Proton Radius, Darwin-Foldy Term and Radiative Corrections

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Muonic Hydrogen and Lamb Shift
Theory of Bound Systems: Three Developments

**Schrödinger Theory:**

\[
E_n = -\frac{(Z\alpha)^2 m}{2n^2} = -\frac{Z^2 \hbar (2\pi R_\infty c)}{n^2} \quad (\hbar = c = \epsilon_0 = 1).
\]

**Dirac Theory:**

\[
E_{nj} = m - \frac{(Z\alpha)^2 m}{2n^2} - \frac{(Z\alpha)^4 m}{n^3} \left[ \frac{1}{2j + 1} + \frac{3}{8n} \right] + \mathcal{O}[(Z\alpha)^6].
\]

**QED:**

Self-energy effects, corrections to the Coulomb force law, So-called recoil corrections, Feynman diagrams ...
Lamb-Shift Phenomenology (Atomic Hydrogen)

Lifts 2S-2P degeneracy:

\[
E(2S_{1/2}) = E(2P_{1/2})
\]

\[
2S_{1/2} \quad \quad +1045 \text{ Mhz}
\]

\[
2P_{1/2} \quad \quad -13 \text{ Mhz}
\]

Dirac-Theory:
Lamb-Shift Phenomenology (Transitions)

Shifts $nS-n'S$ transition frequencies:

**Lamb:**

- $2S_{1/2}$
  - $E(2S_{1/2})$ +1045 Mhz

**Dirac:**

- $1S_{1/2}$
  - $E(1S_{1/2})$ +8173 Mhz
Up to 2010: QED and experiment were essentially in agreement, but then...
The size of the proton

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CODATA: \( r_p = 0.8768(69) \text{ fm} \), muonic H: \( r_p = 0.84184(67) \text{ fm} \)

Proton Structure from the Measurement of 2S-2P Transition Frequencies of Muonic Hydrogen


Accurate knowledge of the charge and Zemach radii of the proton is essential, not only for understanding its structure but also as input for tests of bound-state quantum electrodynamics and its predictions for the energy levels of hydrogen. These radii may be extracted from the laser spectroscopy of muonic hydrogen (μp, that is, a proton orbited by a muon). We measured the $2S_{1/2}^f-2P_{3/2}^f$ transition frequency in $\mu p$ to be $54611.16(1.05)$ gigahertz (numbers in parentheses indicate one standard deviation of uncertainty) and reevaluated the $2S_{1/2}^f-2P_{3/2}^f$ transition frequency, yielding $49881.35(65)$ gigahertz. From the measurements, we determined the Zemach radius, $r_Z = 1.082(37)$ femtometers, and the magnetic radius, $r_M = 0.87(6)$ femtometer, of the proton. We also extracted the charge radius, $r_E = 0.84087(39)$ femtometer, with an order of magnitude more precision than the 2010-CODATA value and at 7σ variance with respect to it, thus reinforcing the proton radius puzzle.
Muonic Hydrogen Transitions Investigated

\[ 2S_{1/2}^F - 2P_{3/2}^F \]

\[ 2S_{1/2}^F - 2P_{3/2}^F \]

[A. Antognini et al., Science 339 (2013) 417]


Lamb shift

\[ \nu_{\text{triplet}} \]

\[ \nu_{\text{singlet}} \]

2S hyperfine splitting
Muonic Hydrogen Puzzle

CODATA: $r_p = 0.8768(69)$ fm

electronic H: $r_p = 0.8802(80)$ fm

Scattering (Mainz, 2010): $r_p = 0.879(8)$ fm

Scattering (Jefferson Lab, 2011): $r_p = 0.875(10)$ fm

(essentially 0.88 fm) BUT

muonic H: $r_p = 0.84184(67)$ fm

(essentially 0.84 fm)

Why Can You Determine Nuclear Radii from Spectroscopy?

You calculate the spectrum.
[Nonrelativistic Theory.]

