Small mass asymptotic for the motion with variable and vanishing friction.

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The Langevin equation for a particle in a fluid is the Newton’s equation of a form
\[
\mu \ddot{q}_t^\mu = b(q_t^\mu) - \lambda \dot{q}_t^\mu + \sigma(q_t^\mu) \dot{W}_t, \quad q_0^\mu = q \in \mathbb{R}^d, \quad \dot{q}_0^\mu = p \in \mathbb{R}^d.
\]

- \( q_t^\mu \) the position of the particle; \( \mu \) is the small mass; \( \lambda > 0 \) is a constant friction; \( b(\bullet) \) is the drift; \( \sigma(\bullet) \) is a diffusion matrix; \( \dot{W}_t \) is a multidimensional Wiener process.
Small mass asymptotic (Smoluchowski-Kramers approximation).

- **Langevin equation**
  \[
  \mu \ddot{q}_t^{\mu} = b(q_t^{\mu}) - \lambda \dot{q}_t^{\mu} + \sigma(q_t^{\mu})\dot{W}_t, \quad q_0^{\mu} = q \in \mathbb{R}^d, \quad \dot{q}_0^{\mu} = p \in \mathbb{R}^d.
  \]

- Let \( \mu = 0 \) we get
  \[
  \dot{q}_t = \frac{1}{\lambda} b(q_t) + \frac{1}{\lambda} \sigma(q_t)\dot{W}_t, \quad q_0 = q_0^{\mu} = q \in \mathbb{R}^d.
  \]

- For any \( \kappa > 0 \) we have
  \[
  \lim_{\mu \downarrow 0} P \left( \max_{0 \leq t \leq T} |q_t^{\mu} - q_t|_{\mathbb{R}^d} > \kappa \right) = 0.
  \]

- The above approximation is called *Smoluchowski-Kramers approximation*. 
Variable friction $\lambda = \lambda(q)$?

▶ Variable friction: $\lambda = \lambda(q)$ is a function of the position.

▶ First suppose that $0 < \lambda_0 \leq \lambda(q) \leq \Lambda < \infty$.

▶ Langevin equation with variable friction

$$\mu \ddot{q}_t^\mu = b(q_t^\mu) - \lambda(q_t) \dot{q}_t^\mu + \sigma(q_t^\mu) \dot{W}_t, \quad q_0^\mu = q \in \mathbb{R}^d, \quad \dot{q}_0^\mu = p \in \mathbb{R}^d.$$

▶ Let $\mu = 0$ we get

$$\dot{q}_t = \frac{b(q_t)}{\lambda(q_t)} + \frac{\sigma(q_t)}{\lambda(q_t)} \dot{W}_t.$$

▶ Is it again true that for any $\kappa > 0$ we have

$$\lim_{\mu \downarrow 0} P \left( \max_{0 \leq t \leq T} |q_t^\mu - q_t|^d \mathbb{R}^d > \kappa \right) = 0 ?$$
Variable friction $\lambda = \lambda(q)$?

- The answer is **NO** in general.
- One has to use a further regularization.
- **Remark.** If friction is variable, the limit still can exist without regularization. It is a bit different from the constant friction case, but it coincides with the regularized result. See [Hottovy et al., *J Stat Phys* (2012) **146**, 762–773].
Approximation of the Wiener process.

Approximation of the Wiener process

\[ W^\delta_t = \frac{1}{\delta} \int_0^\infty W_s \rho \left( \frac{s-t}{\delta} \right) ds = \frac{1}{\delta} \int_0^\delta W_{s+t} \rho \left( \frac{s}{\delta} \right) ds, \]

where \( \rho(\bullet) \) is a smooth \( C^\infty \) function whose support is contained in the interval \([0, 1]\) such that

\[ \int_0^1 \rho(s) ds = 1. \]

\( \dot{W}^\delta_t \) is a small \( \delta \)-correlated noise.
Regularized Smoluchowski-Kramers approximation.

▶ Langevin equation with an approximated Wiener process

\[
\mu \ddot{q}_{t}^{\mu,\delta} = b(q_{t}^{\mu,\delta}) - \lambda(q_{t}^{\mu,\delta}) \dot{q}_{t}^{\mu,\delta} + W_{t}^{\delta}, \quad q_{0}^{\mu,\delta} = q \in \mathbb{R}^d, \quad \dot{q}_{0}^{\mu,\delta} = p \in \mathbb{R}^d.
\]

▶ First let \( \mu \downarrow 0 \) we have

\[
\dot{\tilde{q}}_{t}^{\delta} = \frac{b(\tilde{q}_{t}^{\delta})}{\lambda(\tilde{q}_{t}^{\delta})} + \frac{1}{\lambda(\tilde{q}_{t}^{\delta})} \dot{W}_{t}^{\delta}, \quad \tilde{q}_{0}^{\delta} = q \in \mathbb{R}^d.
\]

▶ Regularized Smoluchowski-Kramers approximation

\[
\lim_{\mu \downarrow 0} P \left( \max_{0 \leq t \leq T} |q_{t}^{\mu,\delta} - \tilde{q}_{t}^{\delta}|_{\mathbb{R}^d} > \kappa \right) = 0.
\]
Regularized Smoluchowski-Kramers approximation:

Second limit as $\delta \downarrow 0$.

