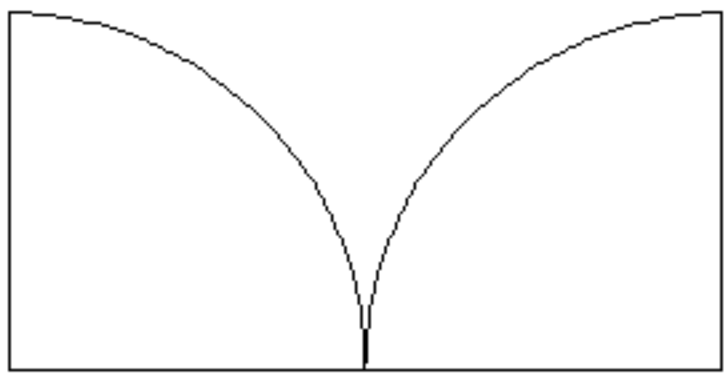
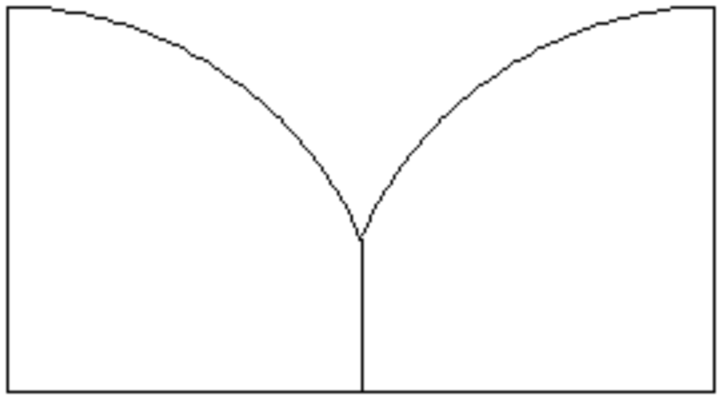
Project: OUTFIELD FENCES

DESCRIPTION: In this project you will work with the equations of projectile motion and use mathematical models to analyze a design problem.

Two softball fields in Rolla, Missouri are situated as shown with the right field foul line of one field becoming the left field foul line of the other. The fences are circular arcs 275 ft. from home plate, and meet on the common foul line.



The city wants to attract more softball tournaments to town, but the tournament organizers prefer fields with fences 300 ft. from home plate. If the fences on the existing fields are extended to 300 ft., parts of the two fields will overlap, clearly an unworkable situation. The Parks and Recreation Department has suggested, as a compromise, that the fences be extended to 300 ft. where possible, and that the straight fence common to both fields be made higher than the standard 6 ft. height to compensate for the shorter distance.



What should be the height of the common fence, and how should this height be determined? The Parks and Recreation Department has proposed that the common fence be 16 ft. high, prompting the sports editor of the *Rolla Daily News* , a softball player himself, to write an editorial about how there will be fewer home runs hit over the high fence. (By the way, in this project, the words "short" and "long" will always refer to horizontal distances and the words "low" and "high" will refer to the vertical heights of the fences being considered.)

BACKGROUND NEEDED: Vector functions ; projectile motion ; Pythagorean Theorem; parametric representation of curves. From a softball rule book: men's slow-pitch softballs "shall be between 11-7/8 inches and 12-1/8 inches in circumference and shall weigh between 6-1/4 ounces and 7 ounces".

OTHER NEEDS: A machine that will numerically solve a system of differential equations with initial conditions, and plot graphs of the results.

SUGGESTED TIME FRAME: 1 week.

PART A: Assume no air resistance

If you have had some experience hitting balls (baseballs, golfballs, tennis balls, softballs, etc.) when the wind was blowing, you will know that air resistance is not insignificant in this problem. Nevertheless, a model that neglects air resistance leads to simple equations for the path of the ball, and can still give good results. Look at the problem as a comparison question: Will a ball that is a home run over a 300 ft. fence also be a home run over a shorter but higher fence? Do you think that making the common fence 16 ft. high is a good solution? (Make a guess, at least.) Also think about what other assumptions you need to make in order to deal with this problem.

