Math 3108: Linear Algebra
Chapter 1

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Chapter 1. Linear Equations in Linear Algebra

1.1 Systems of Linear Equations
1.2 Row Reduction and Echelon Forms.

Our first application of linear algebra is the use of matrices to efficiently solve linear systems of equations.
A linear system of $m$ equations with $n$ unknowns can be represented by a matrix with $m$ rows and $n + 1$ columns:

The system

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$
$$\vdots \quad \vdots \quad \vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m$$

corresponds to the matrix

$$[A|b] = \begin{bmatrix}
a_{11} & \cdots & a_{1n} & b_1 \\
\vdots & \vdots & \vdots & \vdots \\
a_{m1} & \cdots & a_{mn} & b_n
\end{bmatrix}.$$  

Similarly, every such matrix corresponds to a linear system.
A is the $m \times n$ coefficient matrix:

$$A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
\vdots & \ddots & \vdots \\
a_{m1} & \cdots & a_{mn}
\end{bmatrix}$$

$a_{ij}$ is the element in row $i$ and column $j$.

$[A|b]$ is called the augmented matrix.
Example. The 2x2 system

\[3x + 4y = 5\]
\[6x + 7y = 8\]

corresponds to the augmented matrix

\[
\begin{bmatrix}
3 & 4 & 5 \\ 6 & 7 & 8
\end{bmatrix}
\]
To solve linear systems, we manipulate and combine the individual equations (in such a way that the solution set of the system is preserved) until we arrive at a simple enough form that we can determine the solution set.
Example. Let us solve

\[3x + 4y = 5\]
\[6x + 7y = 8.\]

Multiply the first equation by -2 and add it to the second:

\[3x + 4y = 5\]
\[0x - y = -2.\]

Multiply the second equation by 4 and add it to the first:

\[3x + 0y = -3\]
\[0x - y = -2.\]

Multiply the first equation by \(\frac{1}{3}\) and the second by \(-1\):

\[x + 0y = -1\]
\[0x + y = 2.\]
Example. (continued) We have transformed the linear system

\[
\begin{align*}
3x + 4y &= 5 \\
6x + 7y &= 8
\end{align*}
\]

into

\[
\begin{align*}
x + 0y &= −1 \\
0x + y &= 2
\end{align*}
\]

in such a way that the solution set is preserved.

The second system clearly has solution set \(\{(-1, 2)\}\).

Remark. For linear systems, the solution set \(S\) satisfies one of the following:

\begin{itemize}
  \item \(S\) contains a single point (\textit{consistent} system)
  \item \(S\) contains infinitely many points (\textit{consistent} system),
  \item \(S\) is empty (\textit{inconsistent} system).
\end{itemize}
The manipulations used to solve the linear system above correspond to **elementary row operations** on the augmented matrix for the system.

**Elementary row operations.**

- Replacement: replace a row by the sum of itself and a multiple of another row.
- Interchange: interchange two rows.
- Scaling: multiply all entries in a row by a nonzero constant.

Row operations **do not change the solution set** for the associated linear system.
Example. (revisited)

\[
\begin{bmatrix}
  3 & 4 & 5 \\
  6 & 7 & 8 \\
\end{bmatrix}
\xrightarrow{R_2 \rightarrow -2R_1 + R_2}
\begin{bmatrix}
  3 & 4 & 5 \\
  0 & -1 & -2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  3 & 4 & 5 \\
  0 & -1 & -2 \\
\end{bmatrix}
\xrightarrow{R_1 \rightarrow 4R_2 + R_1}
\begin{bmatrix}
  3 & 0 & -3 \\
  0 & -1 & -2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  3 & 0 & -3 \\
  0 & -1 & -2 \\
\end{bmatrix}
\xrightarrow{R_1 \rightarrow \frac{1}{3} R_1}
\begin{bmatrix}
  1 & 0 & -1 \\
  0 & -1 & -2 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 & -1 \\
  0 & -1 & -2 \\
\end{bmatrix}
\xrightarrow{R_2 \rightarrow -R_2}
\begin{bmatrix}
  1 & 0 & -1 \\
  0 & 1 & 2 \\
\end{bmatrix}
\]

(i) it is simple to determine the solution set for the last matrix
(ii) row operations preserve the solution set.
It is always possible to apply a series of row reductions to put an augmented matrix into **echelon form** or **reduced echelon form**, from which it is simple to discern the solution set.

