Abstract
By translating teaching and learning styles into propositional logic, we have established the foundational
knowledge needed for the development of a "smart" computerized tutoring program. Here, we use set
theory to explore the act of learning and, with statements in terms of propositional logic and with
diagrams, we construct many different models of teachers and students. It is stressed that programs
resulting from any future projects using this concept would seek not to replace human teachers, who are
a necessity for society, but to benefit all involved in the world of education. The application of technology
to this work would greatly enhance communication between student and teacher, facilitate the student’s
understanding of material, and give the student more time in an assisted-learning environment without
the usual strain on time or resources.

1 Introductory Motivation
Students and teachers in all levels of education face a dilemma; crowded classrooms and teacher shortages are
increasingly resulting in hurried instruction, rote memorization rather than understanding, and an impaired
sense of communication between instructor and pupil. Frequently, students leave school without a correct
understanding of some of the material. Either this is a fact which the teacher does not always realize, or
it is an unfortunate reality for which there is insufficient time to remedy. The time constraints of a 8am to
4pm school day and the finite number of ways a teacher may explain new material might leave the number
of students which the teacher is able to reach at an acceptable level. However, in the eyes of the parent or
adult student trying to obtain the best education for a child or for himself, this number is sure to be less
than optimal.

So what possibly can be done to create a new generation of more efficient schools? The answer lies in the
use of technology, beginning with the expression of the teaching and learning processes in a different way:
propositional logic. By creating a computer program which keeps track of the information understood or
misunderstood by the student, teachers receive accurate feedback of how effectively their classroom time is
being spent. Students would have an after-hours tutor, correcting the misconceptions as they are discovered
and giving the student an encouraging pat on the back for those concepts which are used correctly. No longer
will teachers have to search the seas of blank faces in their classrooms and wonder if their pupils really do
understand what is being taught. The answer will be in front of them on a computer screen.

Parents and students demand renewal in today’s system. When schools seem to suffer from a lack of
student interest or ability, the problem is actually the result of limited options and boredom. Most often,
the perceived lack of concern from teachers is a manifestation resulting from a lack of time and resources.
If given better chances to absorb material, more study time, and more educational options, students and
teachers may see a new type of school in the future: one that is more enthusiastic about learning, thrives in
the educational process, and grows more successful and efficient with each passing class.

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2 History of CAI/ITS and Previous Work

2.1 CAI

The concepts of Computer-Aided Instruction (or CAI) and, later, Intelligent Tutoring Systems (ITS), were born as contemporaries of the first computer and began to first take root in the 1950's.[3] First came the CAI system, acting very much like a textbook, which simply acquired a body of knowledge. These early systems clearly had shortcomings and limited capabilities. A lack of any type of computerized “intelligence”, for example, meant that human beings alone were responsible for predicting and including in code every possible outcome concerning the inference of concepts or any questions that the reader might have.[4] This task was impossible to complete and made CAI systems very inadequate for what was needed, specifically, in the world of education.

2.2 ITS

With the emergence of Artificial Intelligence (AI) came a breakthrough in computer science: Intelligent Tutoring Systems. Suddenly, an explosion of research and new systems occurred, utilizing the new possibilities of ITS. Systems such as SOPHIE, created by John Seely Brown and Richard Burton, and SCHOLAR, designed by John Carbonell, came to the surface as new innovations which used methodology such as neural nets to simulate inference. Many other systems were created for different uses. Numerous papers were published, and as the number of papers grew, authors such as Etienne Wenger and Lawler and Yazdani began to compile all of the new and exciting information being generated.

It had appeared that the world of education was on the verge of a renaissance. States began investing in computers, as by the 1980’s they had become affordable and smaller. However, by the 1990’s the computerized-education boom was more of a bust. The work being done in computerizing education slowly subsided, and we now see AI only being mentioned, rather than implemented, in education today.[3] There are many theories and reasons as to why the possibilities of ITS never materialized. We hope, by the simplified explanations of the education process, that our current work will renew interest in the subject and show that the promises and destiny of ITS in education have yet to be fulfilled.

2.3 Previous Work

Although our work was done independently, other papers have been authored with visions similar to the ones expressed in this work. As mentioned previously, Robert Lawler and Masoud Yazdani have compiled many interesting and groundbreaking works authored by a myriad of researchers in Artificial Intelligence and Education. In the article On the Mathematical Foundations of Learning, Felipe Crucker and Steve Smale echo the modeling process, but concentrate more on the actual process of extracting a model from a set of knowledge by dealing with the probabilistic and statistical errors that occur because of the extraction. We highly recommend reading the latter works.

