Proton Radius, Darwin-Foldy Term and Radiative Corrections

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APS April Meeting
Denver, Colorado
15-APR-2013

Muonic Hydrogen and Lamb Shift

Theory of Bound Systems: Three Developments

Schrödinger Theory:

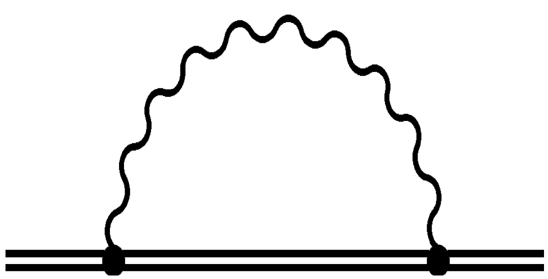
$$E_n = -\frac{(Z\alpha)^2\, m}{2\, n^2} = -\frac{Z^2\, \hbar\, (2\pi R_\infty c)}{n^2} \qquad (\hbar = c = \epsilon_0 = 1)\,.$$

Dirac Theory:

$$E_{nj} = m - \frac{(Z\alpha)^2\, m}{2n^2} - \frac{(Z\alpha)^4\, m}{n^3} \left[\frac{1}{2j+1} + \frac{3}{8\,n}\right] + \mathcal{O}[(Z\alpha)^6].$$

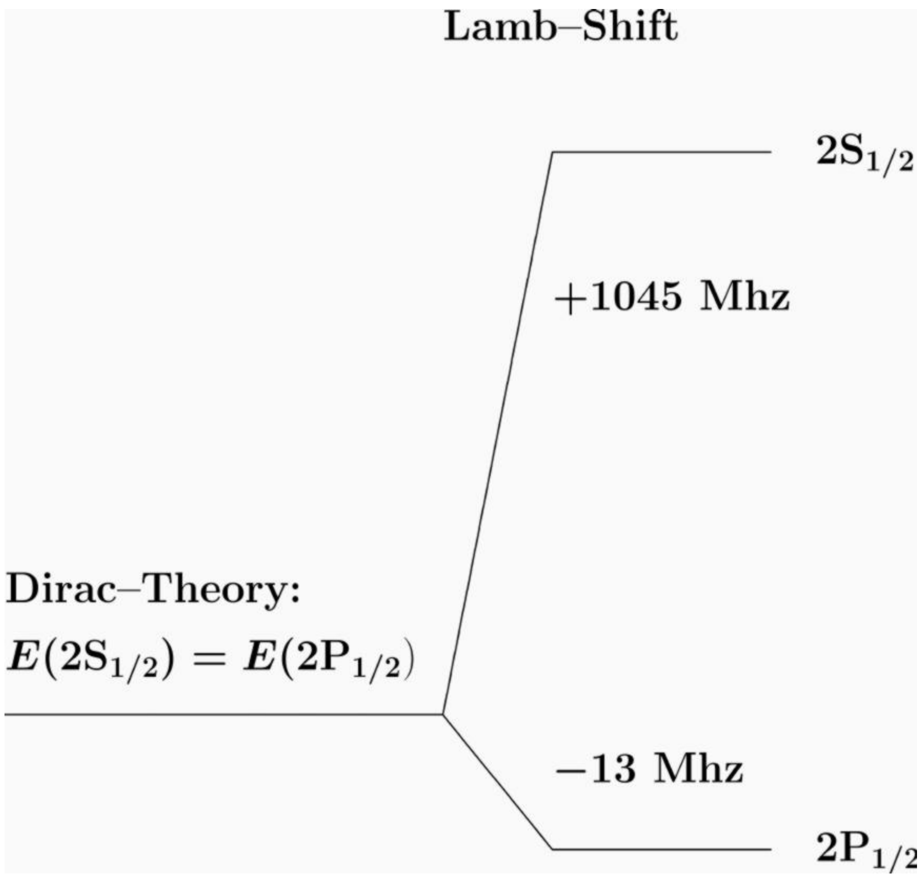
QED:

Self-energy effects,
corrections to the Coulomb force law,
So-called recoil corrections,
Feynman diagrams ...



