Special Functions

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Chapter 3. Elliptic Integrals and Elliptic Functions

3.1. Motivational Examples

Example 3.1.1 (The Pendulum). *A simple, undamped pendulum of length L has motion governed by the differential equation*

$$u'' + \frac{g}{L}\sin u = 0, \tag{3.1.1}$$

where u is the angle between the pendulum and a vertical line, g is the gravitational constant, and $'$ is differentiation with respect to time. Consider the "energy" function (some of you may recognize this as a Lyapunov function):

$$E(u, u') = \frac{(u')^2}{2} + \int_0^u \frac{g}{L}\sin z\,dz = \frac{(u')^2}{2} + \frac{g}{L}(1 - \cos u).$$

For $u(0)$ and $u'(0)$ sufficiently small (3.1.1) has a periodic solution. Suppose $u(0) = A$ and $u'(0) = 0$. At this point the energy is $\frac{g}{L}(1 - \cos A)$, and by conservation of energy, we have

$$\frac{(u')^2}{2} + \frac{g}{L}(1 - \cos u) = \frac{g}{L}(1 - \cos A).$$

Simplifying, and noting that at first u is decreasing, we get

$$\frac{du}{dt} = -\sqrt{\frac{2g}{L}}\sqrt{\cos u - \cos A}.$$

This DE is separable, and integrating from $u = 0$ to $u = A$ will give one-fourth of the period. Denoting the period by T, and realizing that the period depends on A, we get

$$T(A) = 2\sqrt{\frac{2L}{g}}\int_0^A \frac{du}{\sqrt{\cos u - \cos A}}. \tag{3.1.2}$$

If $A = 0$ or $A = \pi$ there is no motion (do you see why this is so physically?), so assume $0 < A < \pi$. Let $k = \sin\frac{A}{2}$, making $\cos A = 1 - 2k^2$, and also let $\sin\frac{u}{2} = k\sin\theta$. Substituting into (3.1.2) gives

$$T(A) = 4\sqrt{\frac{L}{g}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}. \tag{3.1.3}$$

Since $0 < A < \pi$, we have $0 < k < 1$, and the integral in (3.1.3) cannot be evaluated in terms of elementary functions. This integral is the complete elliptic integral of the first kind *and is denoted by K, $K(k)$, or $K(m)$ (where $m = k^2$).*

$$K = K(k) = K(m) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}. \tag{3.1.4}$$

A slight generalization, replacing the upper limit by a variable ϕ, with $0 \le \phi \le \pi/2$, yields the incomplete elliptic integral of the first kind*, denoted $K(\phi \,|\, k)$ or $K(\phi \,|\, m)$.*

$$K(\phi \,|\, k) = K(\phi \,|\, m) = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}. \tag{3.1.5}$$

Exercise 3.1.1. *Fill in all the details in Example 3.1.1.*

Example 3.1.2 (Circumference of an Ellipse). *Let an ellipse be given by* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$*, where we assume that* $a < b$*. In parametric form, the ellipse is* $x = a\,\cos\theta$*,* $y = b\,\sin\theta$*, and the circumference* L *is given by*

$$\begin{aligned} L &= 4\int_0^{\frac{\pi}{2}} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta \\ &= 4b\int_0^{\frac{\pi}{2}} \sqrt{1-(1-\frac{a^2}{b^2})\sin^2\theta}\,d\theta \\ &= 4b\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta, \end{aligned}$$

where $k^2 = 1 - \frac{a^2}{b^2}$*. Again, the integral cannot be evaluated in terms of elementary functions except in degenerate cases (*$k = 0$ *or* $k = 1$*). Integrals of the form*

$$E = E(k) = E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta \tag{3.1.6}$$

are called complete elliptic integrals of the second kind (as before, $m = k^2$*), and integrals of the form*

$$E(\phi\,|\,k) = E(\phi\,|\,m) = \int_0^{\phi} \sqrt{1-m\sin^2\theta}\,d\theta \tag{3.1.7}$$

are called incomplete elliptic integrals of the second kind.