**Echelon form:**
- Nonzero rows are above any row of zeros.
- The **leading entry** (first nonzero element) of each row is in a column to the right of the leading entry of the row above it.
- All entries in a column below a leading entry are zeros.

**Reduced echelon form:** (two additional conditions)
- The leading entry of each nonzero row equals 1.
- Each leading 1 is the only nonzero entry in its column.
Examples.

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 3 & -6 & 6 & 4 & -5
\end{bmatrix}
\text{not in echelon form}
\]

\[
\begin{bmatrix}
3 & -9 & 12 & -9 & 6 & 15 \\
0 & 2 & -4 & 4 & 2 & -6 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}
\text{echelon form, not reduced}
\]

\[
\begin{bmatrix}
1 & 0 & 2 & 3 & 0 & -24 \\
0 & 1 & -2 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 4
\end{bmatrix}
\text{reduced echelon form}
\]
**Remark.** Every matrix can be put into reduced echelon form in a unique manner.

**Definition.**

A **pivot position** in a matrix is a location that corresponds to a leading 1 in its reduced echelon form.

A **pivot column** is a column that contains a pivot position.

**Remark.** Pivot positions lie in columns corresponding to dependent variables for the associated systems.
Row Reduction Algorithm.

1. Begin with the leftmost column; if necessary, interchange rows to put a nonzero entry in the first row.
2. Use row replacement to create zeros below the pivot.
3. Repeat steps 1. and 2. with the sub-matrix obtained by removing the first column and first row. Repeat the process until there are no more nonzero rows.

This puts the matrix into echelon form.

4. Beginning with the rightmost pivot, create zeros above each pivot. Rescale each pivot to 1. Work upward and to the left.

This puts the matrix into reduced echelon form.
Example.

\[
\begin{bmatrix}
 3 & -9 & 12 & -9 & 6 & 15 \\
 3 & -7 & 8 & -5 & 8 & 9 \\
 0 & 3 & -6 & 6 & 4 & -5 \\
\end{bmatrix}
\]

\[
R_2 \mapsto -R_1 + R_2
\]

\[
\begin{bmatrix}
 3 & -9 & 12 & -9 & 6 & 15 \\
 0 & 2 & -4 & 4 & 2 & -6 \\
 0 & 3 & -6 & 6 & 4 & -5 \\
\end{bmatrix}
\]

\[
R_3 \mapsto -\frac{3}{2} R_2 + R_3
\]

\[
\begin{bmatrix}
 3 & -9 & 12 & -9 & 6 & 15 \\
 0 & 2 & -4 & 4 & 2 & -6 \\
 0 & 0 & 0 & 0 & 0 & 1 & 4 \\
\end{bmatrix}
\]

The matrix is now in echelon form.
Example. (continued)

\[
\begin{bmatrix}
  3 & -9 & 12 & -9 & 6 & 15 \\
  0 & 2 & -4 & 4 & 2 & -6 \\
  0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  3 & -9 & 12 & -9 & 0 & -9 \\
  0 & 2 & -4 & 4 & 0 & -14 \\
  0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  3 & -9 & 12 & -9 & 0 & -9 \\
  0 & 1 & -2 & 2 & 0 & -7 \\
  0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  3 & 0 & -6 & 9 & 0 & -72 \\
  0 & 1 & -2 & 2 & 0 & -7 \\
  0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 0 & -2 & 3 & 0 & -24 \\
  0 & 1 & -2 & 2 & 0 & -7 \\
  0 & 0 & 0 & 0 & 1 & 4
\end{bmatrix}
\].

The matrix is now in reduced echelon form.
Solving systems.