- First let $\mu \downarrow 0$ we have
  \[
  \dot{\tilde{q}}^\delta_t = \frac{b(\tilde{q}^\delta_t)}{\lambda(\tilde{q}^\delta_t)} + \frac{1}{\lambda(\tilde{q}^\delta_t)} \tilde{W}^\delta_t, \quad \tilde{q}^\delta_0 = q \in \mathbb{R}^d.
  \]

- Second limit: as $\delta \downarrow 0$ we let
  \[
  \dot{\hat{q}}_t = \frac{b(\hat{q}_t)}{\lambda(\hat{q}_t)} + \frac{1}{\lambda(\hat{q}_t)} \hat{W}_t, \quad \hat{q}_0 = q \in \mathbb{R}^d.
  \]

- Limit as $\delta \downarrow 0$:
  \[
  \lim_{\delta \to 0} \mathbb{E} \max_{t \in [0, T]} |\tilde{q}^\delta_t - \hat{q}_t|_{\mathbb{R}^d} = 0.
  \]

- Because of the Wong-Zakai theorem we have Stratonovich integral instead of Itô integral.

Fast oscillating friction and drift

\[ \mu \ddot{q}_{t}^{\mu,\delta,\varepsilon} = \begin{bmatrix} b \left( \frac{q_{t}^{\mu,\delta,\varepsilon}}{\varepsilon} \right) - \lambda \left( \frac{q_{t}^{\mu,\delta,\varepsilon}}{\varepsilon} \right) \dot{q}_{t}^{\mu,\delta,\varepsilon} + \dot{W}_{t}^{\delta} \end{bmatrix}, \]

\[ q_{0}^{\mu,\delta,\varepsilon} = q \in \mathbb{R}^{d}, \quad \dot{q}_{0}^{\mu,\delta,\varepsilon} = p \in \mathbb{R}^{d}. \]

Using homogenization theory to characterize the limit as first \( \mu \downarrow 0 \) and then \( \delta \downarrow 0 \) and then \( \varepsilon \downarrow 0 \).
Application 2. Friction with jump: gluing condition.

- Friction with a jump, 1-d case:
  \[ \mu q^{\mu,\delta,\varepsilon}_t = b(q_t^{\mu,\delta,\varepsilon}) - \lambda_\varepsilon(q_t^{\mu,\delta,\varepsilon}) q_t^{\mu,\delta,\varepsilon} + \dot{W}_t^\delta, \]
  \[ q^{\mu,\delta,\varepsilon}_0 = q \in \mathbb{R}^1, \quad \dot{q}^{\mu,\delta,\varepsilon}_0 = p \in \mathbb{R}^1. \]

- \( \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon(q) = \lambda_1 > 0 \) for \( q < 0. \)

- \( \lim_{\varepsilon \downarrow 0} \lambda_\varepsilon(q) = \lambda_2 > 0 \) for \( q \geq 0. \)

- \( \lambda_1 \neq \lambda_2. \)

- Gluing condition \( \frac{1}{\lambda_1} f'_-(O) = \frac{1}{\lambda_2} f'_+(O). \) (Such boundary conditions are typical for diffusion processes on graphs!)
gluing condition at $O$: \[ \frac{1}{\lambda_1} f_-(O) = \frac{1}{\lambda_2} f_+(O) \]
Vanishing friction.

- We assumed previously that $0 < \lambda_0 \leq \lambda(q) \leq \Lambda < \infty$.
- What if in some regions we have $\lambda(q) = 0$?
What is the generator and boundary condition?

G

\( \lambda(q) = 0 \)

no time spent here? mixing in G?

Limiting process lives outside G ... so ...

boundary condition?

**Fig.**
Vanishing friction: Regularization.

- We apply a further regularization by adding a small $\varepsilon$ in the friction:

$$\mu \ddot{q}_{t, \mu, \delta, \varepsilon} = b(q_{t, \mu, \delta, \varepsilon}) - (\lambda(q_{t, \mu, \delta, \varepsilon}) + \varepsilon)\dot{q}_{t, \mu, \delta, \varepsilon} + \dot{W}_{t, \delta},$$

$$q_{0, \mu, \delta, \varepsilon} = q, \quad \dot{q}_{0, \mu, \delta, \varepsilon} = p.$$

- We assume that $\lambda(q) = 0$ for $q \in [G] \subset \mathbb{R}^n$ and $\lambda(q) > 0$ for $q \in \mathbb{R}^n \setminus [G]$. Here $G$ is a domain in $\mathbb{R}^n$ and $[G]$ its closure in the standard Euclidean metric.

- For simplicity of presentation we assume that $\sigma(\cdot) = \text{identity}$.