Exercise 3.1.2. *Show that, in the setting of Example 3.1.2,* k *is the eccentricity of the ellipse.*

There are elliptic integrals of the third kind, denoted by Π. As before, if the upper limit in the integral is $\pi/2$, the integral is called complete.

$$\Pi(\phi\,|\,k, N) = \int_0^{\phi} \frac{d\theta}{(1+N\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}. \tag{3.1.8}$$

Unfortunately, I don't know any nice motivating examples for this case. The following is lifted verbatim from Whittaker and Watson, page 523, and and assumes knowledge of things we have not covered. Understanding of it is something to which you are encouraged to aspire.

Example 3.1.3 (Rigid Body Motion). *It is evident from the expression of* $\Pi(u, a)$ *in terms of Theta-functions that if* u*,* a*,* k *are real, the average rate of increase of* $\Pi(u, a)$ *as* u *increases is* $Z(a)$*, since* $\Theta(u \pm a)$ *is periodic with respect to the real period* $2K$*. This result determines the mean precession about the invariable line in the motion of a rigid body relative to its centre [Whittaker and Watson were British] of gravity under forces whose resultant passes through its centre of gravity. It is evident that, for purposes of computation, a result of this nature is preferable to the corresponding result in terms of Sigma-functions and Weierstrassian Zeta-Functions, for the reasons that the Theta-functions have a specially simple behaviour with respect to their real period - the period which is of importance in Applied Mathematics - and that the q-series are much better adapted for computation than the product by which the Sigma-function is most simply defined.*

Before we consider elliptic integrals in general, look back at Example 3.1.1. By either the Binomial Theorem or Taylor's Theorem,

$$\left(1-k^2\sin^2\theta\right)^{-1/2} = 1 + \frac{1}{2}\,k^2\sin^2\theta + \frac{1}{2}\cdot\frac{3}{4}\,k^4\sin^4\theta + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\,k^6\sin^6\theta + \cdots.$$

and so

$$K = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \sum_{n=0}^{\infty} \int_0^{\frac{\pi}{2}} \frac{(2n)!}{2^{2n}(n!)^2}\, k^{2n} \sin^{2n}\theta\, d\theta.$$

By Wallis' formula, we get

$$K = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 k^{2n} = \frac{\pi}{2}\left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 k^6 + \cdots\right].$$

This series can be used to approximate the value of K. Note that $|k| < 1$ is necessary for convergence.

Exercise 3.1.3. *Express E as a power series in powers of k.*

The functions K, E, and Π are tabulated in A&S and are part of Mathematica and Maple.

3.2. General Definition of Elliptic Integrals

If $R(x, y)$ is a rational algebraic function of x and y, the integral $\int R(x, y)\, dx$ can be evaluated in terms of elementary functions if $y = \sqrt{ax+b}$ or $y = \sqrt{ax^2+bx+c}$. Things are not so nice if y^2 is a cubic or quartic, however.

Exercise 3.2.1. *Evaluate $\int xy\, dx$ and $\int \frac{1}{y}\, dx$ when $y = \sqrt{ax+b}$ and $y = \sqrt{ax^2+bx+c}$.*

Definition 3.2.1. *If $R(x, y)$ is a rational function of x and y and y^2 is a cubic or quartic polynomial in x with no repeated factors, then the integral $\int R(x, y)\, dx$ is an elliptic integral.*

So, the trigonometry in the above examples notwithstanding, elliptic integrals are concerned with integrating algebraic functions that you couldn't handle in second-semester calculus. Given an elliptic integral, the problem is to reduce it to a recognizable form.

Example 3.2.1. *Evaluate $I = \int_1^{\infty} \frac{dx}{\sqrt{x^4-1}}$. Here, $y^2 = x^4 - 1$ and $R(x, y) = \frac{1}{y}$. A sequence of substitutions will convert the integral to a form we have seen. First, let $x = \frac{1}{t}$ to get $I = \int_0^1 \frac{dt}{\sqrt{1-t^4}}$. Next, let $t = \sin\phi$, giving $I = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{2-\cos^2\phi}}$. Finally, let $\phi = \frac{\pi}{2} - \theta$ to get*

$$I = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2-\sin^2\theta}} = \frac{1}{\sqrt{2}} \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{1}{\sqrt{2}} K\left(\tfrac{1}{2}\right).$$