You calculate the spectrum more accurately.
[Relativistic Effects.]

You calculate the spectrum even more accurately.
[QED effects.]

At some point the nuclear size becomes important.
[Distortion of Coulomb Potential.]

Someone else measures the spectrum.
[And then you can tell what the nuclear size is.]
Brief Overview of Subtleties of the Theory
[Corrections to the Spectrum without Nuclear Structure]
HFS-FS-Coupling in the Muonic Hydrogen System
[Even without QED, theory is not without subtleties]

If we define the states
\[ |1\rangle = |2P_{1/2}(F = 0)\rangle, \quad |2\rangle = |2P_{1/2}(F = 1)\rangle, \quad |3\rangle = |2P_{3/2}(F = 1)\rangle, \quad |4\rangle = |2P_{3/2}(F = 2)\rangle, \]

and the matrix elements
\[
\beta_{1/2} = E_{\text{hfs}}(2P_{1/2}), \quad \nu = V(2P), \quad \beta_{3/2} = E_{\text{hfs}}(2P_{3/2}), \quad f = E_{\text{hfs}}(2P),
\]

and the zero point of the energy scale to be the hyperfine centroid of the $2P_{1/2}$ levels, then the Breit-Pauli Hamiltonian in the $2P$ state manifold assumes the following matrix form $M_{\text{BP}}$:

\[
M = \begin{pmatrix}
-\frac{3}{4} \beta_{1/2} & 0 & 0 & 0 \\
0 & \frac{1}{4} \beta_{1/2} & \nu & 0 \\
0 & \nu & -\frac{5}{8} \beta_{3/2} + f & 0 \\
0 & 0 & 0 & \frac{3}{8} \beta_{3/2} + f
\end{pmatrix}.
\]

The off-diagonal elements $\nu$ lead to admixtures to the $|2P_{1/2}(F = 1)\rangle$ levels from the $|2P_{3/2}(F = 1)\rangle$ levels and vice versa, and to a repulsive interaction as for any coupled two-level system. In agreement with this general consideration, a diagonalization of $M_{\text{BP}}$ immediately leads to the conclusion, that the $|2P_{1/2}(F = 1)\rangle$ is lowered in energy by
\[
\Delta = 0.145 \text{ meV},
\]

whereas the $|2P_{3/2}(F = 1)\rangle$ energy is increased by $\Delta$. This is in full agreement with Ref. [19].

Vacuum Polarization Effects.

*The Coulomb law is incorrect at small distances.*

Muonic hydrogen is smaller than atomic hydrogen by a factor of 207 (mass ratio of muon to electron).

The vacuum polarization energy shift is 40,000 times larger in muonic hydrogen.

Reason:

Generation of virtual electron-positron pairs in the vicinity of the proton. The quantum vacuum has structure!

(The 2P state is energetically higher, for muonic hydrogen)
For short distances, the Uehling potential only adds a logarithmic divergence to the Coulomb potential.

For long distances, the Uehling term is exponentially suppressed.

\[ V(r) = -\frac{\alpha}{r} = -\frac{\alpha^2 m_r}{\rho}. \]  

(Coulomb Law and scaled radial coordinate \( \rho \))

(Mass Ratio) \( x = \frac{m_e}{\alpha m_r} = 0.73738368 \ldots \)

(Quantum Correction) \[ V_{vp}(r) \sim \frac{\alpha^3 m_r}{\pi \rho} \left[ \frac{2}{3} \left( \ln(\rho x) + \gamma_E \right) - \frac{\pi}{2} \rho x + \frac{5}{9} \right] + O(\rho). \]

\[ V_{vp}(r) \sim -\frac{\alpha^3 m_r}{\sqrt{\pi}} e^{-2\rho x} \left[ \frac{1}{4\rho^{5/2}x^{3/2}} - \frac{29}{64\rho^{7/2}x^{5/2}} + \frac{2225}{2048\rho^{9/2}x^{7/2}} + O\left(\frac{1}{\rho^{11/2}}\right) \right], \]
In muonic hydrogen, one-loop vacuum polarization effects are even larger than relativistic corrections!

