beta$  
   $w = q^3 = 5z$

2. Let $\alpha = \pi/6$ and $\beta = \pi/3$ to simplify the problem.

   $s = 3x(\sqrt{3}/2) + 3y/2$  
   $t = 3x(1/2) + 3y(\sqrt{3}/2)$  
   $w = 5z$

You can determine $A^{-1} = B^T$ and $g^{-1} = B^T B$ easily:

$$
A^{-1} = \begin{bmatrix}
\frac{\sqrt{3}}{2} & \frac{3}{2} & 0 \\
\frac{3}{2} & \frac{\sqrt{3}}{2} & 0 \\
0 & 0 & 5
\end{bmatrix} \quad g^{-1} = B^T B = \begin{bmatrix}
\frac{9}{4} & \frac{\sqrt{3}}{2} & 0 \\
\frac{\sqrt{3}}{2} & 9 & 0 \\
0 & 0 & 25
\end{bmatrix} \quad g = \begin{bmatrix}
\frac{4}{9} & -\frac{\sqrt{3}}{9} & 0 \\
-\frac{\sqrt{3}}{9} & \frac{4}{9} & 0 \\
0 & 0 & \frac{1}{25}
\end{bmatrix}
$$

3. Note that $s \cdot t = u^1 \sqrt{g^{11}} - u^2 \sqrt{g^{22}} = (1/9) \cdot 9 \cdot 3/2 = \sqrt{3}/2 \neq 0$.

   also, $u^1 = 3 \hat{s}$; $u^2 = 3 \hat{t}$; and $u^3 = 5 \hat{w}$.  
   $u_i = g_{1m} u^m$
\[ \begin{align*}
\hat{u}^i &= \sqrt{g_{ii}} \hat{q}^i \quad \text{no sum on } i \\
\hat{u}_k &= g_{kij} \hat{u}^j
\end{align*} \]

**Note** easiest way to find covariant and contravariant components of \( F \):

1) "pick off" \( F_j^i \) as the **coefficients of** \( u^j \) in the expression for \( F \).

2) find \( F^k_i \) from \( g^{kij} F_j \).

**Exercises:**

1. Find the \( u^i \) ( \( i = 1,2,3 \)) in the spherical coordinate system in terms of \( r, \theta, \phi \).

2. Find **covariant** and **contravariant** components of the position vector, \( r \), and \( dr \), in the spherical coordinate system. Leave answer in terms of \( r, \theta, \phi \) and \( dr, d\theta, d\phi \). Hint for \( dr \): recall how \( dq \) are related to the **definition** of contravariant components.
Use of A and B to find $F \cdot \hat{x}^i$

The components of $\mathbf{F}$ in the xyz system can be obtained from the $F^k$ using $A$, or from the $F'^j$ using $B$:

$$
\begin{bmatrix}
\mathbf{F} \cdot \hat{x} \\
\mathbf{F} \cdot \hat{y} \\
\mathbf{F} \cdot \hat{z}
\end{bmatrix} = A
\begin{bmatrix}
F'^1 \\
F'^2 \\
F'^3
\end{bmatrix}
$$

Example: Find $\mathbf{F} \cdot \hat{x}$ if $\mathbf{F} = 2\mathbf{s} + 4\mathbf{t}$ in the example on page II-21.

First, find the $F'^j$: $\mathbf{F} = 2\sqrt{9} \mathbf{u}^1 + 4\sqrt{9} \mathbf{u}^2$ so $F'_1 = 2\sqrt{9}$ and $F'_2 = 4\sqrt{9}$. Then use $B = [A^{-1}]^T$ to relate the covariant components in the xyz and stw system:

$$
\begin{bmatrix}
\mathbf{F} \cdot \hat{x} \\
\mathbf{F} \cdot \hat{y} \\
\mathbf{F} \cdot \hat{z}
\end{bmatrix} =
\begin{bmatrix}
3\sqrt{3} & 3 & 0 \\
2 & 2 & 0 \\
3 & 3\sqrt{3} & 0 \\
2 & 2 & 5 \\
\end{bmatrix}
\begin{bmatrix}
\sqrt{3} + 2 \\
\sqrt{3} \\
\sqrt{3} + 2 \\
\sqrt{3} + 2 \\
\end{bmatrix}
$$

Exercise: In the above example, find $F^k$.  