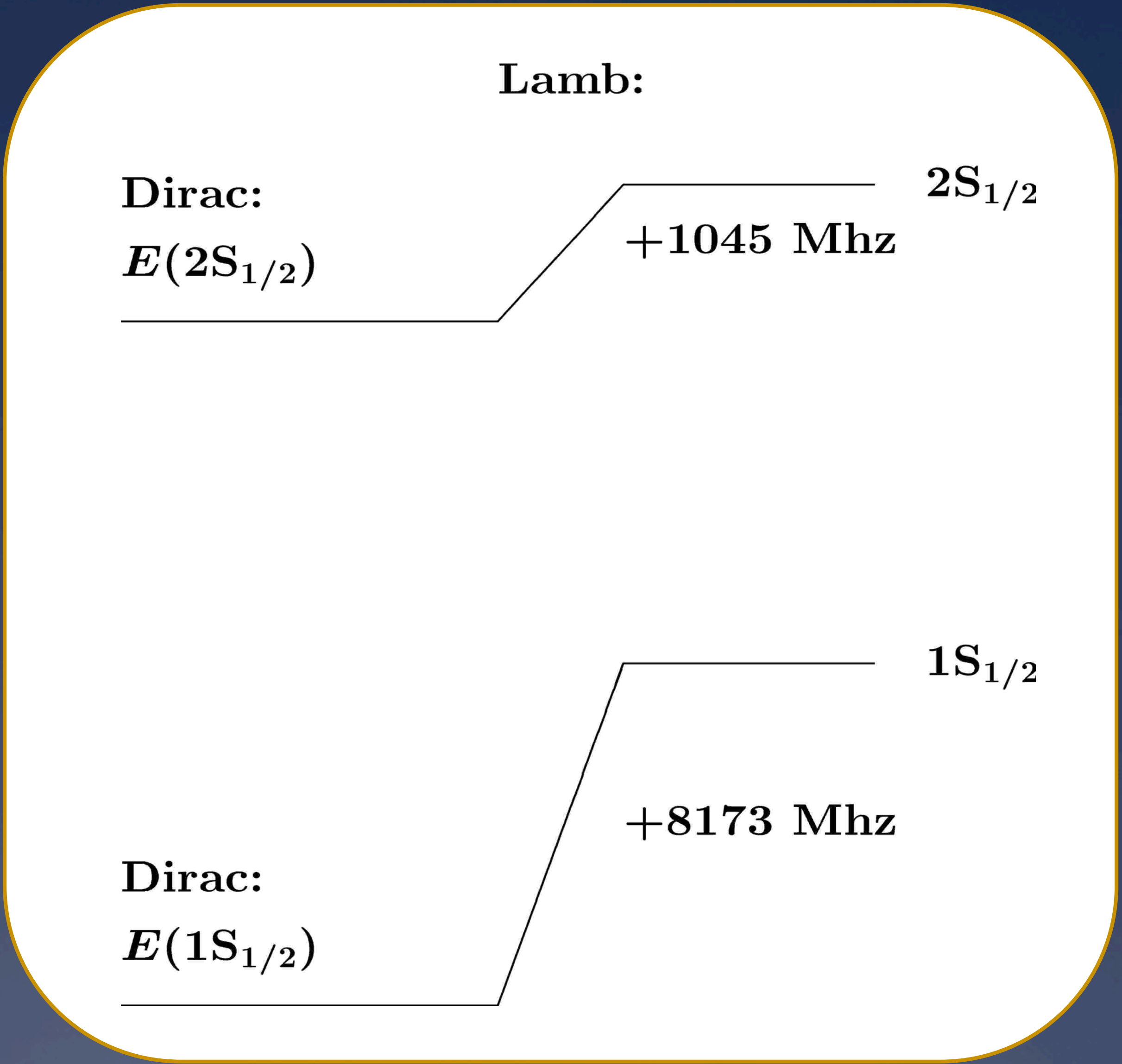
Lamb-Shift Phenomenology (Atomic Hydrogen)

Lifts 2S-2P degeneracy:



Lamb-Shift Phenomenology (Transitions)

Shifts *nS-n'S* transition frequencies:



Up to 2010:
QED and experiment were
essentially in agreement,
but then...

The First Paper...

Nature **466** (2010) 213

Vol 466 | 8 July 2010 | **doi:10.1038/nature09250** nature

LETTERS

The size of the proton

Randolf Pohl[1], Aldo Antognini[1], François Nez[2], Fernando D. Amaro[3], François Biraben[2], João M. R. Cardoso[3], Daniel S. Covita[3,4], Andreas Dax[5], Satish Dhawan[5], Luis M. P. Fernandes[3], Adolf Giesen[6]†, Thomas Graf[6], Theodor W. Hänsch[1], Paul Indelicato[2], Lucile Julien[2], Cheng-Yang Kao[7], Paul Knowles[8], Eric-Olivier Le Bigot[2], Yi-Wei Liu[7], José A. M. Lopes[3], Livia Ludhova[8], Cristina M. B. Monteiro[3], Françoise Mulhauser[8]†, Tobias Nebel[1], Paul Rabinowitz[9], Joaquim M. F. dos Santos[3], Lukas A. Schaller[8], Karsten Schuhmann[10], Catherine Schwob[2], David Taqqu[11], João F. C. A. Veloso[4] & Franz Kottmann[12]

CODATA: $r_p = 0.8768(69)$ fm,   muonic H: $r_p = 0.84184(67)$ fm

[R. Pohl et al., Nature 466 (2010) 213]

The Second Paper...

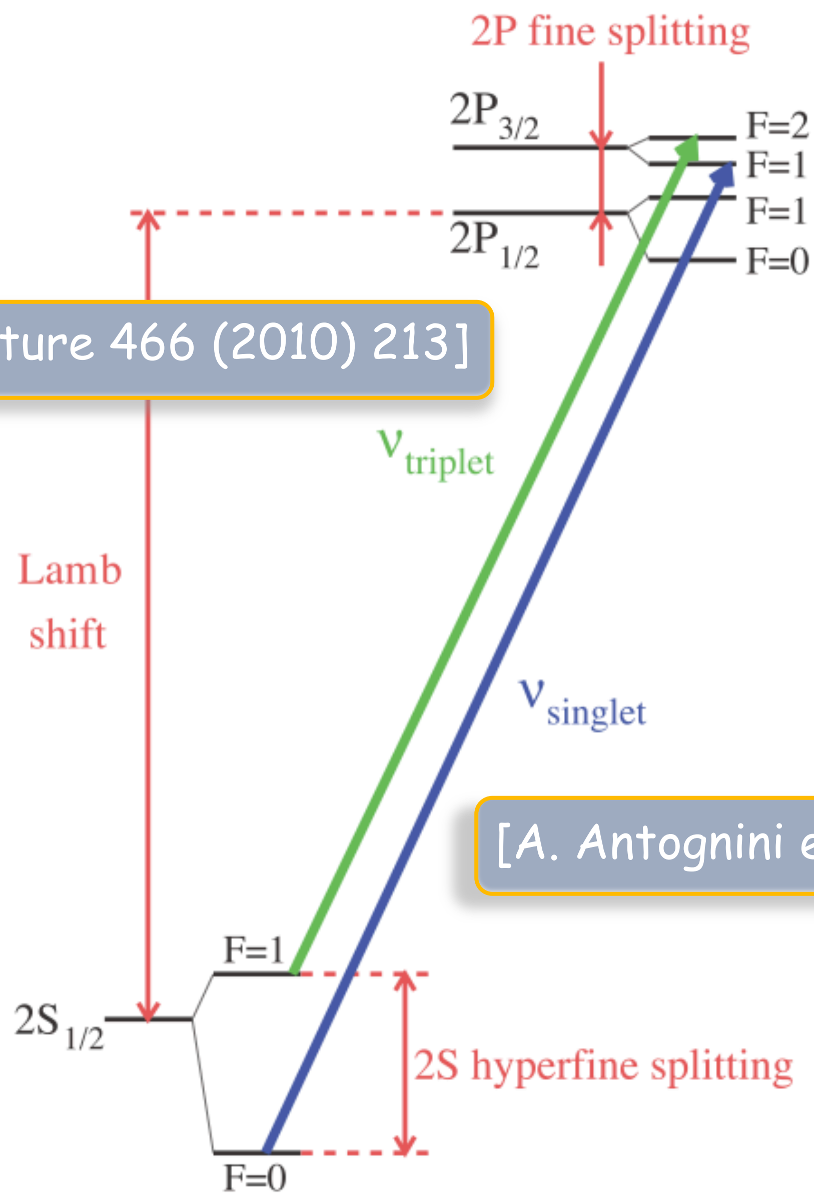
Proton Structure from the Measurement of 2S-2P Transition Frequencies of Muonic Hydrogen