- Find the augmented matrix \([A|b]\) for the given linear system.
- Put the augmented matrix into reduced echelon form \([A'|b']\).
- Find solutions to the system associated to \([A'|b']\). Express dependent variables in terms of free variables if necessary.
Example 1. The system

\[
\begin{align*}
2x - 4y + 4z &= 6 \\
x - 2y + 2z &= 3 \\
x - y + 0z &= 2
\end{align*}
\]

\[
\rightarrow \left[ \begin{array}{ccc|c}
2 & -4 & 4 & 6 \\
1 & -2 & 2 & 3 \\
1 & -1 & 0 & 2
\end{array} \right] \rightarrow \left[ \begin{array}{ccc|c}
1 & 0 & -2 & 1 \\
0 & 1 & -2 & -1 \\
0 & 0 & 0 & 0
\end{array} \right]
\]

\[
\rightarrow \begin{align*}
&x - 2z = 1 \\
&y - 2z = -1.
\end{align*}
\]

The solution set is

\[
S = \{(1 + 2z, -1 + 2z, z) : z \in \mathbb{R}\}.
\]
Example 2. The system

\[
\begin{align*}
2x - 4y + 4z &= 6 \\
x - 2y + 2z &= 4 \\
x - y + 0z &= 2
\end{align*}
\]

\[
\begin{bmatrix}
2 & -4 & 4 & 6 \\
1 & -2 & 2 & 4 \\
1 & -1 & 0 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -2 & 1 \\
0 & 1 & -2 & -1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
x - 2z = 1 \\
\rightarrow \quad y - 2z = -1, \\
0 = 1.
\]

- The solution set is empty—the system is inconsistent.
- This is always the case when a pivot position lies in the last column.
Row equivalent matrices.

Two matrices are **row equivalent** if they are connected by a sequence of elementary row operations.

Two matrices are row equivalent if and only if they have the same reduced echelon form.

We write $A \sim B$ to denote that $A$ and $B$ are row equivalent.
Chapter 1. Linear Equations in Linear Algebra

1.3 Vector Equations
1.4 The Matrix Equation $Ax = b$.  

A matrix with one column or one row is called a vector, for example

\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\quad \text{or} \quad \begin{bmatrix}
1 & 2 & 3
\end{bmatrix}.
\]

By using vector arithmetic, for example

\[
\alpha \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} + \beta \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix} = \begin{bmatrix}
\alpha + 4\beta \\
2\alpha + 5\beta \\
3\alpha + 6\beta
\end{bmatrix},
\]

we can write linear systems as vector equations.
The linear system

\[
x + 2y + 3z = 4 \\
5x + 6y + 7z = 8 \\
9x + 10y + 11z = 12
\]

is equivalent to the vector equation

\[
x \begin{bmatrix} 1 \\ 5 \\ 9 \end{bmatrix} + y \begin{bmatrix} 2 \\ 6 \\ 10 \end{bmatrix} + z \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 12 \end{bmatrix},
\]

in that they have the same solution sets, namely,

\[
S = \{(−2 + z, 3 − 2z, z) : z \in \mathbb{R}\}.
\]
Geometric interpretation.

The solution set $S$ may be interpreted in different ways:

- $S$ consists of the points of intersection of the three planes

$$
x + 2y + 3z = 4$$
$$5x + 6y + 7z = 8$$
$$9x + 10y + 11z = 12.
$$

- $S$ consists of the coefficients of the linear combinations of the vectors

$$
\begin{bmatrix}
1 \\
5 \\
9
\end{bmatrix}, \quad
\begin{bmatrix}
2 \\
6 \\
10
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
3 \\
7 \\
11
\end{bmatrix}
$$

that yield the vector

$$
\begin{bmatrix}
4 \\
8 \\
12
\end{bmatrix}.
$$
Linear combinations and the span.