Exercise 3.2.2. *Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}}$ and $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$ in terms of complete elliptic integrals, and use the tables in A&S to get numerical values. [Hint: Although not the only way, the substitution $\cos(\cdot) = \cos^2 u$ can be used at some stage in both problems.] Express these integrals in terms of the gamma function using Theorems 2.2.2 and 2.2.4. Also try integrating these directly using Mathematica or Maple.*

3.3. Evaluation of Elliptic Integrals

A systematic way to evaluate elliptic integrals is desirable. Since $R(x,y)$ is a rational function of x and y, we can write $R(x,y) = \dfrac{y\,P(x,y)\,Q(x,-y)}{y\,Q(x,y)\,Q(x,-y)}$, where P and Q are polynomials. Now $Q(x,y)Q(x,-y)$ is an even function of y, making it a polynomial in x and y^2, and thus a polynomial in x. When $y\,P(x,y)\,Q(x,-y)$ is expanded, even powers of y can be written as polynomials in x and odd powers of y can be written as (polynomials in x) times y, so the numerator of R is linear in y. We then have

$$R(x,y) = \frac{R_1(x) + y\,R_2(x)}{y} = R_2(x) + \frac{R_1(x)}{y}$$

where R_1 and R_2 are rational functions of x. The integration problem has been reduced to

$$\int R(x,y)\,dx = \int R_2(x)\,dx + \int \frac{R_1(x)}{y}\,dx.$$

The first integral can be done by second-semester calculus methods, and the second one will be studied further. Recall that y^2 is a cubic or quartic in x. Think of a cubic as a quartic with the coefficient of x^4 equal to 0. Then the following factorization is useful.

Theorem 3.3.1. *Any quartic in x with no repeated factors can be written in the form*

$$\left[A_1(x-\alpha)^2 + B_1(x-\beta)^2\right]\left[A_2(x-\alpha)^2 + B_2(x-\beta)^2\right]$$

where, if the coefficients in the quartic are real, then the constants $A_1, B_1, A_2, B_2, \alpha$, and β are real.

Proof. Any quartic $Q(x)$ with real coefficients can be expressed as $Q(x) = S_1(x)\,S_2(x)$ where S_1 and S_2 are quadratics. The complex roots (if any) of the quartic occur in conjugate pairs, so there are three cases.

> **Case 1. Four real roots.** Call the roots $\{r_i\}$, and assume $r_1 < r_2 < r_3 < r_4$. Let $S_1(x) = (x-r_1)(x-r_2)$ and $S_2(x) = (x-r_3)(x-r_4)$, with an appropriate constant multiplier in case the coefficient of x^4 is not 1. Note that the roots of S_1 and S_2 do not interlace.
>
> **Case 2. Two real roots and two complex roots.** Denote the real roots by r_1 and r_2, and the complex roots by $\rho_1 \pm \rho_2 i$. Let $S_1(x) = (x-r_1)(x-r_2)$ and $S_2(x) = x^2 - 2\rho_1\,x + (\rho_1^2+\rho_2^2)$.
>
> **Case 3. Four complex roots.** Call the roots $\rho_1 \pm \rho_2 i$ and $\rho_3 \pm \rho_4 i$. Let $S_1(x) = x^2 - 2\rho_1\,x + (\rho_1^2+\rho_2^2)$ and $S_2(x) = x^2 - 2\rho_3\,x + (\rho_3^2+\rho_4^2)$.

In case y^2 is a cubic, we simply eliminate one real factor in Case 1 or Case 2. Case 3 will not apply if y^2 is a cubic. So, in general, we have

$$S_1(x) = a_1x^2 + 2b_1x + c_1 \text{ and } S_2(x) = a_2x^2 + 2b_2x + c_2.$$

Now we look for constants λ such that $S_1(x) - \lambda S_2(x)$ is a perfect square. Since $S_1(x) - \lambda S_2(x)$ is simply a quadratic in x, it is a perfect square if and only if the discriminant is zero, or $(a_1-\lambda a_2)(c_1-\lambda c_2)-(b_1-\lambda b_2)^2 = 0$. This discriminant is a quadratic in λ, and has two roots, λ_1 and λ_2. We get

$$S_1(x) - \lambda_1 S_2(x) = (a_1-\lambda_1 a_2)\left[x + \frac{b_1-\lambda_1 b_2}{a_1-\lambda_1 b_2}\right]^2 = (a_1-\lambda_1 a_2)(x-\alpha)^2, \tag{3.3.1}$$

$$S_1(x) - \lambda_2 S_2(x) = (a_1-\lambda_2 a_2)\left[x + \frac{b_1-\lambda_2 b_2}{a_1-\lambda_2 b_2}\right]^2 = (a_1-\lambda_2 a_2)(x-\beta)^2, \tag{3.3.2}$$