Aldo Antognini,[1,2*] François Nez,[3] Karsten Schuhmann,[2,4] Fernando D. Amaro,[5] François Biraben,[3] João M. R. Cardoso,[5] Daniel S. Covita,[5,6] Andreas Dax,[7] Satish Dhawan,[7] Marc Diepold,[1] Luis M. P. Fernandes,[5] Adolf Giesen,[4,8] Andrea L. Gouvea,[5] Thomas Graf,[8] Theodor W. Hänsch,[1,9] Paul Indelicato,[3] Lucile Julien,[3] Cheng-Yang Kao,[10] Paul Knowles,[11] Franz Kottmann,[2] Eric-Olivier Le Bigot,[3] Yi-Wei Liu,[10] José A. M. Lopes,[5] Livia Ludhova,[11] Cristina M. B. Monteiro,[5] Françoise Mulhauser,[11] Tobias Nebel,[1] Paul Rabinowitz,[12] Joaquim M. F. dos Santos,[5] Lukas A. Schaller,[11] Catherine Schwob,[3] David Taqqu,[13] João F. C. A. Veloso,[6] Jan Vogelsang,[1] Randolf Pohl[1]

Accurate knowledge of the charge and Zemach radii of the proton is essential, not only for understanding its structure but also as input for tests of bound-state quantum electrodynamics and its predictions for the energy levels of hydrogen. These radii may be extracted from the laser spectroscopy of muonic hydrogen (μp, that is, a proton orbited by a muon). We measured the $2S_{1/2}^{F=0}$-$2P_{3/2}^{F=1}$ transition frequency in μp to be 54611.16(1.05) gigahertz (numbers in parentheses indicate one standard deviation of uncertainty) and reevaluated the $2S_{1/2}^{F=1}$-$2P_{3/2}^{F=2}$ transition frequency, yielding 49881.35(65) gigahertz. From the measurements, we determined the Zemach radius, $r_Z = 1.082(37)$ femtometers, and the magnetic radius, $r_M = 0.87(6)$ femtometer, of the proton. We also extracted the charge radius, $r_E = 0.84087(39)$ femtometer, with an order of magnitude more precision than the 2010-CODATA value and at 7σ variance with respect to it, thus reinforcing the proton radius puzzle.

[A. Antognini et al., Science 339 (2013) 417]

Muonic Hydrogen Transitions Investigated



$2S_{1/2}^{F=0}$-$2P_{3/2}^{F=1}$

$2S_{1/2}^{F=1}$-$2P_{3/2}^{F=2}$

[R. Pohl et al., Nature 466 (2010) 213]

[A. Antognini et al., Science 339 (2013) 417]

Muonic Hydrogen Puzzle

CODATA: r_p = 0.8768(69) fm

electronic H: r_p = 0.8802(80) fm

Scattering (Mainz, 2010): r_p = 0.879(8) fm

Scattering (Jefferson Lab, 2011): r_p = 0.875(10) fm

<u>(essentially 0.88 fm)</u> BUT

muonic H: r_p = 0.84184(67) fm

<u>(essentially 0.84 fm)</u>

[R. Pohl et al., Nature 466 (2010) 213] [A. Antognini et al., Science 339 (2013) 417]

Why Can You Determine Nuclear Radii from Spectroscopy?

You calculate the spectrum.
[Nonrelativistic Theory.]

You calculate the spectrum more accurately.
[Relativistic Effects.]

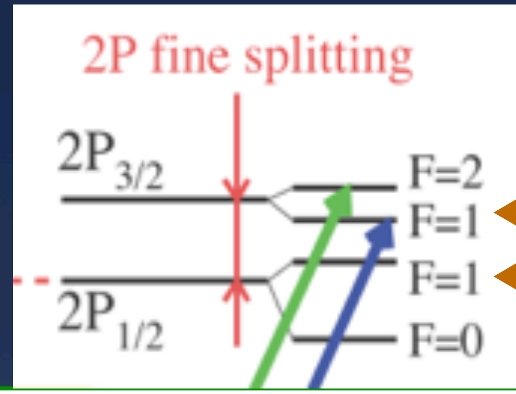
You calculate the spectrum even more accurately.
[QED effects.]