💬
The Determinant in Summation Notation

\[ \det M = \begin{vmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{vmatrix} \]

\[ = (-1)^{1+1} m_{11}(m_{22}m_{33} - m_{23}m_{32}) + (-1)^{2+1} m_{21}(m_{12}m_{33} - m_{13}m_{32}) + (-1)^{3+1} m_{31}(m_{12}m_{23} - m_{13}m_{22}) \]

\[ = \sum_i \sum_j \sum_k m_{1i} m_{2j} m_{3k} \cdot \{ \text{"sign factor"} \} \quad \text{--Note six terms for which } i, j, \text{ and } k \text{ are all different.} \]

**sign factor** =

- +1 for \( i, j, k \) even permutation of 123
- -1 for \( i, j, k \) odd permutation of 123
- 0 otherwise

\[ = \varepsilon^{ijk} \]

Thus the \( \varepsilon^{ijk} \) symbol can be used to form a concise summation notation for \( \det M \).

\[ \det M = \varepsilon^{ijk} m_{1i} m_{2j} m_{3k} \]

**Exercise:** Write the determinant of \( g \) in summation notation.

**Exercise:** Write \( \left| g^{-1} \right| \) in summation notation. Hint, use \( \varepsilon_{ijk} \) rather than \( \varepsilon^{ijk} \).

**Exercise:** Write \( \left| A \right| A^{-1} \right| \) in summation notation.
Exercise: Write out the terms in $\varepsilon_{kmn} \varepsilon^{kij} \mathbf{g}^{mn} \mathbf{g}_{ji}$ and try to find the simplest final result.

Exercise: There is a "similar" expression (to that in the box on page II-24) for the determinant of a 4x4 matrix, using $\varepsilon^{ijkm}$. Using what you know about determinants write down what you think it should be.

An interesting "expression" for $\varepsilon_{ijk}$ is:

$$\varepsilon^{ijk} = \begin{vmatrix}
\delta^i_1 & \delta^i_2 & \delta^i_3 \\
\delta^j_1 & \delta^j_2 & \delta^j_3 \\
\delta^k_1 & \delta^k_2 & \delta^k_3
\end{vmatrix} = \begin{vmatrix}
(\delta^i_1 \delta^j_2 \delta^k_3 + \delta^i_2 \delta^j_3 \delta^k_1 + \delta^i_3 \delta^j_1 \delta^k_2) \\
(\delta^i_1 \delta^j_3 \delta^k_2 + \delta^i_2 \delta^j_1 \delta^k_3 + \delta^i_3 \delta^j_2 \delta^k_1)
\end{vmatrix}$$

3 even perm. 3 odd perm.

A word of warning: in Cartesian systems one can use $\varepsilon^{ijk}$ and $\varepsilon_{ijk}$ in an "equivalent" way. But in the general coordinate system care must be taken. In strict covariant notation, these symbols are usually accompanied by either $\sqrt{|\mathbf{g}|}$ or $1/\sqrt{|\mathbf{g}|}$. In Cartesian systems, since $|\mathbf{g}|=1$, one cannot see the distinction. We shall discuss this later.
Vector Operators in Covariant Form

1. The gradient, $\nabla \phi$

$$
\nabla \phi = \hat{x}^k \frac{\partial \phi}{\partial x^k} \\
= A^k_j \mathbf{u}^j \mathbf{B}^m_k \frac{\partial \phi}{\partial q^m} \\
= A^k_j [A^{-1}]^m_k \frac{\partial \phi}{\partial q^m} \mathbf{u}^j \\
= [A^{-1}]^m_k A^k_j \frac{\partial \phi}{\partial q^m} \mathbf{u}^j \\
= [A^{-1}A]^m_j \mathbf{u}^j \frac{\partial \phi}{\partial q^m} \\
= \delta^m_j \mathbf{u}^j \frac{\partial \phi}{\partial q^m} \\
= \mathbf{u}^j \frac{\partial \phi}{\partial q^m}
$$

$\nabla \phi = \mathbf{u}^j \nabla'_j \phi$

where we define

$$
\nabla'_j \equiv \frac{\partial}{\partial q^j}
$$

**Exercise:** Find $\nabla \phi(r, \theta, \phi)$ in spherical coordinates, as a function of $r$, $\theta$, $\phi$ and $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ only.