The set of linear combinations of the vectors $v_1, \ldots, v_n$ is called the \textbf{span} of these vectors:

$$\text{span}\{v_1, \ldots, v_n\} = \{\alpha_1 v_1 + \cdots + \alpha_n v_n : \alpha_1, \ldots, \alpha_n \in \mathbb{R}\}.$$ 

A vector equation

$$x v_1 + y v_2 = v_3$$

is consistent (that is, has solutions) if and only if

$$v_3 \in \text{span}\{v_1, v_2\}.$$
Example. Determine whether or not

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} \in \text{span} \left\{ \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \begin{bmatrix}
1 \\
3 \\
4
\end{bmatrix}, \begin{bmatrix}
1 \\
4 \\
5
\end{bmatrix} \right\}.
\]

This is equivalent to the existence of a solution to:

\[
x \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix} + y \begin{bmatrix}
1 \\
3 \\
4
\end{bmatrix} + z \begin{bmatrix}
1 \\
4 \\
5
\end{bmatrix} = \begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

The associated system is

\[
x + y + z = 1 \\
2x + 3y + 4z = 1 \\
3x + 4y + 5z = 1.
\]
The augmented matrix is

\[
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 3 & 4 & 1 \\
3 & 4 & 5 & 1 \\
\end{bmatrix}.
\]

The reduced echelon form is

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}.
\]

The system is inconsistent. Thus

\[
\begin{bmatrix}
1 \\
1 \\
1 \\
\end{bmatrix} \not\in \text{span } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \right\}.
\]
**Geometric description of span.**

Let

\[
S = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}, \quad T = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.
\]

Then

- \( S \) is the line through the points \((0, 0, 0)\) and \((0, 1, 1)\).
- \( T \) is the plane through the points \((0, 0, 0)\), \((0, 1, 1)\), and \((1, 0, 1)\).
The following are equivalent:

- Is $\mathbf{v}_3$ spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$?
- Can $\mathbf{v}_3$ be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$?
- Is $\mathbf{v}_3$ in the plane containing the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$?
Cartesian equation for span.

Recall the definition of the plane $T$ above.

A point $(x, y, z)$ belongs to $T$ when the following vector equation is consistent:

$$\alpha \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$ 

The augmented matrix and its reduced echelon form are as follows:

$$\begin{bmatrix} 0 & 1 & x \\ 1 & 0 & y \\ 1 & 1 & z \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & y \\ 0 & 1 & x \\ 0 & 0 & z - x - y \end{bmatrix}.$$ 

Thus the Cartesian equation for the plane is

$$0 = z - x - y.$$
**Matrix equations.** Consider a matrix of the form

\[ A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}, \]

where the \( \mathbf{a}_j \) are column vectors. The product of \( A \) with a column vector is defined by

\[ A \begin{bmatrix} x \\ y \\ z \end{bmatrix} := xa_1 + ya_2 + za_3. \]

Thus all linear systems can be represented by matrix equations of the form \( AX = b \).
Example. (Revisited) The system

\[\begin{align*}
x + y + z &= 1 \\
2x + 3y + 4z &= 1 \\
3x + 4y + 5z &= 1
\end{align*}\]

is equivalent to the matrix equation \(AX = b\), where

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
2 & 3 & 4 \\
3 & 4 & 5
\end{bmatrix}, \quad X = \begin{bmatrix}
x \\
y \\
z
\end{bmatrix}, \quad b = \begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

Remark. \(AX = b\) has a solution if and only if \(b\) is a linear combination of the columns of \(A\).
Question. When does the vector equation $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b} \in \mathbb{R}^m$?

Answer. When the columns of $A$ span $\mathbb{R}^m$.

An equivalent condition is the following: the reduced echelon form of $A$ has a pivot position in every row.

To illustrate this, we study a non-example:
Non-example. Let

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
2 & 3 & 4 \\
3 & 4 & 5
\end{bmatrix}
\xrightarrow{\text{reduced echelon form}}
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{bmatrix}.
\]

This means that for any \(b_1, b_2, b_3 \in \mathbb{R}\), we will have

\[
A = \begin{bmatrix}
1 & 1 & 1 & | & b_1 \\
2 & 3 & 4 & | & b_2 \\
3 & 4 & 5 & | & b_3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -1 & | & f_1(b_1, b_2, b_3) \\
0 & 1 & 2 & | & f_2(b_1, b_2, b_3) \\
0 & 0 & 0 & | & f_3(b_1, b_2, b_3)
\end{bmatrix}
\]

for some linear functions \(f_1, f_2, f_3\). However, the formula

\[
f_3(b_1, b_2, b_3) = 0
\]

imposes a constraint on the choices of \(b_1, b_2, b_3\).