At some point the nuclear size becomes important.
[Distortion of Coulomb Potential.]

Someone else measures the spectrum.
[And then you can tell what the nuclear size is.]

Brief Overview of Subtleties of the Theory
[Corrections to the Spectrum without Nuclear Structure]

HFS-FS-Coupling in the Muonic Hydrogen System [Even without QED, theory is not without subtleties]



Up by 0.145 meV

Down by 0.145 meV

If we define the states

$$|1\rangle = |\,2P_{1/2}(F=0)\rangle, \quad |2\rangle = |\,2P_{1/2}(F=1)\rangle, \quad |3\rangle = |\,2P_{3/2}(F=1)\rangle, \quad |4\rangle = |\,2P_{3/2}(F=2)\rangle, \tag{2.16}$$

and the matrix elements

$$\beta_{1/2} = E_{\rm hfs}(2P_{1/2}), \quad v = V(2P), \quad \beta_{3/2} = E_{\rm hfs}(2P_{3/2}), \quad f = E_{\rm hfs}(2P), \tag{2.17}$$

and the zero point of the energy scale to be the hyperfine centroid of the $2P_{1/2}$ levels, then the Breit–Pauli Hamiltonian in the $2P$ state manifold assumes the following matrix form $M_{\rm BP}$:

$$M = \begin{pmatrix} -\frac{3}{4}\beta_{1/2} & 0 & 0 & 0 \\ 0 & \frac{1}{4}\beta_{1/2} & v & 0 \\ 0 & v & -\frac{5}{8}\beta_{3/2}+f & 0 \\ 0 & 0 & 0 & \frac{3}{8}\beta_{3/2}+f \end{pmatrix}. \tag{2.18}$$

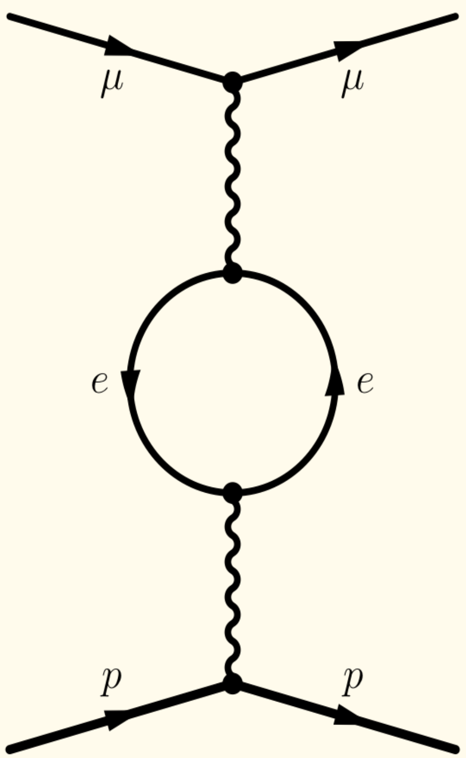
The off-diagonal elements v lead to admixtures to the $|2P_{1/2}(F=1)\rangle$ levels from the $|2P_{3/2}(F=1)\rangle$ levels and vice versa, and to a repulsive interaction as for any coupled two-level system. In agreement with this general consideration, a diagonalization of $M_{\rm BP}$ immediately leads to the conclusion, that the $|2P_{1/2}(F=1)\rangle$ is lowered in energy by

$$\Delta = 0.145\ \text{meV}, \tag{2.19}$$

whereas the $|2P_{3/2}(F=1)\rangle$ energy is increased by Δ. This is in full agreement with Ref. [19].

[U.D.J., Ann.Phys.(N.Y) 326 (2011) 500]

Vacuum Polarization Diagram



Vacuum Polarization Effects.

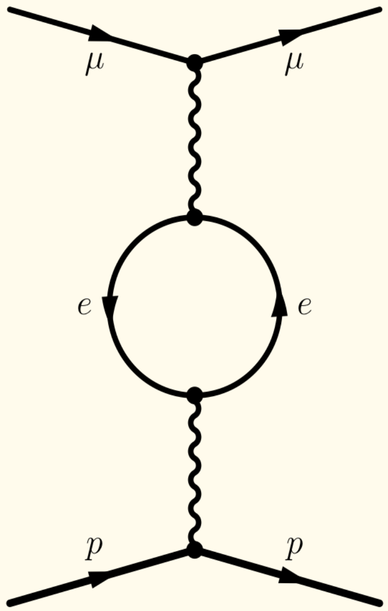
The Coulomb law is incorrect at small distances.

Muonic hydrogen is smaller than atomic hydrogen by a factor of 207 (mass ratio of muon to electron).

The vacuum polarization energy shift is 40,000 times larger in muonic hydrogen.

Reason:
Generation of virtual electron-positron pairs in the vicinity of the proton.
The quantum vacuum has structure!

(The 2P state is energetically higher, for muonic hydrogen)



For short distances, the Uehling potential only adds a logarithmic divergence to the Coulomb potential.

For long distances, the Uehling term is exponentially suppressed.

$$V(r) = -\frac{\alpha}{r} = -\frac{\alpha^2 m_r}{\rho}.$$

(Coulomb Law and scaled radial coordinate ρ)

(Mass Ratio)

$$x = \frac{m_e}{\alpha m_r} = 0.73738368\ldots$$

(Quantum Correction)

$$V_{\rm vp}(r) \sim \frac{\alpha^3 m_r}{\pi\rho}\left[\frac{2}{3}(\ln(\rho x) + \gamma_E) - \frac{\pi}{2}\rho x + \frac{5}{9}\right] + \mathcal{O}(\rho).$$

$$V_{\rm vp}(r) \sim -\frac{\alpha^3 m_r}{\sqrt{\pi}} e^{-2\rho x}\left[\frac{1}{4\rho^{5/2}x^{3/2}} - \frac{29}{64\rho^{7/2}x^{5/2}} + \frac{2225}{2048\rho^{9/2}x^{7/2}} + \mathcal{O}\left(\frac{1}{\rho^{11/2}}\right)\right],$$

In muonic hydrogen, one-loop vacuum polarization effects are even larger than relativistic corrections!