**Exercise:** Find $\nabla \phi(\rho, \theta, z)$ in cylindrical coordinates, as a function of $\rho$, $\theta$, $z$ and $\hat{\rho}$, $\hat{\theta}$, and $\hat{z}$ only.
2. The Divergence, $\nabla \cdot \mathbf{F}$.

1. $\nabla \cdot \mathbf{F} = \sum_k \nabla_k F_k = \nabla_i F^i = \nabla_i F^i$  
\[ = \nabla_i F^i \delta^i_n = \nabla_i F^i \]
\[ = (B_k^i \nabla_i) (A_j^k F^i) \quad \text{Note that } \nabla_i \text{ acts on all that follows.} \]
\[ = B_k^i A_j^k \nabla_i F^i + B_k^i F^i \nabla_i A_j^k \]
\[ = [A^{-1}]_i^j A_j^k \nabla_i F^i + F^i \{ B_k^i \nabla_i A_j^k \} \]
\[ = \delta^i_j \nabla_i F^i + F^i \{ \ldots \ldots \ldots \} \]
\[ = \nabla_i F^i + F^i \{ \ldots \ldots \ldots \} \]

2. We now derive three relationships which will be used to evaluate $\{ \ldots \ldots \ldots \}$.

a) First note that (see page II-12 for $g_{tt}$ in terms of derivatives)
\[ \frac{1}{2} g_{tt} \partial_t g_{tt} / \partial q^i = \frac{1}{2} \sum_a (\partial q^a / \partial x^m)(\partial q^a / \partial x^m) \partial / \partial q^i \left[ \sum_a (\partial x^m / \partial q^a)(\partial x^m / \partial q^a) \right] \quad \text{sum over } i \text{ and } \ell \text{ implied!} \]
\[ = \frac{1}{2} \sum_a (\partial q^a / \partial x^m)(\partial q^a / \partial x^m) \left[ (\partial^2 x^m / (\partial q^a \partial q^a))(\partial x^m / \partial q^a) + (\partial x^m / \partial q^a)(\partial^2 x^m / (\partial q^a \partial q^a)) \right] \quad \text{rearrange terms:} \]
\[ = \frac{1}{2} \sum_a (\partial q^a / \partial x^m)(\partial^2 x^m / (\partial q^a \partial q^a))(\partial x^m / \partial q^a) + (\partial q^a / \partial x^m)(\partial^2 x^m / (\partial q^a \partial q^a)) \]
\[ = \frac{1}{2} \sum_a (\partial q^a / \partial x^m)(\partial^2 x^m / (\partial q^a \partial q^a))(\delta_{\ell m}^n) + (\partial q^a / \partial x^m)(\partial^2 x^m / (\partial q^a \partial q^a))(\delta_{\ell m}^n) \quad \text{-- use } \delta_{\ell m}^n \text{ to do } n \text{ sum} \]
\[ = \frac{1}{2} \sum_a (\partial q^a / \partial x^m)(\partial^2 x^m / (\partial q^a \partial q^a)) \delta_{\ell m}^n + (\partial q^a / \partial x^m)(\partial^2 x^m / (\partial q^a \partial q^a)) \delta_{\ell m}^n \quad \text{-- add terms} \]
\[ = \sum_m (\partial q^m / \partial x^m) \nabla_i (\partial x^m / \partial q^i) + (\partial q^m / \partial x^m) \nabla_i (\partial x^m / \partial q^i) \quad \text{i, } \ell \text{ dummy indices } \Rightarrow \text{ add terms} \]
\[ = \sum_m (\partial q^m / \partial x^m) \nabla_i (\partial x^m / \partial q^i) \]
\[ = B_m^i \nabla_i A_j^m = B_k^i \nabla_i A_j^k \]

so,
\[ \frac{1}{2} g_{tt} \nabla^t_j g_{tt} = B_k^i \nabla^t_i A_j^k \]

and $\{ \ldots \ldots \ldots \} = \frac{1}{2} g_{tt} \nabla^t_j g_{tt}$.
b) Next we consider \( \det g = g = g_{ik} [g^{\text{cof}}]_{ik} \) sum on \( k \), no sum on \( i \): \( i \) can be 1,2,or 3.