That is, we cannot solve \(AX = b\) for arbitrary choices of \(b\).
If instead the reduced echelon form of $A$ had a pivot in any row, then we could use the reduced echelon form for the augmented system to find a solution to $AX = b$. 
Chapter 1. Linear Equations in Linear Algebra

1.5 Solution Sets of Linear Systems
The system of equations $AX = b$ is

- **homogeneous** if $b = 0$,
- **inhomogeneous** if $b \neq 0$.

For homogeneous systems:

- The augmented matrix for a homogeneous system has a column of zeros.
- Elementary row operations will not change this column.

Thus, for homogeneous systems it is sufficient to work with the coefficient matrix alone.
Example.

\[
\begin{align*}
2x - 4y + 4z &= 6 \\
x - 2y + 2z &= 3 \\
x - y &= 2
\end{align*}
\rightarrow
\begin{bmatrix}
2 & -4 & 4 & | & 6 \\
1 & -2 & 2 & | & 3 \\
1 & -1 & 0 & | & 2
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -2 & | & 1 \\
0 & 1 & -2 & | & -1 \\
0 & 0 & 0 & | & 0
\end{bmatrix}.
\]

The solution set is

\[
x = 1 + 2z, \quad y = -1 + 2z, \quad z \in \mathbb{R}.
\]

On the other hand,

\[
\begin{align*}
2x - 4y + 4z &= 0 \\
x - 2y + 2z &= 0 \\
x - y &= 0
\end{align*}
\rightarrow
x = 2z, \quad y = 2z, \quad z \in \mathbb{R}.
\]
The solution set for the previous inhomogeneous system \( AX = b \) can be represented in **parametric vector form**:

\[
\begin{align*}
2x - 4y + 4z &= 6 \\
x - 2y + 2z &= 3 \\
x - y &= 2
\end{align*}
\]

\[
\rightarrow x = 1 + 2z, \quad y = -1 + 2z
\]

\[
\rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}.
\]

The parametric form for the homogeneous system \( AX = 0 \) is given by

\[
\begin{align*}
x &= 2z \\
y &= 2z
\end{align*}
\]

\[
\rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}.
\]

Both solution sets are parametrized by the free variable \( z \in \mathbb{R} \).
Example. Express the solution set for $AX = b$ in parametric vector form, where

$$[A|b] = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & -2 & -2 & 2 \end{bmatrix}$$

Row reduction leads to

$$\begin{bmatrix} 1 & 1 & 0 & -2 & 0 & -2 \\ 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

which means the solution set is

$$x_1 = -2 - x_2 + 2x_4, \quad x_3 = 1 - x_4 + x_5, \quad x_2, x_4, x_5 \in \mathbb{R}.$$
Example. (continued) In parametric form, the solution set is given by

\[ X = \begin{bmatrix} -2 - x_2 + 2x_4 \\ x_2 \\ 1 - x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \]

where \( x_2, x_4, x_5 \in \mathbb{R} \).

To solve the corresponding homogeneous system, simply erase the first vector.
General form of solution sets. If $X_p$ is any particular solution to $AX = b$, then any other solutions to $AX = b$ may be written in the form

$$X = X_p + X_h,$$

where $X_h$ is some solution to $AX = 0$.

Indeed, given any solution $X$,

$$A(X - X_p) = AX - AX_p = b - b = 0,$$

which means that $X - X_p$ solves the homogeneous system.

Thus, to find the general solution to the inhomogeneous problem, it suffices to

1. Find the general solution to the homogeneous problem,
2. Find any particular solution to the inhomogeneous problem.

Remark. Something similar happens in linear ODE.
**Example.** (Line example) Suppose the solution set of $AX = b$ is a line passing through the points

$$p = (1, -1, 2), \quad q = (0, 3, 1).$$

Find the parametric form of the solution set.