Small size of muonic hydrogen:
Sensitive to nuclear structure!

...and genuine two-body system, unlike high-Z muonic ions...

LARGE MASS RATIOS

$$\xi_p = \frac{m_\mu}{m_p} = 0.112609\ldots \approx \frac{1}{9},$$

$$\xi_d = \frac{m_\mu}{m_d} = 0.0563327\ldots \approx \frac{1}{18},$$

Definition of Coulomb Gauge: G_{00} is Static or Instantaneous

For massive photon exchange:

- $$G_{00}(\vec{q}) = -\frac{1}{\vec{q}^2 + \lambda^2}$$

- $$G_{ij}(\vec{q}) = -\frac{1}{\vec{q}^2 + \lambda^2}\left[\delta^{ij} - \frac{q^i q^j}{\vec{q}^2 + \lambda^2}\right]$$

[K. Pachucki, Phys.Rev.A 53 (1996) 2092]

[U.D.J., Phys.Rev.A 84 (2011) 012505]

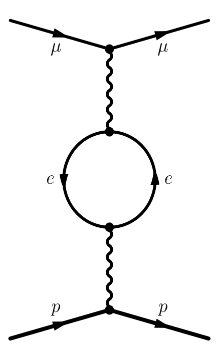
[S.G. Karshenboim, V.G. Ivanov and E.Yu. Korzinin, Phys.Rev.A 85 (2012) 032509]

One-Loop Vacuum Polarization with Magnetic Photon Exchange, and Radiatively Corrected Breit Hamiltonian (with Reduced Mass)

TABLE I. Detailed breakdown of the first-order and second-order individual contributions $\delta E_i^{(1)}$ and $\delta E_j^{(2)}$ to the relativistic Breit correction δE_{vp} of vacuum polarization for μH, μD, and muonic helium ions. All units are meV.

	μH	μD	μHe^3	μHe^4
		$2P_{1/2}$ [meV]		
$\delta E_1^{(1)}$	−0.000558	−0.000679	−0.020331	−0.020970
$\delta E_2^{(1)}$	0.000064	0.000038	0.000467	0.000360
$\delta E^{(2)}$	−0.049149	−0.060227	−1.621122	−1.675776
δE_{vp}	−0.024245	−0.028149	−0.821668	−0.840404
		$2P_{1/2}$-$2S_{1/2}$ [meV] and comparison to other work		
ΔE_{vp} (this work)	0.018759	0.021781	0.509344	0.521104
Ref. [17]	0.0169			
Ref. [11]				−0.202
Ref. [16][a]	0.0169	0.0214	0.495	0.508

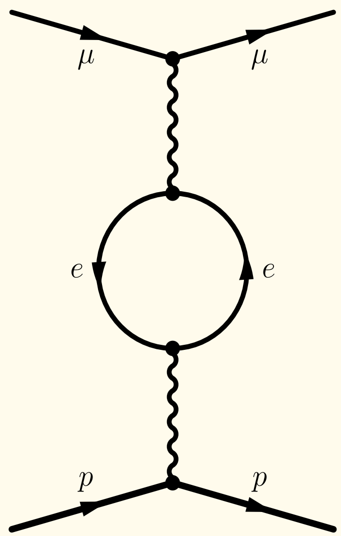
[a]A conceptually different approach is used in Ref. [16].



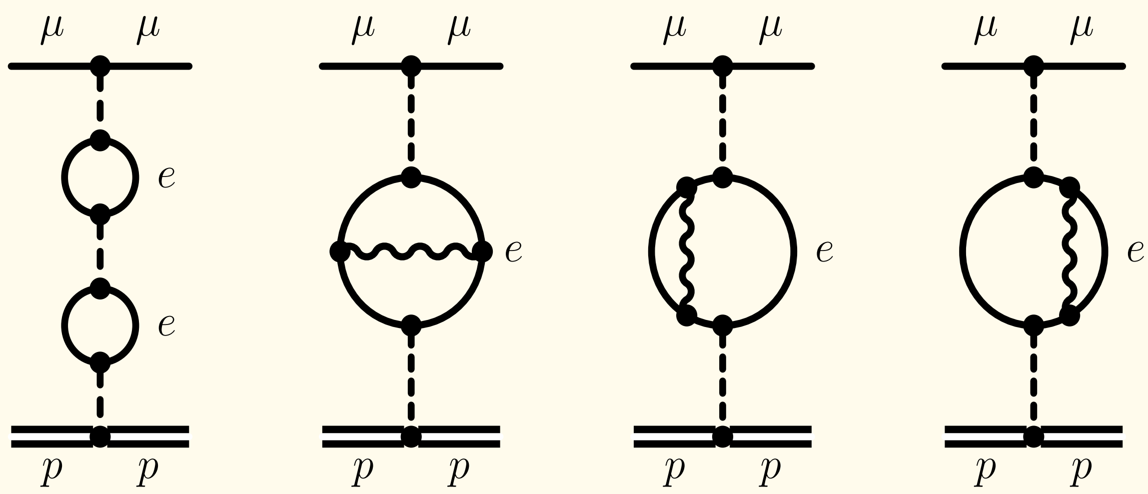
[U.D.J., Phys.Rev.A 84 (2011) 012505]

