On stochasticity in nearly-elastic systems

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1 dimensional elastic system

- Hamiltonian System: \( H = \frac{p^2}{2} + F(q) \).
- \( X_t = (q_t, p_t) \).
- Elastic reflection at boundary points.
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Figure: The 1-dimensional mechanical model
nearly-elastic system

Coefficient of restitution:
\[ p_{new} = -p_{old}(1 - \varepsilon c_i(p_{old})) \]
for \( i = 1, 2 \), \( q = a_1, a_2 \), nearly-elastic reflection.

Long-time behavior: \( X^\varepsilon_t = (q^\varepsilon_t, p^\varepsilon_t) \), rescale time: \( X^\varepsilon_{t/\varepsilon} \), fast motion, slow motion, projection onto the graph \( \Gamma \):
\[ Y^\varepsilon_t = (H(q^\varepsilon_t, p^\varepsilon_t), K(q^\varepsilon_t, p^\varepsilon_t)). \]

Distribution of \( q^\varepsilon_\infty \) as \( \varepsilon \downarrow 0 \)? No limiting distribution!

Regularization needed: small perturbation of intensity \( \delta \),
\[ Y^\varepsilon,\delta_t = Y(q^\varepsilon_t,\delta, p^\varepsilon_t,\delta) \], double limit \( \lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} Y^\varepsilon,\delta_t = Y_t \), weak convergence.
Model problems

- $H = \frac{p^2}{2}$.
- $q_2, q_3, \ldots, q_{n-1} \in (q_1, q_n)$, potential has sharp maxima at $q_i, 2 \leq i \leq n - 1$ and is close to a constant between $q_i$ and $q_{i-1}$.
- nearly-elastic: $p_{\text{new}} = -p_{\text{old}}(1 - \varepsilon c_k(p_{\text{old}}))$. 

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Figure: 1-dimensional model problem
Regularization by perturbation of the initial condition

- consider the case when system has only two wells
- averaging: fast motion: $q^\varepsilon_t/\varepsilon$; slow motion: $H^\varepsilon_t/\varepsilon$

\[
\frac{dH}{dt} = -2 \frac{c_1(\sqrt{2H}) + c_2(\sqrt{2H})}{T_3(H)} H, \quad H(0) = H_0 \quad \text{on } l_3
\]

\[
\frac{dH}{dt} = -2 \frac{c_1(\sqrt{2H}) + c_3(\sqrt{2H})}{T_1(H)} H, \quad \text{on } l_1
\]

\[
\frac{dH}{dt} = -2 \frac{c_2(\sqrt{2H}) + c_3(\sqrt{2H})}{T_2(H)} H, \quad \text{on } l_2
\]

Here $T_3(H) = \frac{2(a_1 + a_2)}{\sqrt{2H}}, \quad T_1(H) = \frac{2a_1}{\sqrt{2H}}, \quad T_2(H) = \frac{2a_2}{\sqrt{2H}}$

are the corresponding periods of motion on phase picture $\Gamma$ along the energy level $H$ for each edge of $\Gamma$.
Figure: The case when the model system has only two wells
Regularization by perturbation of the initial condition

- perturbation of the initial condition: \( X_0^{\varepsilon,\delta} = x + \xi_\delta \), where \( \xi_\delta \) is a two dimensional random variable with a continuous density \( f_\delta(x) \) such that \( f_\delta(x) > 0 \) for \( |x| < \delta \) and \( f_\delta(x) = 0 \) for \( |x| \geq \delta \).

- \( \mathcal{U}(x, \delta) = \{ y \in \cap : |x - y| < \delta \}, x \in \cap, \delta > 0 \),
  \( \mathcal{U}_i^{\varepsilon} = \{ x \in \cap : X^{\varepsilon,x} \text{ eventually enters the well } \mathcal{E}_i \}, i \in \{1, 2\} \)

\[
\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} \frac{\mu(\mathcal{U}_1^{\varepsilon} \cap \mathcal{U}(x,\delta))}{\mu(\mathcal{U}_2^{\varepsilon} \cap \mathcal{U}(x,\delta))} = \frac{c_1(H(O))}{c_2(H(O))}.
\]

- branching probabilities 
  \[
p_1 = \frac{c_1(H(O))}{c_1(H(O)) + c_2(H(O))},
\]
  \[
p_1 = \frac{c_2(H(O))}{c_1(H(O)) + c_2(H(O))}
\]
**Figure:** Regularization by stochastic perturbation of the initial condition
Regularization by perturbation of the initial condition

- multi-well case: does not work!
- example: \( c_1 \) and \( c_2 = c_3 = c_4 = c_5 = c (\neq c_1) \), strip A goes alternatively into wells 1 and 2 or wells 1 and 3

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Figure: The case when the system has more than two trapping wells
Regularization by perturbation of the dynamics

- assume $p_{\text{new}} = -p_{\text{old}}(1 - \varepsilon c_1 - \varepsilon \delta \eta_k)$,
  $p_{\text{new}} = -p_{\text{old}}(1 - \varepsilon c_2 - \varepsilon \delta \xi_k)$, $p_{\text{new}} = -p_{\text{old}}(1 - \varepsilon c_3 - \varepsilon \delta \zeta_k)$.