First note that $v = p - q$ is parallel to this line.

As $q$ belongs to the solution set, the solution set is therefore

$$X = q + tv, \quad t \in \mathbb{R}.$$ 

Note that we may also write this as

$$X = (1 - t)q + tp, \quad t \in \mathbb{R}.$$ 

Note also that the solution set to $AX = 0$ is simply $tv, t \in \mathbb{R}$. 

**Example.** (Plane example) Suppose the solution set of $AX = b$ is a plane passing through

$$p = (1, -1, 2), \quad q = (0, 3, 1), \quad r = (2, 1, 0).$$

This time we form the vectors

$$v_1 = p - q, \quad v_2 = p - r.$$  

(Note that $v_1$ and $v_2$ are linearly independent, i.e. one is not a multiple of the other.)

Then the plane is given by

$$X = p + t_1 v_1 + t_2 v_2, \quad t_1, t_2 \in \mathbb{R}.$$  

(The solution set to $AX = 0$ is then the span of $v_1$ and $v_2$.)
Chapter 1. Linear Equations in Linear Algebra

1.7 Linear Independence
Definition. A set of vectors

\[ S = \{v_1, \ldots, v_n\} \]

is (linearly) independent if

\[ x_1v_1 + \cdots + x_nv_n = 0 \implies x_1 = \cdots = x_n = 0 \]

for any \( x_1, \ldots, x_n \in \mathbb{R} \).

Equivalently, \( S \) is independent if the only solution to \( AX = 0 \) is \( X = 0 \), where \( A = [v_1 \cdots v_n] \).

Otherwise, we call \( S \) (linearly) dependent.
Example. Let

\[ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \quad A := \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}. \]

Then

\[ A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \]

In particular, the equation \( AX = 0 \) has a nontrivial solution set, namely

\[ \mathbf{X} = z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad z \in \mathbb{R}. \]

Thus the vectors are dependent.
Dependence has another useful characterization:

The vectors \( \{v_1, \ldots, v_n\} \) are dependent if and only if (at least) one of the vectors can be written as a linear combination of the others.

Continuing from the previous example, we found that

\[
AX = 0, \quad \text{where} \quad X = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}
\]

(for example). This means

\[
v_1 - 2v_2 + v_3 = 0, \quad \text{i.e.} \quad v_1 = 2v_2 - v_3.
\]
Some special cases.

▶ If $S = \{ \mathbf{v}_1, \mathbf{v}_2 \}$, then $S$ is dependent if and only if $\mathbf{v}_1$ is a scalar multiple of $\mathbf{v}_2$ (if and only if $\mathbf{v}_1$ and $\mathbf{v}_2$ are co-linear).

▶ If $0 \in S$, then $S$ is always dependent. Indeed, if

$$S = \{ 0, \mathbf{v}_1, \cdots, \mathbf{v}_n \},$$

then a nontrivial solution to $A\mathbf{X} = 0$ is

$$0 = 1 \cdot 0 + 0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n.$$
Pivot columns. Consider

\[
A = [v_1 v_2 v_3 v_4] = \begin{bmatrix}
1 & 2 & 3 & 4 \\
-2 & -4 & -5 & -6 \\
3 & 6 & 7 & 8
\end{bmatrix} \sim \begin{bmatrix}
1 & 2 & 0 & -2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

In particular, the vector equation \( A\mathbf{x} = 0 \) has solution set

\[
\mathbf{x} = x_2 \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
2 \\
0 \\
-2
\end{bmatrix}, \quad x_2, x_4 \in \mathbb{R}.
\]

Thus \( \{v_1, v_2, v_3, v_4\} \) are dependent.

The pivot columns of \( A \) are relevant: By considering

\( (x_2, x_4) \in \{ (1, 0), (0, 1) \} \),

we find that \( v_1 \) and \( v_3 \) can be combined to produce \( v_2 \) or \( v_4 \):

- \( v_2 = 2v_1 \)
- \( v_4 = -2v_1 + 2v_3 \).
Let $A$ be an $m \times n$ matrix. We write $A \in \mathbb{R}^{m \times n}$.