[S.G. Karshenboim, V.G. Ivanov and E.Yu. Korzinin, Phys.Rev.A 85 (2012) 032509]

From One-Loop to Two-Loop Vacuum Polarization

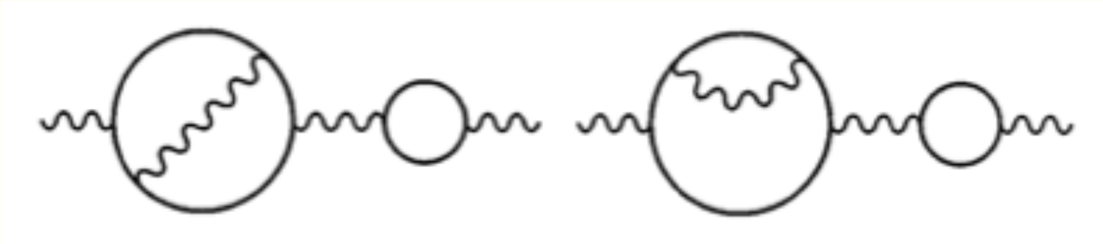


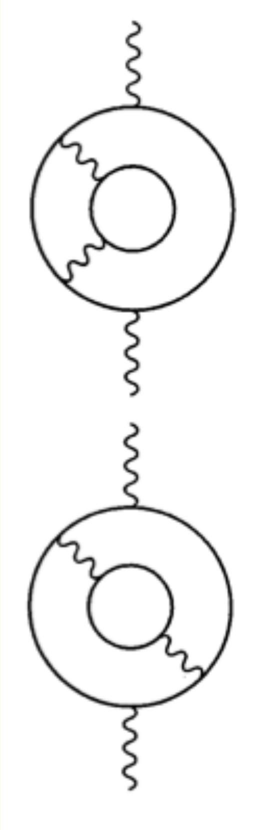
One- and Two-Loop Effects



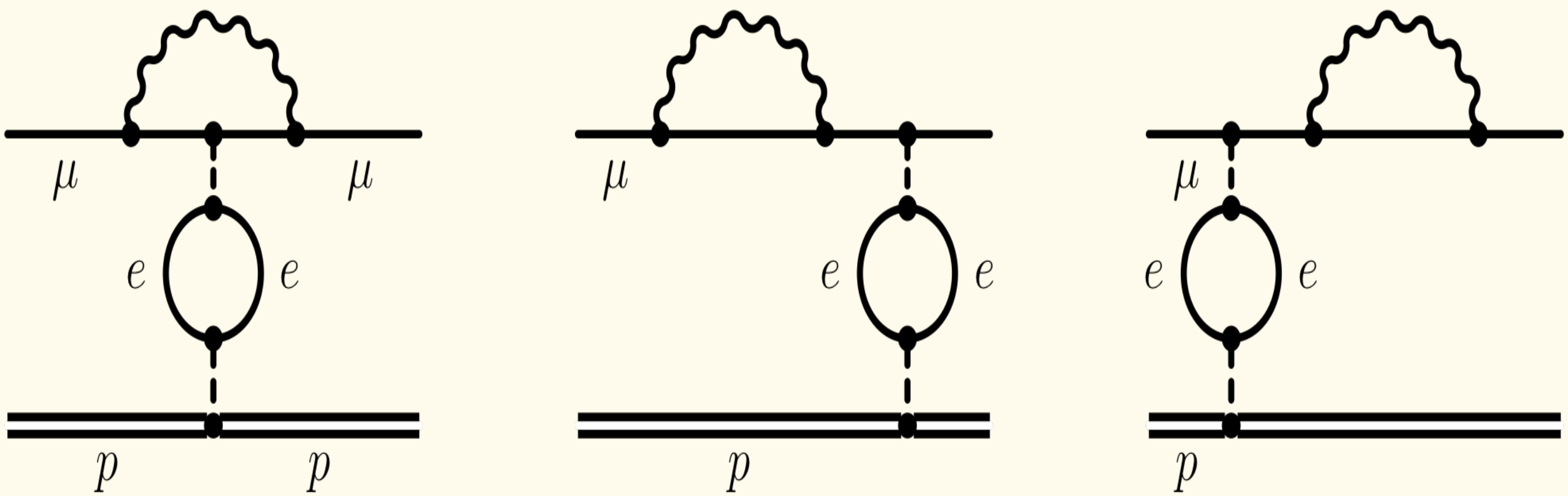
Even three-loop vacuum polarization:

[T. Kinoshita and M. Nio,
Phys. Rev. Lett. 82 (1999) 3240]





Two-Loop Self-Energy Vacuum-Polarization



The final numerical values for the contributions to the $2P_{1/2}$–$2S_{1/2}$ Lamb shift are given as