Since \( g_{ik} \) is symmetric and \( \det g = \det g^T = g \), \( [g^{\text{cof}}]_{ik} = [g^{\text{cof}}]_{ki} \). Thus,

\[
\frac{\partial g}{\partial g_{ik}} = [g^{\text{cof}}]_{ik}
\]

\( \text{c) Also, we know that,} \)

\[
[g^{-1}]_{ik} = [g^{\text{cof}}]^T_{ik}/g, \text{ note: } g = \det g \]

\[
g^{ik} = [g^{\text{cof}}]_{ki}/g. = \frac{1}{g} \frac{\partial g}{\partial g_{ik}} = \frac{1}{g} \frac{\partial g}{\partial g_{ik}} \text{ using result above for cofactor.} \]

\[
\frac{\partial}{\partial g} [\ln g] = \frac{1}{g} \frac{\partial g}{\partial g_{ik}}
\]

3. Finally, using the results in 2 a), b), and c):

\[
F^j \{\ldots\} = F^j \left[ \frac{1}{2} g_{it} \nabla^i j g_{it} \right] \text{ note sums on } i, t, j
\]

\[
= F^j \left[ \frac{1}{2} (\partial [\ln g]/\partial g_{it}) (\partial g_{it}/\partial q^j) \right]
\]

\[
= F^j \left[ [1/\sqrt{g}] (\partial [\ln g]/\partial q^j) \right]
\]

4. So,

\[
\nabla \cdot F = \sqrt{g}[\sqrt{g}] \nabla_j F^i + F^j [(1/\sqrt{g}) \partial \sqrt{g}/\partial q^j] = (1/\sqrt{g})[\nabla_j \sqrt{g} F^i]
\]

and,

\[
\nabla \cdot F = \frac{1}{\sqrt{g}} \nabla_j [\sqrt{g} F^i]
\]
Exercise: Find $\nabla \cdot \mathbf{F}$ in the spherical coordinate system, using the generalized formula. Put answer in terms of $r, \theta, \phi$ and $\mathbf{r}, \mathbf{\theta}, \mathbf{\phi}$. Use $\mathbf{F} = f_r \mathbf{r} + f_\theta \mathbf{\theta} + f_\phi \mathbf{\phi}$.

Exercise: Repeat the above exercise for the cylindrical coordinate system.

Exercise: Given $\sqrt{g} = st \sin w$ in the $s,t,w$ ($= q_1,q_2,q_3$) system, find $\nabla \cdot \mathbf{F}$ for $\mathbf{F} = [st/w^2]u_1 + [(\sin w)/t]u_2 -wu_3$. 
3. The Laplacian, $\nabla \cdot \nabla \phi = \nabla^2 \phi$.

\[
\nabla \cdot \nabla \phi = \nabla \cdot [\nabla \phi] \quad \text{use form for divergence and gradient:}
\]

\[
= \nabla \cdot [u^i \nabla_i \phi] = \nabla \cdot [u^i u^k (u^j \nabla_j \phi)]
\]

\[
= [1/\sqrt{g}] \nabla_k [\sqrt{g} u^i u^k \nabla_i \phi] \quad \text{note contravariant components of } \nabla \phi
\]

\[
= [1/\sqrt{g}] \nabla_k [\sqrt{g} g^{kj} \nabla_j \phi]
\]

\[
\frac{1}{\sqrt{g}} \nabla' \ n_k [\sqrt{g} g^{kj} \nabla'_j \phi]
\]

Exercise: Find $\nabla^2 [r^2 e^{i\phi}]$ by working totally in the $r, \theta, \phi$ system.
4. The Curl, $\nabla \times \mathbf{F}$

1. Before deriving the general expression for the curl, we need to consider how $\varepsilon^{ijk}$ "transforms" under the generalized coordinate transformation.