Small size of muonic hydrogen:
Sensitive to nuclear structure!

...and genuine two-body system, unlike high-Z muonic ions...

LARGE MASS RATIOS

\[
\xi_p = \frac{m_\mu}{m_p} = 0.112609 \ldots \approx \frac{1}{9},
\]

\[
\xi_d = \frac{m_\mu}{m_d} = 0.0563327 \ldots \approx \frac{1}{18},
\]
Definition of Coulomb Gauge: $G_{00}$ is Static or Instantaneous

For massive photon exchange:

\[ G_{00}(\vec{q}) = -\frac{1}{\vec{q}^2 + \lambda^2} \]

\[ G_{ij}(\vec{q}) = -\frac{1}{\vec{q}^2 + \lambda^2} \left[ \delta^{ij} - \frac{q^i q^j}{\vec{q}^2 + \lambda^2} \right] \]


### Table I

Detailed breakdown of the first-order and second-order individual contributions $\delta E_i^{(1)}$ and $\delta E_j^{(2)}$ to the relativistic Breit correction $\delta E_{vp}$ of vacuum polarization for $\mu H$, $\mu D$, and muonic helium ions. All units are meV.

<table>
<thead>
<tr>
<th></th>
<th>$\mu H$</th>
<th>$\mu D$</th>
<th>$\mu$He$^3$</th>
<th>$\mu$He$^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta E_i^{(1)}$</td>
<td>-0.000558</td>
<td>-0.000679</td>
<td>-0.020331</td>
<td>-0.020970</td>
</tr>
<tr>
<td>$\delta E_j^{(1)}$</td>
<td>0.000064</td>
<td>0.000038</td>
<td>0.000467</td>
<td>0.000360</td>
</tr>
<tr>
<td>$\delta E_{vp}^{(2)}$</td>
<td>-0.049149</td>
<td>-0.060227</td>
<td>-1.621122</td>
<td>-1.675776</td>
</tr>
<tr>
<td>$\delta E_{vp}$</td>
<td>-0.024245</td>
<td>-0.028149</td>
<td>-0.821668</td>
<td>-0.840404</td>
</tr>
<tr>
<td>$\Delta E_{vp}$   (this work)</td>
<td>$2P_{1/2}-2S_{1/2}$ [meV]</td>
<td>$0.018759$</td>
<td>$0.021781$</td>
<td>$0.509344$</td>
</tr>
<tr>
<td>Ref. [17]</td>
<td>0.0169</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ref. [11]</td>
<td></td>
<td>0.0214</td>
<td>0.495</td>
<td></td>
</tr>
<tr>
<td>Ref. [16]$^a$</td>
<td>0.0169</td>
<td>0.0214</td>
<td>0.495</td>
<td>0.508</td>
</tr>
</tbody>
</table>

$^a$A conceptually different approach is used in Ref. [16].

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From One-Loop to Two-Loop Vacuum Polarization

One- and Two-Loop Effects
Even three-loop vacuum polarization:

The final numerical values for the contributions to the 2P_{1/2}–2S_{1/2} Lamb shift are given as

\[ \Delta E_{\text{svp}}(\mu H) = -0.00254 \text{ meV}, \]  
(29a)

\[ \Delta E_{\text{svp}}(\mu D) = -0.00306 \text{ meV}, \]  
(29b)

\[ \Delta E_{\text{svp}}(\mu ^3\text{He}^+) = -0.06269 \text{ meV}, \]  
(29c)

\[ \Delta E_{\text{svp}}(\mu ^4\text{He}^+) = -0.06462 \text{ meV}. \]  
(29d)

Recoil Correction to Vacuum Polarization

[Vacuum-Polarization Insertion in Two-Photon Exchange]