3 The Act of Learning

Let the following rules apply:

1. $A \subseteq K$

   The set of axioms $A$ is part of the given theory $K$.

2. $B_t \subseteq A \cup R, t \in N$

   The set of blackboard knowledge $B_t$ on a certain day $t$ is contained in the union of the axiom set $A$ with the set of inference rules $R$. 
Understanding originates from formulae presented, correct inference rules, or incorrect inference rules.

The goals of the instructor lie within the set of knowledge for the given theory or for the inference rules used to generate another theory from the axioms. The instructor’s goals are ultimately the union of goals for each session, or $G_t$:

$$G_\infty = \bigcup_{t \in \mathbb{N}} G_t.$$  

Using the above, it is possible to begin modeling instructors and students.

4 Modeling Various Instruction Styles

We now explore different styles of instruction and how each may be modeled by propositional logic. Implementing the logic into the computerized tutoring program would enable it to offer each student a myriad of different teaching styles to match his or her own individual learning style (learning styles are addressed in the next section).

- **A Goal-Oriented Lecturer**
  
  We begin by providing a model of a simple lecturer who presents his goals to the students. We assume that, in general, a lecturer attempts to get his or her goals across to the pupils only by way of a blackboard in this case. To model this in propositional logic we may equate the blackboard items to the instructors goals, or by simply stating that the instructor is writing the goals of the day on the board:

  $$B_t = G_t.$$  

- **A Lecturer with Feedback**
  
  Likewise, we may add the event of a teacher giving feedback to students by modeling what the instructor
might place on the blackboard during the next class session, here called $t+1$. These items respond to how well the students understand the covered material. Here, $\Phi$ is a formula; $\Gamma$ is the set of unsound inference rules; and $K$ is the set of formulae within a chosen theory. Also used is $\gamma$, which is defined as being any incorrect notions the student has derived from using unsound inference rules.

With this we may discuss how an instructor using feedback in a class might be translated into propositional logic. Using feedback might involve giving the pupils a quiz, asking them informal questions, or having them work review problems on the board. By assessing the understanding of the knowledge offered the day before, the instructor may then proceed in either continuing on to the next set of goals, which is $G_{t+1}$; correcting any mistakes the students may have made in the technical definition of a formula; or repairing any misunderstandings they may have in how to use or incorporate the concepts.

In terms of logic:

$$B_{t+1} = G_{t+1} \cup \{\neg \Phi: \Phi \in U_t \backslash K\} \cup \{\neg \gamma: \gamma \in U_t \cap \Gamma\}.$$ 

This concept may be useful for keeping in check students who may be unsure of their understanding of material and may not express it in class. These are the students who many times fall further and further behind because of not wanting to feel singled out in a room full of their peers. This bit of logic may give the tutoring program the capability to help them in sorting things out without embarrassment and to give them more confidence in their abilities.

• A Lecturer with Eventual Feedback

We now turn to a more time-generic version of the previous propositional model. This instruction style is allowed a bit more leeway in that it uses any time (not necessarily the next session, $t + 1$) after the current session to correct misunderstandings. Therefore, $t + 1$ is replaced by the current session plus any constant, or $t + k$.

We must first set an initial value for the blackboard. We state that there are already some items on the board, and that these items are considered to be the initial goals of the instructor:

$$B_0 = G_0.$$ 

We ensure that the teacher has no goals which are not stated; all goals of the instructor are written on the blackboard. Also, there is always enough room on the blackboard for all goals that the instructor could possibly wish to cover in a session:

$$G_t \subseteq B_t.$$ 

Now we allow an unspecified amount of time to pass before a student is corrected, which also ensures that all misunderstandings will be covered at some point. In the computerized program, this propositional statement would enable a student to be taken back a few chapters, if needed, to review a forgotten topic:

$$\Phi \in U_t \backslash K \text{ implies } (\exists k)(\neg \Phi \in B_{t+k})$$

and

$$\gamma \in U_t \cap \Gamma \text{ implies } (\exists k)(\neg \gamma \in B_{t+k}).$$
• **A Lecturer with No Stated Theorems as Goals**

The final lecturer model is one of an instructor who helps the students to derive theorems themselves, rather than directly giving them to the students. In this model, the transformation $T_t$ maps goals as a part of the student’s understanding, as below:

$$T(G_\infty) \subseteq U_\infty.$$ 

The instructor’s main job is to be a passive presence of assistance, pointing the students in the right direction. He does not give theorems; he only presents definitions or axioms and addresses those concepts which the students misunderstand:

$$B_{t+1} = A_{t+1} \cup \{\neg \Phi : \Phi \in U_t \setminus K\} \cup \{\neg \gamma : \gamma \in U_t \cap \Gamma\}.$$ 

Finally, set of goals in this model are not static, but increase with each session:

$$G_t \subseteq G_{t+1}.$$ 

There exist other details which could be translated into this type of language. However, we cover only those which we believe would be most useful to the development of a computer program.