$$\Delta E_{\rm svp}(\mu \mathrm{H}) = -0.00254\,\mathrm{meV}, \qquad (29a)$$

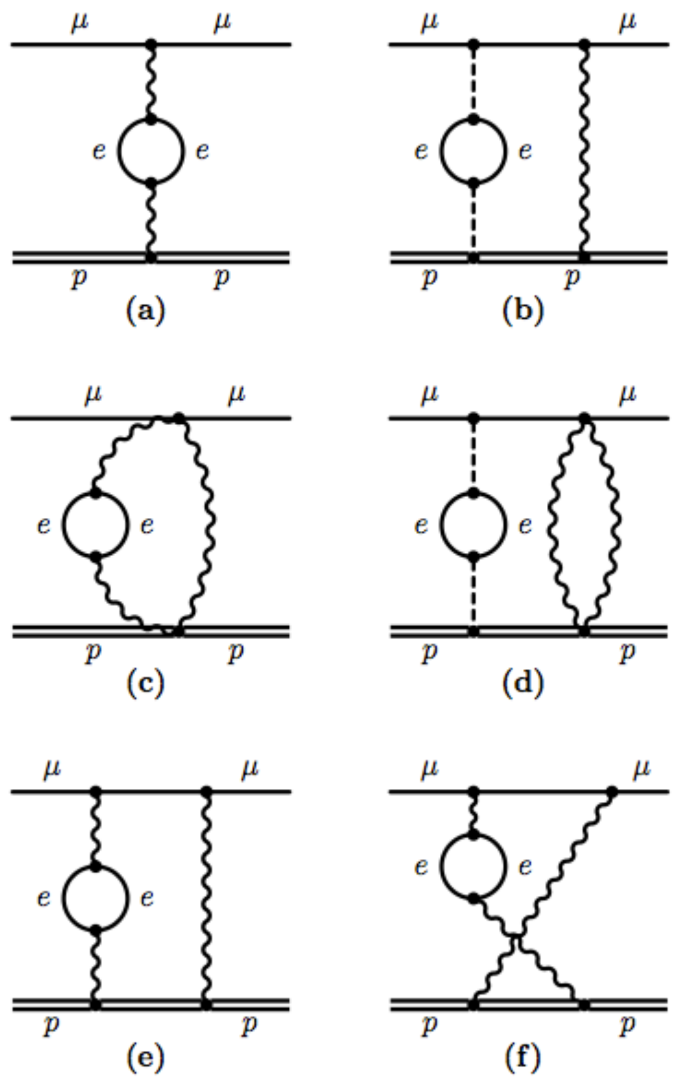
$$\Delta E_{\rm svp}(\mu \mathrm{D}) = -0.00306\,\mathrm{meV}, \qquad (29b)$$

$$\Delta E_{\rm svp}(\mu\, {}^3\mathrm{He}^+) = -0.06269\,\mathrm{meV}, \qquad (29c)$$

$$\Delta E_{\rm svp}(\mu\, {}^4\mathrm{He}^+) = -0.06462\,\mathrm{meV}. \qquad (29d)$$

[U.D.J. Eur. Phys. J. D 65 (2011) 357]

Recoil Correction to Vacuum Polarization
[Vacuum-Polarization Insertion in Two-Photon Exchange]



For the Lamb shift, the total logarithmic radiative-recoil correction is given as the sum

$$\Delta E_{RR} = \Delta E_L + \Delta E_S + \Delta E_W . \qquad (45)$$

It evaluates to

$$\Delta E_{RR}(\mu\mathrm{H}) = 0.000136\,\mathrm{meV}, \qquad (46a)$$

$$\Delta E_{RR}(\mu\mathrm{D}) = 0.000093\,\mathrm{meV}, \qquad (46b)$$

$$\Delta E_{RR}(\mu\,{}^3\mathrm{He}^+) = 0.004941\,\mathrm{meV}, \qquad (46c)$$

$$\Delta E_{RR}(\mu\,{}^4\mathrm{He}^+) = 0.003867\,\mathrm{meV}. \qquad (46d)$$

The results are numerically small and suppressed with respect to the leading recoil correction given in equation (32) by a factor α.

[U.D.J. Eur. Phys. J. D 65 (2011) 357]

Proton Radius Definition And Darwin-Foldy Term

[Now It's Getting a Little Technical]

Nuclear-Size Correction (Leading Order)

(Leading-Order)
Finite-Size Hamiltonian
[Affects S States with a Nonvanishing
Probability Density at the Origin]

(proportional to the Dirac-δ function,
measures probability density of the
electronic wave function at the origin)

$$\Delta H_{\rm fs} = \sum_i \frac{2}{3} \langle r^2 \rangle \, \left[\pi Z \alpha \delta^3(\vec{r_i}) \right]$$

Very well, but how is $\langle r^2 \rangle$ defined?

Barker-Glover Term

$$E_{\mathrm{BG}} = \frac{(Z\alpha)^4 m_r^3\, c^2}{2n^3 m_N^2} \left(\frac{1}{j+1/2} - \frac{1}{\ell+1/2} \right) (1-\delta_{\ell 0}), \quad (5)$$

Darwin-Foldy Term [Just The One Proportional to the Dirac-δ]

$$\begin{aligned} E_{\mathrm{DF}} &= -\frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2} \left(\frac{1}{j+1/2} - \frac{1}{\ell+1/2} \right) \delta_{\ell 0} \\ &= \frac{(Z\alpha)^4 m_r^3 c^2}{2n^3 m_N^2}\, \delta_{\ell 0}, \end{aligned}$$

Can Write the Darwin-Foldy Term As Follows

$$E_{\mathrm{DF}} = \frac{2}{3} \left(\frac{m_r}{m_e} \right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3\, \lambda_C^2} \left\{ \frac{3\hbar^2}{4 m_N^2 c^2} \right\} \delta_{\ell 0}.$$

Nuclear Size Correction

$$E_{\rm NS} = \frac{2}{3}\left(\frac{m_r}{m_e}\right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3\,\lambda_C^2}\,\langle r^2\rangle_N\,\delta_{\ell 0},$$

Darwin-Foldy Term

$$E_{\rm DF} = \frac{2}{3}\left(\frac{m_r}{m_e}\right)^3 \frac{(Z\alpha)^4 m_e c^2}{n^3\,\lambda_C^2}\left\{\frac{3\hbar^2}{4m_N^2c^2}\right\}\delta_{\ell 0}.$$