For the Lamb shift, the total logarithmic radiative-recoil correction is given as the sum

$$\Delta E_{RR} = \Delta E_L + \Delta E_S + \Delta E_W.$$  \hspace{1cm} (45)

It evaluates to

$$\Delta E_{RR}(\mu \text{H}) = 0.000136 \text{ meV},$$  \hspace{1cm} (46a)
$$\Delta E_{RR}(\mu \text{D}) = 0.000093 \text{ meV},$$  \hspace{1cm} (46b)
$$\Delta E_{RR}(\mu ^3\text{He}^+) = 0.004941 \text{ meV},$$  \hspace{1cm} (46c)
$$\Delta E_{RR}(\mu ^4\text{He}^+) = 0.003867 \text{ meV}.$$  \hspace{1cm} (46d)

The results are numerically small and suppressed with respect to the leading recoil correction given in equation (32) by a factor $\alpha$. 

Proton Radius Definition
And Darwin-Foldy Term

[Now It’s Getting a Little Technical]
Nuclear-Size Correction (Leading Order)

(Leading-Order)
Finite-Size Hamiltonian
[Affects S States with a Nonvanishing Probability Density at the Origin]

(proportional to the Dirac-δ function, measures probability density of the electronic wave function at the origin)

\[ \Delta H_{fs} = \sum_{i} \frac{2}{3} \langle r^2 \rangle \left[ \pi Z \alpha \delta^3 (r_i^2) \right] \]

Very well, but how is \( \langle r^2 \rangle \) defined?
Barker-Glover Term

\[ E_{BG} = \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \left( \frac{1}{j + 1/2} - \frac{1}{\ell + 1/2} \right) (1 - \delta_{\ell 0}), \]  \[ \tag{5} \]

Darwin-Foldy Term [Just The One Proportional to the Dirac-\( \delta \)]

\[ E_{DF} = - \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \left( \frac{1}{j + 1/2} - \frac{1}{\ell + 1/2} \right) \delta_{\ell 0} \]
\[ = \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \delta_{\ell 0}, \]

Can Write the Darwin-Foldy Term As Follows

\[ E_{DF} = \frac{2}{3} \left( \frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \left\{ \frac{3\hbar^2}{4m_N^2 c^2} \right\} \delta_{\ell 0}. \]
Nuclear Size Correction

\[ E_{NS} = \frac{2}{3} \left( \frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \langle r^2 \rangle_N \delta_{\ell_0}, \]

Darwin-Foldy Term

\[ E_{DF} = \frac{2}{3} \left( \frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3 \lambda_C^2} \left\{ \frac{3\hbar^2}{4m_N^2 c^2} \right\} \delta_{\ell_0}. \]

Comparison: Darwin-Foldy Correction to Radius Square

\[ \langle r^2 \rangle_{N}^{DF} = \frac{3\hbar^2}{4m_N^2 c^2}. \]

Correspondence of Darwin-Foldy Correction to Energy and Radius Square

\[ E_{NS} \rightarrow E_{NS} + E_{DF} \iff \langle r^2 \rangle_{N}^{\text{ATP}} \rightarrow \langle r^2 \rangle_{N}^{\text{ATP}} + \langle r^2 \rangle_{N}^{DF}, \]

ATP = “Atomic-Physics Conventions”
Definition of the Radius Square in Atomic-Physics Conventions (ATP)

\[ \langle r^2 \rangle_{p}^{\text{ATP}} = \langle r^2 \rangle_{E}^p = 6\hbar^2 \frac{\partial G_E(q^2)}{\partial q^2} \bigg|_{q^2=0}, \]

Decomposition of the Sachs Form Factor

\[ G_E(q^2) = F_1(q^2) + \frac{q^2}{4(m_pc)^2} F_2(q^2), \]
\[ G_M(q^2) = F_1(q^2) + F_2(q^2), \quad F_2(0) = \pi_p, \]

Friar, Martorell and Sprung Use Different Conventions (FMS)
[Darwin-Foldy Term Sits Inside the Radius]

\[ \tilde{G}_E(q^2) = \frac{G_E(q^2)}{\sqrt{1 - q^2/4m_p^2}}, \]
\[ \langle r^2 \rangle_{p}^{\text{FMS}} = \langle r^2 \rangle_{p}^{\text{ATP}} + \langle r^2 \rangle_{p}^{\text{DF}}. \]

Is There a Problem with Different Conventions?