Now solve equations (3.3.1) and (3.3.2) for $S_1(x)$ and $S_2(x)$ to get the required forms. ♠

Example 3.3.1. *Consider* $Q(x) = 3x^4 - 16x^3 + 24x^2 - 16x + 4 = (3x^2 - 4x + 2)(x^2 - 4x + 2)$*. Mathematica or Maple is handy for the factorization. For* $S_1(x) - \lambda S_2(x)$ *to be a perfect square, we need* $(3 - \lambda)(2 - 2\lambda) - (-2 + 2\lambda)^2 = 0$*, which has solutions* $\lambda_1 = -1$ *and* $\lambda_2 = 1$*. So we get* $S_1(x) = 2(x-1)^2 + (x-0)^2$ *and* $S_2(x) = 2(x-1)^2 - (x-0)^2$*.*

Exercise 3.3.1. *Fill in all the details in the proof of Theorem 3.3.1 and in Example 3.3.1.*

If $b_1 = b_2 = 0$, there appears to be a breakdown in the form specified in Theorem 3.3.1. In this case you can set up S_1 and S_2 with complex coefficients to get the form in the theorem. In practice, this will not be necessary, however, because the form of the integral will already be one toward which you are working.

In the integral $\int \frac{R_1(x)}{y}\, dx$, with the denominator written as in Theorem 3.3.1, make the substitution

$$t = \frac{x - \alpha}{x - \beta}\ ,\ dx = (x - \beta)^2 (\alpha - \beta)^{-1}\, dt.$$

This gives

$$y^2 = \left[A_1(x-\alpha)^2 + B_1(x-\beta)^2\right]\left[A_2(x-\alpha)^2 + B_2(x-\beta)^2\right] = (x-\beta)^4\left(A_1t^2 + B_1\right)\left(A_2t^2 + B_2\right)$$

and the integrand becomes

$$R_1(x)\left[\frac{(\alpha-\beta)^{-1}dt}{\sqrt{(A_1t^2+B_1)\,(A_2t^2+B_2)}}\right].$$

Finally, $R_1(x)$ can be written as $\pm(\alpha - \beta)R_3(t)$ where R_3 is a rational function of t.

Lemma 3.3.1. *There exist rational functions* R_4 *and* R_5 *such that* $R_3(t) + R_3(-t) = 2\,R_4(t^2)$ *and* $R_3(t) - R_3(-t) = 2t\,R_5(t^2)$*. Therefore,* $R_3(t) = R_4(t^2) + t\,R_5(t^2)$*.*

Exercise 3.3.2. *Use Mathematica or Maple to verify Lemma 3.3.1 for several rational functions of your choice, including at least one with arbitrary coefficients. Mathematica commands which might be useful are Denominator, Expand, Numerator, and Together. Then prove the lemma. [Hint: Check for even and odd functions.]*

The integral $\int \frac{R_1(x)}{y}\, dx$ is now reduced to

$$\int \frac{R_4(t^2)\,dt}{\sqrt{(A_1t^2+B_1)\,(A_2t^2+B_2)}} + \int \frac{t\,R_5(t^2)\,dt}{\sqrt{(A_1t^2+B_1)\,(A_2t^2+B_2)}}.$$

The substitution $u = t^2$ allows the second integral to be evaluated in terms of elementary functions. If $R_4(t^2)$ is expanded in partial fractions[1] the first integral is reduced to sums of integrals of the following type:

$$\int t^{2m}\left[\left(A_1t^2+B_1\right)\left(A_2t^2+B_2\right)\right]^{-1/2} dt \qquad (3.3.3)$$

$$\int (1+Nt^2)^{-m}\left[\left(A_1t^2+B_1\right)\left(A_2t^2+B_2\right)\right]^{-1/2} dt \qquad (3.3.4)$$

[1] The Apart command in Mathematica does this.