Comparison: Darwin-Foldy Correction to Radius Square

$$\langle r^2\rangle_N^{\rm DF} = \frac{3\hbar^2}{4m_N^2c^2}.$$

Correspondence of Darwin-Foldy Correction to Energy and Radius Square

$$E_{\rm NS} \to E_{\rm NS} + E_{\rm DF} \;\Leftrightarrow\; \langle r^2\rangle_N^{\rm ATP} \to \langle r^2\rangle_N^{\rm ATP} + \langle r^2\rangle_N^{\rm DF},$$

ATP = "Atomic-Physics Conventions"

Definition of the Radius Square in Atomic-Physics Conventions (ATP)

$$\langle r^2 \rangle_p^{\mathrm{ATP}} = \langle r^2 \rangle_E^p = 6\hbar^2 \left. \frac{\partial G_E(q^2)}{\partial q^2} \right|_{q^2=0},$$

Decomposition of the Sachs Form Factor

$$G_E(q^2) = F_1(q^2) + \frac{q^2}{4(m_p c)^2} F_2(q^2),$$

$$G_M(q^2) = F_1(q^2) + F_2(q^2), \qquad F_2(0) = \varkappa_p,$$

Friar, Martorell and Sprung Use Different Conventions (FMS)
[Darwin-Foldy Term Sits Inside the Radius]

$$\widetilde{G}_E(q^2) = \frac{G_E(q^2)}{\sqrt{1 - q^2/4\,m_p^2}},$$

$$\langle r^2 \rangle_p^{\mathrm{FMS}} = 6\hbar^2 \left. \frac{\partial \widetilde{G}_E(q^2)}{\partial q^2} \right|_{q^2=0}.$$

$$\langle r^2 \rangle_p^{\mathrm{FMS}} = \langle r^2 \rangle_p^{\mathrm{ATP}} + \langle r^2 \rangle_p^{\mathrm{DF}}.$$

[J. L. Friar, J. Martorell and D. W. L. Sprung, Phys.Rev.A 56 (1997) 4579]

Is There a Problem with Different Conventions?

A closer inspection [for details see U.D.J., Eur. Phys. J. D 61 (2011) 7] reveals that all atomic-physics determinations (including the CODATA adjustments) as well as all relevant scattering experiment use ATP conventions.
The Darwin-Foldy term is subtracted before the radius square is determined.

But we are not done just yet.

There is a further subtlety connected with the internal-structure contributions to the form factors and the pure QED contributions to the proton form factors. QED contributions are ***excluded*** from the proton radius:

$$
\begin{aligned}
G_E(q^2) &= \overline{G}_E(q^2) + G_E^{\mathrm{QED}}(q^2),\\
F_1(q^2) &= \overline{F}_1(q^2) + F_1^{\mathrm{QED}}(q^2),\\
F_2(q^2) &= \overline{F}_2(q^2) + F_2^{\mathrm{QED}}(q^2),\\
\varkappa_p &= \overline{\varkappa}_p + \varkappa_p^{\mathrm{QED}}.
\end{aligned}
$$

$$
\begin{aligned}
\langle r^2 \rangle_p^{\mathrm{ATP}} &= 6\hbar^2 \left. \frac{\partial \overline{G}_E(q^2)}{\partial q^2} \right|_{q^2=0}\\
&= 6\hbar^2 \left. \frac{\partial \overline{F}_1(q^2)}{\partial q^2} \right|_{q^2=0} + \frac{3\hbar^2}{2\,(m_p c)^2}\,\overline{\varkappa}_p,
\end{aligned}
$$

=> We must make sure that we do not artificially add QED contributions to the
=> proton structure, e.g., by an improper definition of the nuclear self-energy.

A closer inspection reveals that the energy correction induced by the F_1 form factor slope of the proton and the anomalous magnetic moment correction from the proton line precisely add up to a nuclear-size correction determined by the slope of the Sachs form factor (internal-structure terms).

"We are doing everything right in leading order!"

In order not to mess things up in higher order, we should define the nuclear self-energy correction as follows,

$$E_{\rm NSE} = \frac{Z(Z\alpha)^5}{\pi n^3} \left(\frac{m_r}{m_p}\right)^2 m_r c^2 \times \left\{ \left[\frac{4}{3} \ln\left(\frac{m_p}{m_r (Z\alpha)^2}\right) + \frac{10}{9}\right] \delta_{\ell 0} - \frac{4}{3} \ln k_0(n,\ell) \right\}. \tag{28}$$

"No internal proton structure in the nuclear self-energy, please!"

⇒ There is not a problem with the conventions, at least not on the level
⇒ relevant to the comparison of theory and experiment.

[U.D.J., Eur. Phys. J. D 61 (2011) 7]

So...

Status Regarding 2S-2P Lamb Shift in mH

Muonic Hydrogen Discrepancy: 0.420 meV.

Largest Conceivable Uncertainty within Standard Model: +/- 0.010 meV.

All theorists agree.

The fact that no theoretical explanation for the observed discrepancy exists was pointed out in

[U.D.J., Ann.Phys.(N.Y) 326 (2011) 500]
[U.D.J., Ann.Phys.(N.Y) 326 (2011) 516]

and various other papers authored by the theoretical community since then.

Conundrum remains unsolved!

New fundamental forces!?!

Conclusions

Conclusions

Interesting muonic hydrogen discrepancies
[Very technical experiment]

Theorists have done a lot of work
[including recoil corrections to vacuum polarization
and radiative-recoil]
The effect of the large mass ratio seems to be
well under control.

Darwin-Foldy term is excluded from
the proton radius.
[...and we can define the nuclear self-energy so that
the internal proton structure and QED are well separated]

Thanks!