- We study the limit as first $\mu \downarrow 0$ then $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$. 
Vanishing friction: first $\mu \downarrow 0$ and then $\delta \downarrow 0$.

As first $\mu \downarrow 0$ and then $\delta \downarrow 0$ previous results apply: the limiting process looks like

$$\dot{q}_t^\varepsilon = \frac{1}{\lambda(q_t^\varepsilon) + \varepsilon} b(q_t^\varepsilon) + \frac{1}{\lambda(q_t^\varepsilon) + \varepsilon} \circ \dot{W}_t$$

and we study the limit of $q_t^\varepsilon$ as $\varepsilon \downarrow 0$.

In Itô’s form it is

$$\dot{q}_t^\varepsilon = \frac{1}{\lambda(q_t^\varepsilon) + \varepsilon} b(q_t^\varepsilon) - \frac{\nabla \lambda(q_t^\varepsilon)}{2(\lambda(q_t^\varepsilon) + \varepsilon)^3} + \frac{1}{\lambda(q_t^\varepsilon) + \varepsilon} \dot{W}_t$$
Vanishing friction: general theme.

- Limiting process lives outside the domain $G$.
- **Glue all points of $[G]$** and introduce a projection $\pi$.
- The projected space (image of $\pi$) is appropriate for a continuous version of the limiting process to live in.
- We have to show Markov property of the limiting process.
- We have to identify the generator of the limiting process (in Hille-Yosida sense).
- We have to specify the boundary condition (=domain of definition of the generator) at the image of $\pi([G])$. 
Vanishing friction: 1-d case.

\[ \dot{q}_t^\varepsilon = \frac{b(q_t^\varepsilon)}{\lambda(q_t^\varepsilon) + \varepsilon} - \frac{\lambda'(q_t^\varepsilon)}{2(\lambda(q_t^\varepsilon) + \varepsilon)^3} + \frac{1}{\lambda(q_t^\varepsilon) + \varepsilon} \dot{W}_t, \quad q_0^\varepsilon = q_0 \in \mathbb{R}. \]

- Feller’s $D_{m^\varepsilon}D_{s^\varepsilon}$ process. $m^\varepsilon$-speed measure; $s^\varepsilon$-scale function.
- $\lambda(q) = 0$ for $q \in [-1, 1]$. Outside $[-1, 1]$ we have $m^\varepsilon(\bullet) \to m(\bullet)$ and $s^\varepsilon(\bullet) \to s(\bullet)$ as $\varepsilon \downarrow 0$.

- Gluing condition:

\[ D_m^- f(O) = D_m^+ f(O). \]
original process generator $= D_m \varepsilon D_s \varepsilon$

limiting process is Markov
generator (in Hille-Yosida sense) is $D_m D_s$
gluing condition at $O$: $D^-_m f(O) = D^+_m f(O)$
Vanishing friction : 2-d model problem.

- Assume $\lambda(x, y) = \lambda(y)$ and $b(\bullet) = 0$.
- $\lambda(y) = 0$ for $y \in [-1, 1]$.
- The equation
  \[
  \dot{q}^\varepsilon_t = -\frac{\nabla \lambda(q^\varepsilon_t)}{2(\lambda(q^\varepsilon_t) + \varepsilon)^3} + \frac{1}{\lambda(q^\varepsilon_t) + \varepsilon} \dot{W}_t, \quad q^\varepsilon_0 = q_0 \in \mathbb{R}^2, \varepsilon > 0,
  \]
  becomes
  \[
  \begin{cases}
  \dot{x}^\varepsilon_t = \frac{1}{\lambda(y^\varepsilon_t) + \varepsilon} \dot{W}^1_t, \quad x^\varepsilon_0 = x_0 \in \mathbb{R}, \\
  \dot{y}^\varepsilon_t = -\frac{\lambda'(y^\varepsilon_t)}{2(\lambda(y^\varepsilon_t) + \varepsilon)^3} + \frac{1}{\lambda(y^\varepsilon_t) + \varepsilon} \dot{W}^2_t, \quad y^\varepsilon_0 = y_0 \in \mathbb{R}.
  \end{cases}
  \]
- Let $x \sim \theta \pmod{1}$.
In the domain (1) mixing;

(2) spend very short time;

(3) exit at a uniform distribution in \( \theta \).
Vanishing friction : 2-d model problem.

- Limiting process lives in a double cone.
- Existence in the sense of Hille-Yosida.
- Once it hits $O$ it immediately leaves $O$.
- Gluing condition is a generalized Feller’s boundary condition:

$$\int_0^{2\pi} \lim_{\theta' \to \theta, \tilde{y} \to 0-} D_m f(\theta', \tilde{y}) d\theta = \int_0^{2\pi} \lim_{\theta' \to \theta, \tilde{y} \to 0+} D_m f(\theta', \tilde{y}) d\theta .$$

(It is similar to that of a Walsh BM !)
Vanishing friction: general case.

- We still do not know.
- First difficulty is to establish *existence* of the limiting process with a specified boundary condition.
The end

Thank you for your attention!