Claim:

\[
\varepsilon^{ijk} = A^i_A^m A^k_n \varepsilon^{\prime mn} \sqrt{|g|}
\]

where $g_x = \text{xyz system metric tensor}$

Derivation:

Note that $g_x = \text{metric tensor in the Cartesian system and } g_x = |g_x| = 1$. Consider the following:

\[
\varepsilon^{ijk} |A| = A^i_A^m A^k_n \varepsilon^{\prime mn} = |A| \text{ if } ijk \text{ is cyclic (non-cyclic)}
\]

\[
= 0 \text{ if } ijk \text{ is not a permutation of 123.}
\]

where $\varepsilon^{\prime mn}$ is numerically equal to $\varepsilon^{mn}$

\[
\varepsilon^{ijk} = A^i_A^m A^k_n \varepsilon^{\prime mn} |A|^{-1}
\]

\[
= A^i_A^m A^k_n \varepsilon^{\prime mn} / |A|
\]

\[
= A^i_A^m A^k_n \varepsilon^{\prime mn} / |A|^2
\]

\[
\varepsilon^{ijk} / \sqrt{g} = A^i_A^m A^k_n \varepsilon^{\prime mn} / \sqrt{g}
\]

where one notes that in all cases (e.g., when $|A| < 0$):

\[
\sqrt{g} = |A|
\]

2. To derive the form for the curl, start in the Cartesian system:

\[
\nabla \times \mathbf{F} = \varepsilon^{ijk} \mathbf{x}_i \mathbf{\nabla} j \mathbf{F}_k + \varepsilon^{ijk} \mathbf{x}_j \mathbf{\nabla} i \mathbf{F}_k + \varepsilon^{ijk} \mathbf{x}_k \mathbf{\nabla} i \mathbf{F}_j
\]

\[
= \varepsilon^{ijk} \mathbf{x}_i \mathbf{\nabla} j \mathbf{F}_k + \varepsilon^{ijk} \mathbf{x}_j \mathbf{\nabla} i \mathbf{F}_k + \varepsilon^{ijk} \mathbf{x}_k \mathbf{\nabla} i \mathbf{F}_j
\]

\[
= \varepsilon^{ijk} \mathbf{x}_i \mathbf{\nabla} j \mathbf{F}_k
\]

\[
= [1 / g_x] \varepsilon^{ijk} \mathbf{x}_i \mathbf{\nabla} j \mathbf{F}_k \text{ where } g_x = 1.
\]

That is, the curl can be written like a determinant.
In fact, any cross product in the Cartesian system can be written like a determinant:

\[
\mathbf{F} \times \mathbf{G} = \mathbf{\epsilon}^{ijk} \mathbf{x}^i \mathbf{x}^j \mathbf{F}^k \mathbf{G}^m \mathbf{G}^n = \mathbf{\epsilon}_{lmn} \mathbf{x}^l \mathbf{F}^m \mathbf{G}^n
\]

Cartesian systems only

3. Now we transform the "determinant" form of the curl:

\[
\nabla \times \mathbf{F} = \mathbf{\epsilon}^{ijk} \hat{x}^i \nabla_j F_k = [1/\sqrt{g}] \mathbf{\epsilon}^{ijk} \hat{x}^i \nabla_j F_k
\]

\[
= [1/\sqrt{g}] \mathbf{A}^j_t \mathbf{A}^j_m \mathbf{A}^k_n \mathbf{\epsilon}^{tlnm} \mathbf{B}^r_i \mathbf{F}^r_i \mathbf{B}^s_j \nabla_i \mathbf{B}^t_k \mathbf{F}^t_k
\]

\[
= [1/\sqrt{g}] \mathbf{B}^r_i \mathbf{A}^j_t \mathbf{B}^s_j \mathbf{A}^j_m \mathbf{B}^t_k \mathbf{A}^k_n \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{F}^r_i + \{ \ldots \} \nabla'' \mathbf{B}^t_k \}
\]

\[
= [1/\sqrt{g}] \mathbf{A}^{t1} \mathbf{A}^i_t \mathbf{A}^{1j} \mathbf{A}^{j1} \mathbf{A}^{k1} \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{F}^r_i + \{ \ldots \}
\]

\[
= [1/\sqrt{g}] \mathbf{A}^{t1} \mathbf{A}^i_t \mathbf{A}^{1j} \mathbf{A}^{j1} \mathbf{A}^{k1} \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{F}^r_i + \{ \ldots \}
\]

\[
= [1/\sqrt{g}] \mathbf{\delta}^r_i \mathbf{\delta}^s_m \mathbf{\delta}^t_n \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{F}^r_i + \{ \ldots \}
\]

\[
= [1/\sqrt{g}] \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{F}^r_i + \{ \ldots \}
\]

where we shall next show that the second term, \{ \ldots \} = 0.