- averaging:
  
  $H^\delta(t) = \left( \frac{t}{\sqrt{2}} \frac{c_1 + c_2 + \delta (\mathbb{E}\xi + \mathbb{E}\eta)}{a_1 + a_2} + H_0^{-1/2} \right)^{-2}$, on $l_3$
  
  $H^\delta(t) = \left( \frac{t-t_0}{\sqrt{2}} \frac{c_1 + c_3 + \delta (\mathbb{E}\zeta + \mathbb{E}\eta)}{2a_1} + H(O)^{-1/2} \right)^{-2}$, on $l_1$
  
  $H^\delta(t) = \left( \frac{t-t_0}{\sqrt{2}} \frac{c_2 + c_3 + \delta (\mathbb{E}\xi + \mathbb{E}\zeta)}{2a_2} + H(O)^{-1/2} \right)^{-2}$, on $l_2$

- Here $t_0^\delta = \frac{2(a_1 + a_2)(\sqrt{H_0} - \sqrt{H(O)})}{c_1 + c_2 + \delta \mathbb{E}(\xi + \eta)}$ is the time for the motion $H^\delta(t)$ to come to the interior vertex $O$. 

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branching probabilities: \( \lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} P\{\tilde{X}_t^{\epsilon,\delta} \text{ finally falls into the well } \mathcal{E}_i\} = p_i \), \( p_i = \frac{c_i}{c_1 + c_2} \), \( i = 1, 2 \), proof based on random walk lemma.
Regularization by perturbation of the dynamics

- random walk lemma

\[ S_0^x = x, \quad S_{2m}^x = \sum_{k=1}^{m} (\xi_k + \eta_k), \quad S_{2m+1}^x = S_{2m}^x + \xi_{m+1}. \]

Define \( \tau_{\lambda}^{x,\lambda} \) as the first time \( m \) when \( S_m^x \) is greater than \( \lambda n \):

\[ \tau_{\lambda}^{x,\lambda} = \min\{ m : S_m^x > n\lambda \}. \]

\[
\lim_{n \to \infty} P_0\{\tau_{\lambda}^n \text{ is even}\} = \frac{\mathbb{E}\eta}{\mathbb{E}\xi + \mathbb{E}\eta},
\]

\[
\lim_{n \to \infty} P_0\{\tau_{\lambda}^n \text{ is odd}\} = \frac{\mathbb{E}\xi}{\mathbb{E}\xi + \mathbb{E}\eta}.
\]
Figure: Regularization by stochastic perturbation of the dynamics
Regularization by perturbation of the dynamics

- multi-well case: still works!

- averaging: \( \frac{dH}{dt} = -2 \frac{c_1^{(i)} + c_2^{(i)}}{T_i(H)} H, \quad T_i(H) = \frac{\sqrt{2m_i}}{\sqrt{H}} \)

- branching probabilities: \( p_1^{(l)} = \frac{c_1^{(i)}}{c_1^{(i)} + c_2^{(i)}}, \quad p_2^{(l)} = \frac{c_2^{(i)}}{c_1^{(i)} + c_2^{(i)}} \)
Figure: The case when the system has more than two trapping wells
only have to consider the case when $F(q)$ has just one maximum within $[a_1, a_2]$.

only have to use regularization by perturbation of the initial condition.

limiting process $Y_t$
General Potential

\[ T_3(H) = 2 \int_{a_1}^{a_2} \frac{dq}{\sqrt{2(H-F(q))}}, \quad T_1(H) = 2 \int_{a_1}^{a_2} \frac{dq}{\sqrt{2(H-F(q))}}, \]
\[ T_2(H) = 2 \int_{a_1}^{a_2} \frac{dq}{\sqrt{2(H-F(q))}}, \]
\[ H_0 = H(q_0, p_0) > H(O), \]
\[ K_0 = 3, \quad \dot{H}_t = 2 \frac{c_k(H_t)}{T_k(H_t)} \text{ inside } I_k, \quad k \in \{1, 2, 3\}, \]
where \( c_1(H) \) and \( c_2(H) \) are given and \( c_3(H) = \frac{1}{2}(c_1(H) + c_2(H)) \). The
trajectory \( Y_t \) hits \( O \) in a finite time \( t_0 = \int_{H(O)}^{H(q_0, p_0)} \frac{2 T_3(z)}{c_3(z)} dz \).