- The number of pivots is bounded above by $\min\{m, n\}$.
- If $m < n$ (‘short’ matrix), the columns of $A$ are necessarily dependent.
- If $m > n$ (‘tall’ matrix), the rows of $A$ are necessarily dependent.

**Example.**

\[
A = [v_1 v_2 v_3 v_4] = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 5 \\
2 & 3 & 4 & 5
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

The columns of $A$ are necessarily dependent; indeed, setting the free variable $x_3 = 1$ yields the nontrivial combination

\[
v_1 - 2v_2 + v_3 = 0.
\]
Example. (continued)

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 5 \\
2 & 3 & 4 & 5 \\
\end{bmatrix}
\]

Are the rows of \(A\) dependent or independent?

The rows of \(A\) are the columns of the transpose of \(A\), denoted \(A'\)

\[
A' = \begin{bmatrix}
1 & 1 & 2 \\
1 & 2 & 3 \\
1 & 3 & 4 \\
1 & 5 & 5 \\
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{bmatrix}.
\]

Now note that:

- Each column of \(A'\) is a pivot column.
- \(\iff\) the solution set of \(A'X = 0\) is \(X = 0\).
- \(\iff\) the columns of \(A'\) are independent.
- \(\iff\) the rows of \(A\) are independent.
Chapter 1. Linear Equations in Linear Algebra

1.8 Introduction to Linear Transformations
1.9 The Matrix of a Linear Transformation
Definition. A linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a function $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that

- $T(u + v) = T(u) + T(v)$ for all $u, v \in \mathbb{R}^n$,
- $T(\alpha v) = \alpha T(v)$ for all $v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Note that for any linear transformation, we necessarily have

$$T(0) = T(0 + 0) = T(0) + T(0) \implies T(0) = 0.$$ 

Example. Let $A \in \mathbb{R}^{m \times n}$. Define $T(X) = AX$ for $X \in \mathbb{R}^n$.

- $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$
- $T(X + Y) = A(X + Y) = AX + AY = T(X) + T(Y)$
- $T(\alpha X) = A(\alpha X) = \alpha AX = \alpha T(X)$

We call $T$ a matrix transformation.
Definition. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The range of $T$ is the set

$$R(T) := \{ T(X) : X \in \mathbb{R}^n \}.$$ 

Note that $R(T) \subset \mathbb{R}^m$ and $0 \in R(T)$.

We call $T$ onto (or surjective) if $R(T) = \mathbb{R}^m$. 
Example. Determine if $b$ is in the range of $T(X) = AX$, where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 7 \\ 11 \end{bmatrix}. $$

This is equivalent to asking if $AX = b$ is consistent. By row reduction:

$$[A|b] = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 3 & 0 & 4 & 7 \\ 5 & 6 & 0 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 | 1 \\ 0 & 0 & 1 | 1 \end{bmatrix}. $$

Thus $b \in R(T)$, indeed

$$T(X) = b, \quad \text{where} \quad X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. $$
Example. Determine if $T(X) = AX$ is onto, where

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \\ 3 & 2 & 1 \end{bmatrix}.$$ 

Equivalently, determine if $AX = b$ is consistent for every $b \in \mathbb{R}^m$.

Equivalently, determine if the reduced form of $A$ has a pivot in every row:

$$A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Thus $T$ is not onto.
Example. (Continued) In fact, by performing row reduction on \([A|\mathbf{b}]\) we can describe \(R(T)\) explicitly:

\[
\begin{bmatrix}
0 & 1 & 2 & b_1 \\
2 & 3 & 4 & b_2 \\
3 & 2 & 1 & b_3
\end{bmatrix}
\sim
\begin{bmatrix}
1 & 0 & -1 & -\frac{3}{2}b_1 + \frac{1}{2}b_2 \\
0 & 1 & 2 & b_1 \\
0 & 0 & 0 & \frac{5}{2}b_1 - \frac{3}{2}b_2 + b_3
\end{bmatrix}
\]

Thus

\[R(T) = \{\mathbf{b} \in \mathbb{R}^3 : \frac{5}{2}b_1 - \frac{3}{2}b_2 + b_3 = 0\}.\]
Definition. A linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ is one-to-one (or injective) if

$$T(\mathbf{X}) = 0 \implies \mathbf{X} = 0.$$ 

More generally, a function $f$ is one-to-one if

$$f(x) = f(y) \implies x = y.$$ 

For linear transformations, the two definitions are equivalent. In particular, $T$ is one-to-one if:

- for each $b$, the solution set for $T(\mathbf{X}) = b$ has at most one element.