5 **Simple Student Descriptions and Learning**

Now we proceed to modeling different types of students. The personal habits, personality and learning capabilities of the student must be taken into consideration. This is essential in the process of depicting the teaching and learning processes, for the effort a student puts forth to learn determines how much will be understood and retained. Naturally, a student who does not have a very good memory will forget just as much as a student who does not or cannot concentrate during the lecture. The following discusses some simple examples, modeled by propositional logic, in which the reader should be well-versed by this point.

• **The Perfect Student**

The ideal student understands everthing correctly; he has no misnotions. He notes, studies, and understands correctly all knowledge placed on the blackboard. Therefore, in set theory, each element in the set of goals is eventually mapped intact to the set of understanding, or:

$$U_t = G_t.$$ 

• **The Temporary Learner**

This honest student tries diligently to learn by studying, working on coursework, and attending class, but forgets many of the topics in the end. At some point, he understands some things:

$$U_t \neq \emptyset.$$ 

Additionally, the student has a correct understanding of some or all things presented during a single session:

$$U_t \subseteq G_t.$$ 

However, his understanding eventually fades with time, so that his set of understanding by the end of the course empties:

$$\lim_{t \to \infty} U_t = \emptyset.$$
• **Perfectly Attentive Student**

   An attentive student is one who pays attention in class and takes notes. Therefore, a perfectly attentive student understands and may infer other concepts from whatever is placed on the blackboard during a session:
   \[ B_t \subseteq U_t. \]

• **Monotone Learners**

   Monotone learners may not learn at the same pace as do the Perfect Students; they could be called “slower learners”. Absorption of all knowledge for a given session may be difficult for them, or they may grasp certain concepts long after the teacher has moved on. However, this student’s understanding of the material is in constant growth nonetheless. Here, we begin with a student who leaves the first session after having learned something, which means that at \( t = 0 \) the set of understanding is not empty:
   \[ U_0 \neq \emptyset. \]

   Now assume that the student correctly learns from the given goals:
   \[ U_t \subseteq G_t. \]

   This understanding will be retained in the next session, \( t + 1 \). In other words, this student will not forget in the next session what he has learned in the current session:
   \[ U_t \subseteq U_{t+1}. \]

   Now, as we shift to the long-term view of this student, we establish that the student will continue to learn something at some future session.

   \[ (\forall t)(\exists k) \{ U_{t+k} \setminus U_t \neq \emptyset \}. \]
• **The Progressive Learner**
A Progressive Learner is one whose set of understanding is constantly increasing. He learns concepts, corrects any misunderstandings, and ends up learning more *with each session*, which suggests that he remembers more than he forgets from one session to the next.

We now invite the reader, at this point, to apply the above description to the model below, which is in sets of propositional variables:

1. \( \lim_{t \to \infty} U_t \subseteq \bigcup_{t \in \mathbb{N}} G_t = G_\infty \).
2. \( U_0 \neq \emptyset \).
3. \( (U_t \cap G_\infty) \subseteq (U_{t+1} \cap G_\infty) \).
4. \( (U_{t+1} \cap G_\infty) \setminus (U_t \cap G_\infty) \neq \emptyset \).

### 6 Expanded Student Descriptions and Learning

We may also take the simple models from above and add more characteristics onto them, or we may also combine them. These models may be expanded to as specific a description as the computer scientist would like. This ability to customize the program to the student would be particularly useful for different populations of students, such as remedial versus advanced students.

For example, we have the ability to create a model for an Attentive, Temporary Learner. This student has the same attributes as the Attentive Student, except this student retains none of the concepts of which he has taken note or has seen explained on the blackboard over time (which is the main characteristic of the Temporary Learner).

The same may be done for any number of combinations of characteristics, which could greatly increase the power of the ITS to match the student’s needs in terms of frequency of reviewing and quizzing of material.