where in (3.3.3) m is an integer, and in (3.3.4) m is a positive integer and $N \neq 0$. Reduction formulas can be derived to reduce (3.3.3) or (3.3.4) to a combination of known functions and integrals in the following canonical forms:

$$\int \left[\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)\right]^{-1/2} dt, \tag{3.3.5}$$

$$\int t^2 \left[\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)\right]^{-1/2} dt, \tag{3.3.6}$$

$$\int (1 + Nt^2)^{-1} \left[\left(A_1 t^2 + B_1\right)\left(A_2 t^2 + B_2\right)\right]^{-1/2} dt. \tag{3.3.7}$$

Equations (3.3.5), (3.3.6), and (3.3.7) are the *elliptic integrals of the first, second, and third kinds*, so named by Legendre.

Exercise 3.3.3. *By differentiating* $t\sqrt{(A_1t^2 + B_1)(A_2t^2 + B_2)}$, *obtain a reduction formula for (3.3.3) when* $m = 2$. *Do you see how this process can be extended to all positive* m*?*

Exercise 3.3.4. *Find a reduction formula for (3.3.3) when* $m = -1$.

Exercise 3.3.5. *Show that the transformation* $t = \sin\theta$ *applied to (3.1.4), (3.1.7), or (3.1.8) yields integrals of the form (3.3.5), (3.3.6), and (3.3.7).*

For real elliptic integrals, all the essentially different combinations of signs in the radical are given in the following table.

A_1	+	+	−	+	+	−
B_1	+	−	+	−	−	+
A_2	+	+	+	+	−	−
B_2	+	+	+	−	+	+

Table 3.3.1.

Example 3.3.2. *Find the appropriate substitution for the second column of Table 3.3.1. Assume that* A_1, B_1, A_2, *and* B_2 *are all positive, so the radical is* $\sqrt{(A_1t^2 - B_1)(A_2t^2 + B_2)}$. *Substitute* $t = \sqrt{\frac{B_1}{A_1}}\sec\theta$, $dt = \sqrt{\frac{B_1}{A_1}}\sec\theta\tan\theta\, d\theta$ *so that*

$$\frac{dt}{\sqrt{(A_1t^2 - B_1)(A_2t^2 + B_2)}} = \frac{d\theta}{\sqrt{A_2B_1 + A_1B_2\cos^2\theta}} = \frac{1}{\sqrt{A_2B_1 + A_1B_2}} \frac{d\theta}{\sqrt{1 - \frac{B_2A_1}{B_2A_1 + B_1A_2}\sin^2\theta}}.$$

Here, $m = k^2 = \frac{B_2A_1}{B_2A_1 + B_1A_2} < 1$, *and we have the form of (3.1.4) or (3.1.5).*

Exercise 3.3.6. *Fill in the details of Example 3.3.2, and find the value of* $\displaystyle\int_{\frac{1}{\sqrt{2}}}^{\infty} \frac{dt}{\sqrt{(2t^2 - 1)(t^2 + 2)}}$.

Exercise 3.3.7. *Pick another column of Table 3.3.1 and find the appropriate substitution. Consult with your classmates so that every column is considered by someone.*

3.4. The Jacobian Elliptic Functions

Trigonometric functions, even though they can do the job, are not the best things to use when reducing elliptic integrals. The ideal substitution in, say, (3.3.5), would be one where we have functions f, g, and h such that

$$t = f(v), \quad dt = g(v)\,h(v)\,dv, \quad A_1t^2 + B_1 = g^2(v), \quad \text{and} \quad A_2t^2 + B_2 = h^2(v).$$

Adjustments for the constants would have to be made, of course, but such a substitution would reduce (3.3.5) to $\int dv$. Fortunately, such functions exist! They are called the *Jacobian elliptic functions* and can be thought of as extensions of the trigonometric functions.