For matrix transformations $T(\mathbf{X}) = A\mathbf{X}$, injectivity is equivalent to:

- the columns of $A$ are independent
- the reduced form of $A$ has a pivot in every column
Example. Let $T(X) = AX$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 3 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

- $T$ is \textbf{not} one-to-one, as every column does not have a pivot.
- $T$ \textbf{is} onto, as every row has a pivot.
Summary.

For a matrix transformation $T(X) = AX$.

- Let $B$ denote the reduced echelon form of $A$.
- $T$ is onto if and only if $B$ has a pivot in every row.
- $T$ is one-to-one if and only if $B$ has a pivot in every column.
Matrix representations.

- Not all linear transformations are matrix transformations.
- However, each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ has a matrix representation.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$. Let $\{e_1, \ldots, e_n\}$ denote the standard basis vectors in $\mathbb{R}^n$, e.g.

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n.$$

Define the matrix $[T] \in \mathbb{R}^{m \times n}$ by

$$[T] = [T(e_1) \cdots T(e_n)].$$

- We call $[T]$ the matrix representation of $T$.
- Knowing $[T]$ is equiavalent to knowing $T$ (see below).
Matrix representations. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear,

$$[T] = [T(e_1) \cdots T(e_n)], \quad X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 e_1 + \cdots + x_n e_n.$$

By linearity,

$$T(X) = x_1 T(e_1) + \cdots + x_n T(e_n) = [T]X.$$

Example. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^4$, with

$$T(e_1) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad T(e_2) = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \quad T(e_3) = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Then

$$[T] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \\ 4 & 4 & 4 \end{bmatrix}.$$
Matrix representations.

If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then $[T] \in \mathbb{R}^{m \times n}$, and so:

- $[T] \in \mathbb{R}^{m \times n}$ has $m$ rows and $n$ columns.
- $T$ onto $\iff [T]$ has pivot in every row.
- $T$ one-to-one $\iff [T]$ has pivot in every column.
- If $m > n$, then $T$ cannot be onto.
- If $m < n$, then $T$ cannot be one-to-one.
Linear transformations of the plane $\mathbb{R}^2$.

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear. Then

$$T(X) = [T]X = [T(e_1)\ T(e_2)]X = xT(e_1) + yT(e_2).$$

We consider several types of linear transformations with clear geometric meanings, including:

- shears,
- reflections,
- rotations,
- compositions of the above.
Example. (Shear) Let $\lambda \in \mathbb{R}$ and consider

$$[T] = \begin{bmatrix} 1 & \lambda \\ 0 & 1 \end{bmatrix}, \quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + \lambda y \\ y \end{bmatrix}.$$

Then

$$[T]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$[T]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix},$$
$$[T]\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -\lambda \\ -1 \end{bmatrix}.$$
Example. (Reflection across the line $y = x$)

Let

$$T(X) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}.$$ 

Note

$$[T]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad [T]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$[T]\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad [T]\begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$
Example. (Rotation by angle $\theta$) Let

$$T(X) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$ 

Then

$$[T]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix},$$

$$[T]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}.$$
Example. (Composition) Let us now construct $T$ that (i) reflects about the $y$-axis ($x = 0$) and then (ii) reflects about $y = x$.

(i) \[ [T_1] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(ii) } [T_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

We should then take

\[ T(X) = T_2 \circ T_1(X) = T_2(T_1(X)), \]

that is,

\[ T(X) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}. \]

Note that $T = T_2 \circ T_1$ is a linear transformation, with

\[ [T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]