### 7 Nomenclature

*The following is an explanation of the preliminary material used in the preceding work.*
7.1 Some Rudiments of Sentential Calculus

Sentential calculus is the symbolic language of propositional calculus. Propositional calculus paraphrases a type of logic which depends only on the setup of the phrases within a sentence and not on the validity of the phrases alone. Sentential calculus is broken up into three types of symbols: propositional variables, logical connectives, and parentheses.[1]

- Propositional variables: Any elements of the infinite set \( P = \{ p_n : n \in \omega \} \).
- Logical connectives:
  - \( \neg \) is read as “not”,
  - \( \land \) is read as “and”,
  - \( \lor \) is read as “or”,
  - \( \rightarrow \) is read as “implies” or “only if”,
  - \( \iff \) is read as “if and only if” or “iff”.
- Parentheses:
  The right- and left-handed parentheses or brackets are simply used to separate and relate the “phrases” of S.C. for meaning and clarity.

7.2 Auxiliary Operators and Symbols

In addition to the variables and connectives listed above, quantifiers and set theory operators, which are not a part of sentential calculus, are used to clarify and communicate to the reader. The following discusses the use of each.

- Quantifiers:
  - \( \exists \) is translated as “for some” or “for a certain”,
  - \( \forall \) is translated as “for every” or “for all”,
- Set Theory Operators:
  In set theoretical discussions, we use usual symbols from set theory, such as the following:
  - \( \cup \) is used for union
  - \( \cap \) is used for intersection
  - \( \subseteq \) is used for “is a subset of”
  - \( \setminus \) is translated as set complementation.

7.3 The Teaching and Learning Processes

We expand the concept of Fig. 1 in the beginning of the paper into the following definitions and comments.

\( K_t \): The subset of theoretical knowledge in terms of well-formed formulae chosen at a particular time \( t \). The summation of all \( K_t \) makes up the set \( K \), which consists of all theoretical knowledge to be presented or derived in the course.

\( B_t \): The subset of items from set \( B \), which is any item placed on the blackboard for the student, in a specified period of time. The reader should realize that the blackboard knowledge is in the range of the set of all well-formed formulae, which means that \( B_t \) results from \( K_t \):

\[
B_t = T_t(K_t).
\]

The transformation \( T_t \) is addressed in the next definition.
Additionally, several different $B_t$ may overlap; this occurs when the instructor reviews for exams or simply refreshes the students memories of concepts covered the day before by rewriting them on the blackboard. The figure does well in describing this, as some subsets $B_t$ are partly or wholly contained in other subsets of $B$.

$T_t$ : The teaching process, a transformation of sorts, which transfers knowledge on a particular day $K_t$ to the blackboard items $B_t$. In other words, $T_t$ maps $K_t$ to $B_t$. As seen below:

$$T_t : K_t \rightarrow B_t.$$  

$U_t$ : The subset of items understood by the student. The entire set $U$ is within the range of all information which the student understands at some point on each day for an infinite period of time, or as below:

$$U = \bigcup_{t=0}^{\infty} U_t.$$  

The set $U$, and therefore the subset $U_t$, may change in content and size at any time. For example, the way in which a concept is understood today may change its meaning tomorrow. Also, some concepts may be forgotten in the future. Referring again to the diagram shows the subset $U_t$ having, at one time, a larger boundary than the current one; this subset has shrunk in size, as is shown by the current solid boundary of $U$.

Also, note that $U$ is not necessarily a subset of $B$ (and, likewise, of $K$). For example, the student may either derive correct and logical information on his or her own which may not have been presented on the blackboard; these conclusions, since they are a correct derivation of presented ideas, are part of $K$, but not of $B$. On the other hand, the student may have some misunderstandings about the material presented; these incorrect notions were derived from $B$, but are of incorrect derivation and, thus, are not part of $K$.

Symbolically, we have:

$$U \not\subset B \subset K.$$  

This reality presents a bit of a problem. In this paper, our remedy is to be sure that whatever $U$ the student ends up with is part of $K$, or $U \subset K$.

$S_t$ : The process of studying and learning, another transformation takes the items from the blackboard on a certain day, $B_t$, and transfers it to the set of understood information $U$. As below:

$$S_t : B_t \rightarrow U_t.$$  

### 7.4 Notation Involved in Learning

Here another aspect of teaching and learning is explored: that of true and false knowledge.

**Definitions:**

$F$ : The set of all formulae.

$K$ : The chosen theory.
$R$: A complete set of sound inference rules (i.e., modus ponens).

$\Gamma$: A set of unsound inference rules (where the day is indicated by an integer).

**Modus Ponens**: An inference rule in which, for example, we know $p$ to be true and $p \rightarrow q$ to be true, and we therefore may infer that $q$ is also true. Modus ponens is the only inference rule the reader need know for this subject because of the soundness and completeness theorem for propositional logic.

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References