1. Start by finding where the two 300 ft. fences intersect, and then find the length of the common straight fence.
2. Now let u be the distance from the foul line along the common fence. The value of u will vary between zero and the length you found above. Find the distance s(u) from home plate to a point on the common fence in terms of u. What is the range of values for s(u)?
3. Turning to the flight of the ball, a "minimum effort" home run will have an initial angle of elevation of 45°, as discussed in § 8.?. Assume the ball is hit at a point 3 ft. above home plate. Find parametric equations for the path of the ball. Use x for the horizontal component and z for the vertical component. These equations will be in terms of time, t, and initial speed, v_0. Use 32 ft/sec^2 for g.
4. Find the initial speed required for a home run that just clears a 6 ft. high 300 ft. fence, and find the time when the ball goes over this

fence. You can now write parametric equations, with t as the only variable, for a minimum effort home run. Do it.

5. How high will the home run, whose flight path you found in 4., be when it passes over (or hits) the common straight fence? This height will depend on the variable u, so your answer will be a function of u; call it h(u).
6. Draw the graph of h(u). Draw appropriate conclusions. What would you tell the sports editor?
7. When the fences were redone, and the common fence was built, it was 16 ft. high except for the last 39 ft. 4 in., where the top of the common fence angled down in a straight line to meet the 6 ft. high fence. Modify your conclusions based on this fact.

PART B: Cheap home runs or expensive doubles?

In Part A we only considered one initial angle and one initial speed. Batters hit softballs at many different angles and speeds, and a perfect solution to the problem of the height of the common fence would guarantee that: (a) every home run over the 6 ft. high 300 ft. fence would also be a home run over the higher but shorter common fence, and (b) no ball that would not have gone over the 300 ft. fence goes over the common fence. Do you think a perfect solution is possible? Why or why not? For 1, 2, and 3 which follow, assume that the top of the common fence is given by h(u), found in Part A, and assume the same initial speed you used in Part A.

1. Will a ball hit at an initial angle of 50° go over the 300 ft. fence? Will this ball go over any part of the common fence? If so, which

part? Without doing any calculations, describe what happens for angles between 45° and 50°.

2. Now let the initial angle of elevation be α. Find a function, depending on both u and α, which gives the height of the ball when it reaches the common fence.
3. What is the smallest angle of elevation (greater than 45°) for which a home run is impossible? Will any ball hit at an initial angle less than 45° be a home run?
4. Still assume that the common fence is described by h(u), but now determine the initial speed required to hit a home run over the 6 ft. high 300 ft. fence if the initial angle of elevation is 35°.
5. Using the initial speed you found in 4, discuss the effect of the common fence on balls hit with different initial angles of elevation.
6. Do you have a better idea (better than h(u)) for the common fence? If so, explain why your idea is better, and if not, explain why you support h(u).

PART C: Effects of air resistance

Air resistance does play a significant role in the flight of actual softballs, one effect being that fly balls descend at a steeper angle than they ascend. This has relevance to the height of our fence. Air resistance is made up of two main parts: drag and lift. Drag is the retarding force due to the ball moving through the air, and is in the direction opposite the velocity vector. Lift is the force due to the spin of the ball (balls will be assumed to have backspin) and has direction perpendicular to both the velocity vector and the axis of spin. Experiments indicate that the magnitudes of both drag and lift are proportional to the square of the speed of the ball. Assume that the weight of a softball is 0.4 lb (6.4 oz.), its circumference is 12 in., and that the spin rate is 2000 rpm. If we denote the magnitude of the drag force by D and the magnitude of the lift force by L, Newton's second law leads to the following system of differential equations:

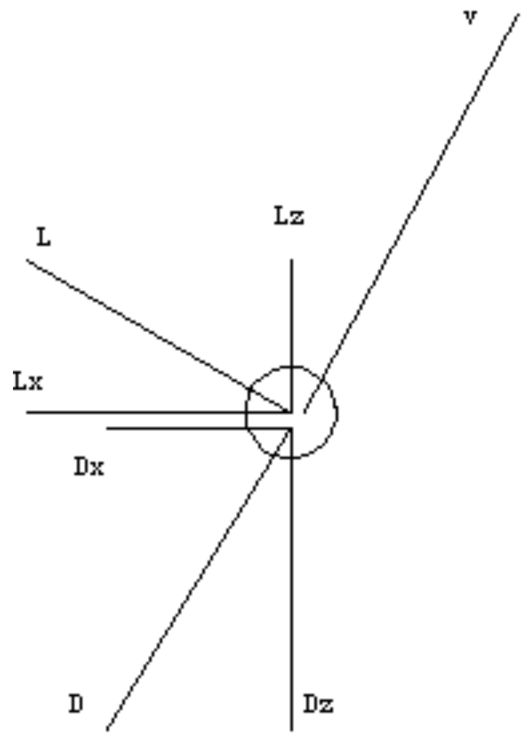
$$m x''(t) = -D_x - L_x$$
$$m z''(t) = -m g - D_z + L_z$$

where the subscripts x and z indicate the components of the forces in the horizontal and vertical directions. The quantities D and L can be calculated from

$$D = \tfrac{1}{2}\rho v^2 \pi R^2 C_D$$
$$L = \tfrac{1}{2}\rho v^2 \pi R^2 C_L$$

in which ρ is the mass density of the air, v is the speed of the ball, R is the radius of the ball, and C_D and C_L are the drag and lift coefficients.

1. Using $\rho = 0.0023$ and $C_D = 0.3$, calculate D in terms of v. A good approximation to C_L is $C_L \approx \frac{R\omega}{v}$, where ω is the spin rate in radians per second. Use this to calculate L in terms of v.
2. The forces on a softball in flight are shown in the diagram.



Note that **L** is perpendicular to **D** (and to **v**). Show that the magnitudes of the components of the forces are

$$D_x=\tfrac{1}{2}\rho\pi R^2 C_D\sqrt{x'(t)^2+z'(t)^2}\,x'(t)$$

$$D_z=\tfrac{1}{2}\rho\pi R^2 C_D\sqrt{x'(t)^2+z'(t)^2}\,z'(t)$$

$$L_x=\tfrac{1}{2}\rho\pi R^3\omega^* z'(t)$$

$$L_z=\tfrac{1}{2}\rho\pi R^3\omega^* x'(t)$$

3. Show that the differential equations for the motion of the ball, using the given values for all the constants, but not (yet) substituting for ω^*, the spin rate in rpm, are

$$x''(t)=-2.2E-3\sqrt{x'(t)^2+z'(t)^2}\,x'(t)-1.22E-4\,\omega^* z'(t)$$

$$z''(t)=1.22E-4\,\omega^* x'(t)-2.2E-3\sqrt{x'(t)^2+z'(t)^2}\,z'(t)-32$$

What initial conditions correspond to the ball being hit 3 ft. above home plate at an angle of elevation α and initial speed v_0?

4. The system of differential equations in 3. is both coupled (both x and z appear in both equations) and nonlinear, and cannot be solved in closed form. With appropriate initial values, the system can be solved numerically, however. Do numerical experiments with initial speeds between 100 ft/sec and 120 ft/sec, spin rates between 1000 rpm and 2000 rpm, and angles of elevation between 30° and 40° to determine conditions that will produce a home run that just clears the 6 ft. high 300 ft. fence.
5. How high will the home run, whose flight path you found in 4., be when it passes over (or hits) the common straight fence? This height will again depend on u, but now must be computed numerically. Plot a graph of these heights using values of u at 8 ft intervals. Compare this determination of the common fence with the one in Part A. Draw more conclusions. Will the air resistance results make the sports editor feel better or worse?
6. Do some more numerical experiments to find the optimal angle of elevation (to the nearest degree) for this model. If the spin rate changes, does the optimal angle change too?

More details about the flight of baseballs can be found in the book *Keep Your Eye on the Ball: The Science and Folklore of Baseball*, by Robert G. Watts and A. Terry Bahill, W. H. Freeman and Co., New York,1990.