4. \{ \ldots \} = [1/\sqrt{g}] \mathbf{B}^r_i \mathbf{A}^i_t \mathbf{B}^s_j \mathbf{A}^j_m \mathbf{A}^k_n \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{B}^t_k

\[
= [1/\sqrt{g}] \mathbf{\delta}^r_i \mathbf{\delta}^s_m \mathbf{\delta}^t_n \mathbf{\epsilon}^{tlnm} \mathbf{u}_i \nabla_i \mathbf{B}^t_k
\]

\[
= [1/\sqrt{g}] \mathbf{\delta}^r_i \mathbf{\delta}^s_m \mathbf{u}_i \mathbf{F}^r_i \mathbf{\epsilon}^{tlnm} \mathbf{A}^k_n \nabla_i \mathbf{B}^t_k
\]

\[
= [1/\sqrt{g}] \mathbf{u}_i \mathbf{F}^r_i \mathbf{\epsilon}^{tlnm} \mathbf{A}^k_n \nabla_i \mathbf{B}^t_k
\]

\[
= [1/\sqrt{g}] \mathbf{u}_i \mathbf{F}^r_i \mathbf{\epsilon}^{tlnm} \mathbf{F}^r_i \mathbf{\epsilon}^{tlnm} \mathbf{e}_n \mathbf{e}_m \nabla_i \mathbf{e}_k
\]

\[
= [1/\sqrt{g}] \mathbf{u}_i \mathbf{F}^r_i \mathbf{\epsilon}^{tlnm} \mathbf{\partial} x^l / \partial q^n \nabla_i \mathbf{\partial} q^m / \partial x^k
\]
\[ \{...\} = [1/\sqrt{g}] uF_i \epsilon^{lmn}[\partial_i \partial^m (\partial^q/\partial x^k)(\partial x^k/\partial q^m) - (\partial q^i/\partial x^k)\partial_j \partial^m (\partial x^k/\partial q^m)] \]
\[ = [1/\sqrt{g}] uF_i \epsilon^{lmn}[\partial_i \partial^m (\partial^q/\partial x^k) - B_k^t \partial_j \partial^m (\partial x^k/\partial q^m)] \]
\[ = [1/\sqrt{g}] uF_i \epsilon^{lmn}[\partial_i \partial^m (\delta^l_n) - B_k^t \nabla_m \nabla_n x^k] \]
\[ = [1/\sqrt{g}] uF_i \epsilon^{lmn}[0 \epsilon B_k^t \nabla_m \nabla_n x^k] \]
\[ = \epsilon [1/\sqrt{g}] F_i B_k^t \epsilon^{lmn} u \nabla_m \nabla_n x^k \]
\[ = [1/\sqrt{g}] F_i B_k^t \epsilon^{lmn} u \nabla_m \nabla_n x^k \quad \text{since } \epsilon^{lmn} = -\epsilon^{lnm} \]
\[ = [1/\sqrt{g}] F_n B_k^t \epsilon^{lmn} u \nabla_n \nabla_m x^k \quad \text{since } \nabla_n \nabla_m = \nabla_m \nabla_n \]
\[ = [1/\sqrt{g}] F_n B_k^t \epsilon^{lmn} u \nabla_m \nabla_n x^k \quad \text{since } m \text{ and } n \text{ are dummy indices.} \]
\[ = 0 \quad \text{(note: we showed } \{...\} = - \{...\} \text{ so } \{...\} = 0) \]

Thus,

\[ \nabla \times F = [1/\sqrt{g}] \epsilon^{ijk} u_j \nabla' i F_k \]

(general q^i system)

Note:

\[ \epsilon^{ijk} x_i x_j F_k = 0 \implies (rxr)F = 0 \]
\[ \epsilon^{ijk} \nabla \nabla F_k = 0 \implies (\nabla \nabla)F = 0 \quad \text{if the } \nabla \quad \text{operators do not act on a function following } F. \]
\[ \epsilon^{ijk} F_i x_j x_k = 0 \implies F(rxr) = 0 \]