A closer inspection [for details see U.D.J., Eur. Phys. J. D 61 (2011) 7] reveals that all atomic-physics determinations (including the CODATA adjustments) as well as all relevant scattering experiment use ATP conventions. The Darwin-Foldy term is subtracted before the radius square is determined.

But we are not done just yet.

There is a further subtlety connected with the internal-structure contributions to the form factors and the pure QED contributions to the proton form factors. QED contributions are excluded from the proton radius:

\[
\begin{align*}
G_E(q^2) &= \bar{G}_E(q^2) + G_{E}^{QED}(q^2), \\
F_1(q^2) &= \bar{F}_1(q^2) + F_{1}^{QED}(q^2), \\
F_2(q^2) &= \bar{F}_2(q^2) + F_{2}^{QED}(q^2), \\
\kappa_p &= \bar{\kappa}_p + \kappa_{p}^{QED}.
\end{align*}
\]

\[
\langle r^2 \rangle_{\text{ATP}} = 6\hbar^2 \frac{\partial G_{E}(q^2)}{\partial q^2} \bigg|_{q^2=0}
= 6\hbar^2 \frac{\partial F_{1}(q^2)}{\partial q^2} \bigg|_{q^2=0} + \frac{3\hbar^2}{2 (m_p c)^2} \kappa_p,
\]

\Rightarrow \text{We must make sure that we do not artificially add QED contributions to the proton structure, e.g., by an improper definition of the nuclear self-energy.}
A closer inspection reveals that the energy correction induced by the $F_1$ form factor slope of the proton and the anomalous magnetic moment correction from the proton line precisely add up to a nuclear-size correction determined by the slope of the Sachs form factor (internal-structure terms).

“We are doing everything right in leading order!”

In order not to mess things up in higher order, we should define the nuclear self-energy correction as follows,

\[
E_{NSE} = \frac{Z(Z\alpha)^5}{\pi n^3} \left( \frac{m_r}{m_p} \right)^2 m_r c^2 \\
\times \left\{ \left[ \frac{4}{3} \ln \left( \frac{m_p}{m_r(Z\alpha)^2} \right) + \frac{10}{9} \right] \delta_{\ell 0} - \frac{4}{3} \ln k_0(n, \ell) \right\}. \tag{28}
\]

“No internal proton structure in the nuclear self-energy, please!”

⇒ There is not a problem with the conventions, at least not on the level relevant to the comparison of theory and experiment.

So...
Muonic Hydrogen Discrepancy: 0.420 meV.

Largest Conceivable Uncertainty within Standard Model: +/- 0.010 meV.

All theorists agree. The fact that no theoretical explanation for the observed discrepancy exists was pointed out in [U.D.J., Ann.Phys.(N.Y) 326 (2011) 500] [U.D.J., Ann.Phys.(N.Y) 326 (2011) 516] and various other papers authored by the theoretical community since then.

Conundrum remains unsolved!

New fundamental forces!?!
Conclusions
Conclusions

Interesting muonic hydrogen discrepancies
[Very technical experiment]

Theorists have done a lot of work
[including recoil corrections to vacuum polarization
and radiative-recoil]
The effect of the large mass ratio seems to be
well under control.

Darwin-Foldy term is excluded from
the proton radius.
[...and we can define the nuclear self-energy so that
the internal proton structure and QED are well separated]
Thanks!