After hitting vertex \( O \), \( Y_t \) leaves \( O \) immediately and goes to
\( I_1 \) or \( I_2 \) with probabilities \( p_1 = \frac{c_1(H(O))}{c_1(H(O)) + c_2(H(O))} \) and
\( p_2 = \frac{c_2(H(O))}{c_1(H(O)) + c_2(H(O))} \).
Figure: The case of general potential
2-dimensional problem

- 2 dimensional problem, general potential
**Figure:** A 2-dimensional problem
billiard model

- phase space $\Lambda: (g, \varphi), g \in G \cup \partial G, 0 \leq \varphi \leq 2\pi$.
- Poincare section $\Pi: (s, \theta), 0 \leq s < L, 0 \leq \theta \leq \pi$.
- billiard map $f: \Pi \rightarrow \Pi, f(s, \theta) = (S(s, \theta), \Theta(s, \theta))$.
- Liouville measure: $m(s, \theta)dsd\theta = \sin \theta dsd\theta$.
- invariant measure on $\Pi$ not unique!
Figure: The billiard model

\[ f(s, \theta) = (S(s, \theta), \Theta(s, \theta)) \]
billiard model

- energy loss: $H - \varepsilon c(x)$.
- scheme of perturbation: diffusion process $l^\theta_t, l^\theta_0 = \theta, 0 \leq \theta \leq \pi$, governed by operator $Lu = \frac{1}{2 \sin \theta} \frac{d}{d\theta} (a(\theta) \frac{du}{d\theta})$, instantaneous reflection at the boundary points.

- scheme of perturbation: $Z^x_\delta = (s, l^\theta_\delta)$.
- 3 processes: $(H^\varepsilon, \delta, g^\varepsilon, \delta, \varphi^\varepsilon, \delta); (\hat{H}^\varepsilon, \delta, g^\varepsilon, \delta, \varphi^\varepsilon, \delta); X^\delta = (s^\delta, \theta^\delta)$.

- Define $\Lambda^\varepsilon, \delta_t = (g^\varepsilon, \delta_t, \varphi^\varepsilon, \delta_t)$;
  $M^\varepsilon, \delta_t = (H^\varepsilon, \delta_t, g^\varepsilon, \delta_t), \hat{M}^\varepsilon, \delta_t = (\hat{H}^\varepsilon, \delta_t, g^\varepsilon, \delta_t)$;
  $N^\varepsilon, \delta_t = (H^\varepsilon, \delta_t, \Lambda^\varepsilon, \delta_t), \hat{N}^\varepsilon, \delta_t = (\hat{H}^\varepsilon, \delta_t, \Lambda^\varepsilon, \delta_t)$. 

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billiard model

- After regularization: ergodicity of $X^\delta_n$ on $\Pi$.
- Single out the unique invariant measure: Liouville measure $m(x) dsd\theta$.
- Averaging: fast motion: billiard flow; slow motion: energy loss.

$$\lim_{\delta \downarrow 0} \lim_{\varepsilon \downarrow 0} H_{t/\varepsilon}^{\varepsilon, \delta} = H_t, \quad \frac{dH_t}{dt} = -\frac{\int_{\Pi} c(x) m(x) \, dx}{\int_{\Pi} T(x, H_t) m(x) \, dx}, \quad H_0 = H_0^{\varepsilon, \delta}.$$ 

- Above equation simplified: 
  $$\frac{dH_t}{dt} = -\sqrt{2H_t} \frac{1}{2\pi A} \int_{\Pi} c(x) m(x) \, dx$$

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Figure: The billiard model with energy loss and perturbation
billiard model

- Geometric fact: $dpd\varphi = \sin \theta dsd\theta$.
- Equation of the chord: $X \cos \varphi + Y \sin \varphi = p$.
- $\int_{\Pi} L(x)m(x)dx = 2\pi A$. 

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Figure: A problem in integral geometry
branching probabilities?

step 1: piecewise linear modification.

step 2: time change: $d\tilde{t} = \frac{L(x)}{c(f(x))} dt$.

step 3: "recover" invariant measure on $\wedge$: the formula of Khasminskii,

$$\hat{\mu}(B) = \frac{1}{2L} \int_\Pi m(x)dx \mathbb{E}_x \int_0^{T_x^1} \chi_B(\Lambda_{t}^{\varepsilon,\delta}) dt,$$

Lebesgue measure.

step 4: geometric calculation
The geometric calculation:

$$\lim_{\varepsilon \downarrow 0} P\{ \hat{N}_{t}^{\varepsilon, \delta} \text{ finally falls into well 1 } \}$$

$$= D \int_{\Lambda_1} \frac{c(f(x(g, \varphi)))}{L(x(g, \varphi))} dgd\varphi$$

$$= D \int_{f^{-1}(\Pi_1)} \frac{c(f(s, \theta))}{L(s, \theta)} L(s, \theta) \sin \theta dsd\theta$$

$$= D \int_{f^{-1}(\Pi_1)} c(f(s, \theta)) m(s, \theta) dsd\theta$$

$$= D \int_{\Pi_1} c(s, \theta) m(s, \theta) dsd\theta.$$
billiard model

$\text{conclusion: } p_1 = \frac{\int_{\Pi_1} c(x)m(x)dx}{\int_{\Pi} c(x)m(x)dx}, p_2 = \frac{\int_{\Pi_2} c(x)m(x)dx}{\int_{\Pi} c(x)m(x)dx}$

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On stochasticity in nearly-elastic systems
The end

Thank you!