Consider the function defined by an integral as follows:

$$u = g(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt = \int_0^\phi d\theta = \phi = \sin^{-1} x.$$

(Here we used the substitution $t = \sin\theta,\ \ x = \sin\phi$, but we really don't want to focus on this part.) Assume that "sin" and "$\sin^{-1}$" are simply names we came up with for the functions involved here, making "$\sin^{-1}$" nothing more than another name for "g", which is *defined by the integral*. Thinking similarly, we can say that $g^{-1}(u) = \sin u$. Thus, we have *defined* the function "sin" as the inverse of the function g which is given in terms of the integral. A table of values for g can be calculated by numerical integration and this table with the entries reversed produces a table for g^{-1}. Now a new function can be defined by $h^{-1}(u) = \sqrt{1-[g^{-1}(u)]^2}$. Clearly, $\left[g^{-1}(u)\right]^2 + \left[h^{-1}(u)\right]^2 = 1$, and we might want to give "h^{-1}" a new name, such as "cos", for instance. All of trigonometry can be developed in this way, without reference to angles, circles, or any of the usual stuff associated with trigonometry. We won't do trigonometry this way, because you already know that subject, but we will use this method to study the Jacobian elliptic functions.

In the spirit of the last paragraph, but assuming you know all about trigonometry, consider, for $0 \le m \le 1$,

$$u = \int_0^x \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\,dt = \int_0^\phi \frac{d\theta}{\sqrt{1-m\,\sin^2\theta}} \tag{3.4.1}$$

where the integrals are related by the substitution $t = \sin\theta,\ \ x = \sin\phi$. Note that, for fixed m, the integrals define u as a function of x or ϕ, and so a table of values for this function and its inverse can be constructed by numerical integration as discussed above.

Definition 3.4.1. *The Jacobian elliptic functions sn, cn, and dn are*

$$sn\,u = \sin\phi = x, \quad cn\,u = \cos\phi = \sqrt{1-x^2}, \quad dn\,u = \sqrt{1-m\,\sin^2\phi} = \sqrt{1-m\,x^2}$$

where u and ϕ or x are related by (3.4.1). In particular, sn is the inverse of the function of x defined by the first integral in (3.4.1). Sometimes specific dependence on the parameter m is denoted by $sn(u\,|\,m)$, $cn(u\,|\,m)$, and $dn(u\,|\,m)$.

Some immediate consequences of Definition 3.4.1 are, for any m,

$$sn^2u + cn^2u = 1, \quad m\,sn^2u + dn^2u = 1, \quad sn(0) = 0, \quad cn(0) = dn(0) = 1$$

Exercise 3.4.1. *Show that sn is an odd function and cn is an even function.*

The following notation and terminology applies to all Jacobian elliptic functions:

$$\begin{aligned} u &: \textit{argument} \\ m &: \textit{parameter} \\ m_1 = 1 - m &: \textit{complementary parameter} \\ \phi &: \textit{amplitude}, \quad \textit{denoted} \quad \phi = am\, u \\ k = \sqrt{m} &: \textit{modulus} \\ k' = \sqrt{1 - k^2} &: \textit{complementary modulus} \end{aligned}$$

Exercise 3.4.2. *Show that* $\dfrac{d\phi}{du} = dn\, u$, *and that* $\dfrac{d}{du} sn\, u = cn\, u\, dn\, u$. *Also find* $\dfrac{d}{du} cn\, u$ *and* $\dfrac{d}{du} dn\, u$.

Exercise 3.4.3. *Derive Formulas* **17.4.44**, **17.4.45**, *and* **17.4.52** *on page 596 of A&S. This verifies that sn, cn, and dn are among the kinds of functions we wanted at the beginning of this section.*

Exercise 3.4.4. *Show that* $sn(u\,|\,0) = \sin u$, $cn(u\,|\,0) = \cos u$, *and* $dn(u\,|\,0) = 1$. *Also show that* $sn(u\,|\,1) = \tanh u$ *and* $cn(u\,|\,1) = dn(u\,|\,1) = sech\, u$.

There are nine other Jacobian elliptic functions, all of which can be expressed in terms of sn, cn, and dn in much the same way all other trig functions can be expressed in terms of sin and cos. The notation uses only the letters s, c, d, and n according to the following rules. First, quotients of two of sn, cn, and dn are denoted by the first letter of the numerator function followed by the first letter of the denominator function. Second, reciprocals are denoted by writing the letters of the function whose reciprocal is taken in reverse order. Thus

$$\begin{aligned} ns(u) &= \frac{1}{sn(u)}, & nc(u) &= \frac{1}{cn(u)}, & nd(u) &= \frac{1}{dn(u)} \\ sc(u) &= \frac{sn(u)}{cn(u)}, & sd(u) &= \frac{sn(u)}{dn(u)}, & cd(u) &= \frac{cn(u)}{dn(u)} \\ cs(u) &= \frac{cn(u)}{sn(u)}, & ds(u) &= \frac{dn(u)}{sn(u)}, & dc(u) &= \frac{dn(u)}{cn(u)} \end{aligned} \tag{3.4.2}$$