In general, in an expression involving \( \epsilon^{ijk} \) or \( \epsilon_{ijk} \):

if the remainder of the expression is symmetric under the interchange of any two of the indices ijk then the total expression is zero. No conclusion can be made if the remainder of the expression is antisymmetric under the interchange of two indices.
Writing the curl as a determinant can sometimes make a calculation easier:

\[
\nabla \times F = \begin{vmatrix}
\frac{1}{\sqrt{g}} & u_1 & u_2 & u_3 \\
\nabla'_1 & \nabla'_2 & \nabla'_3 \\
F'_1 & F'_2 & F'_3
\end{vmatrix}
\]

Transforming the general expression for the curl using the metric tensor:

1. First, we note that:

\[
\varepsilon^{ijk} \left| g^{-1} \right| = g^{il} g^{jm} g^{kn} \varepsilon'_{(mn} = g^{il} g^{mj} g^{nk} \varepsilon'_{(mn}
\]

where \( \varepsilon'_{(mn} \) is equal numerically to \( \varepsilon_{(mn} \)

\[
\varepsilon^{ijk} = g^{il} g^{mj} g^{nk} \left| g^{-1} \right| \varepsilon'_{(mn}
\]

2. Thus starting with the general form given on page II-33,

\[
\nabla \times F = \left[ \frac{1}{\sqrt{g}} \right] \varepsilon^{ijk} u_i \nabla_j \mathbf{F}_k \text{ use the form for } \varepsilon^{ijk} \text{ given above,}
\]

\[
= \left[ \frac{1}{\sqrt{g}} \right] \left| g^{-1} \right| \varepsilon'_{(mn} \mathbf{u}^i \nabla_j \mathbf{F}_k
\]

\[
= \left[ \frac{1}{\sqrt{g}} \right] \varepsilon'_{(mn} \mathbf{u}^i \nabla_j \mathbf{F}_k
\]

You don't have to follow this proof. It essentially shows that one cannot transform the expression into one in which all the covariant indices are transformed into contravariant indices (and all the contravariant indices are transformed into covariant indices).

\[
\neq \int g \varepsilon'_{(mn} \mathbf{u}^i \nabla^m \mathbf{F}_{im}
\]

The expression in this box can not be used for the curl!

Note that \( \nabla'^m = g^{mi} \nabla'_i \neq \nabla'_m \) and \( \nabla \times \mathbf{F} \neq \int g \varepsilon'_{(mn} \mathbf{u}^i \nabla^m \mathbf{F}_{im} \)
In general,
\[
\mathbf{F} \times \mathbf{G} = \sqrt{g} \, \varepsilon'_{\ell mn} \, u^\ell \, F^m \, G^n = [1/\sqrt{g}] \varepsilon^{\ell jk} \, u^i \, F'_j \, G'_k
\]
\[
\mathbf{W} \cdot (\mathbf{F} \times \mathbf{G}) = u^\ell \, \mathbf{W}_m \cdot [1/\sqrt{g}] \varepsilon^{\ell jk} \, u_i \, F'_j \, G'_k = [1/\sqrt{g}] \varepsilon^{\ell jk} \, W_i \, F'_j \, G'_k
\]
(recall \( u^m \cdot u_i = \delta^m_i \))

**Exercise care with such formula when any vector is replaced by \( \nabla \).**

Some other expressions which might be useful:

\[
\varepsilon'_{\ell mn} = \frac{g_{\ell i} \, g_{mj} \, g_{nk}}{|g|} \varepsilon^{ijkl}
\]
\[
\varepsilon_{ijkl} = \frac{B^i_j \, B^m_j \, B^n_k}{|B|} \varepsilon'_{\ell mn}
\]
\[
\varepsilon^{ijkl} = \frac{A^i_j \, A^m_j \, A^n_k}{|A|} \varepsilon'_{\ell mn}
\]

**Exercise:**

In the stw coordinate system where \( \mathbf{g} \) and \( \mathbf{g}^{-1} \) are given by

\[
\mathbf{F} = s^t w s^l \omega^t \omega^l \cdot \Sigma (s w) \omega
\]

and \( \mathbf{F} = s^t w s^l \omega^t \omega^l \cdot \Sigma (s w) \omega \) find \( u^i \cdot \nabla x \mathbf{F} \).

\[
\mathbf{g} = \begin{bmatrix} 7 & 0 & -1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix} ; \quad \mathbf{g}^{-1} = \frac{1}{81} \begin{bmatrix} 12 & 0 & 3 \\ 0 & 27 & 0 \\ 3 & 0 & 21 \end{bmatrix}
\]