Quotients of any two Jacobian elliptic functions can be reduced to a quotient involving sn, cn, and/or dn. For example

$$\frac{sc(u)}{sd(u)} = sc(u) \cdot ds(u) = \frac{sn(u)}{cn(u)} \cdot \frac{dn(u)}{sn(u)} = \frac{dn(u)}{cn(u)} = dc(u).$$

3.5. Addition Theorems

The Jacobian elliptic functions turn out to be not only periodic, but doubly periodic. Viewed as functions of a complex variable, they exhibit periodicity in both the real and imaginary directions. The following theorem will be quite useful in the next section for studying periodicity. Think about this theorem in the light of Exercise 3.4.4.

Theorem 3.5.1 (Addition Theorem). *For a fixed m,*

$$sn(u+v) = \frac{sn(u)\, cn(v)\, dn(v) + sn(v)\, cn(u)\, dn(u)}{1 - m\, sn^2(u)\, sn^2(v)}. \tag{3.5.1}$$

Proof. Let α be a constant, and suppose u and v are related by $u + v = \alpha$, so that $\dfrac{dv}{du} = -1$. Denote differentiation with respect to u by $\dot{}$, and let $s_1 = sn(u)$, $s_2 = sn(v)$. Then, keeping in mind that derivatives of s_2 require the chain rule,

$$\dot{s}_1^2 = (1 - s_1^2)(1 - m\, s_1^2) \quad \text{and} \quad \dot{s}_2^2 = (1 - s_2^2)(1 - m\, s_2^2).$$

If we differentiate again and divide by $2\dot{s}_1$ and $2\dot{s}_2$, we get

$$\ddot{s}_1 = -(1+m)s_1 + 2m\, s_1^3 \quad \text{and} \quad \ddot{s}_2 = -(1+m)s_2 + 2m\, s_2^3.$$

Thus

$$\frac{\ddot{s}_1 s_2 - \ddot{s}_2 s_1}{\dot{s}_1^2 s_2^2 - \dot{s}_2^2 s_1^2} = \frac{-2m\, s_1 s_2}{1 - m\, s_1^2 s_2^2}$$

and multiplying both sides by $\dot{s}_1 s_2 + \dot{s}_2 s_1$ gives

$$\frac{\frac{d}{du}(\dot{s}_1 s_2 - \dot{s}_2 s_1)}{\dot{s}_1 s_2 - \dot{s}_2 s_1} = \frac{\frac{d}{du}(1 - m\, s_1^2 s_2^2)}{1 - m\, s_1^2 s_2^2}.$$

Integration and clearing of logarithms gives

$$\frac{\dot{s}_1 s_2 - \dot{s}_2 s_1}{1 - m\, s_1^2 s_2^2} = \frac{sn(u)\, cn(v)\, dn(v) + sn(v)\, cn(u)\, dn(u)}{1 - m\, sn^2(u)\, sn^2(v)} = C. \tag{3.5.2}$$

Equation (3.5.2) may be thought of as a solution of the differential equation $du + dv = 0$. But $u + v = \alpha$ is also a solution of this differential equation, so the two solutions must be dependent, i.e., there exists a function f such that

$$f(u+v) = \frac{sn(u)\, cn(v)\, dn(v) + sn(v)\, cn(u)\, dn(u)}{1 - m\, sn^2(u)\, sn^2(v)}.$$

If $v = 0$ we see from Exercise 3.4.4 that $f(u) = sn(u)$, so $f = sn$ and the theorem is proved. ♠

Exercise 3.5.1. *Derive the following addition theorems for cn and dn.*

$$cn(u+v) = \frac{cn(u)\, cn(v) - sn(u)\, sn(v)\, dn(u)\, dn(v)}{1 - m\, sn^2(u)\, sn^2(v)} \tag{3.5.3}$$

$$dn(u+v) = \frac{dn(u)\, dn(v) - m\, sn(u)\, sn(v)\, cn(u)\, cn(v)}{1 - m\, sn^2(u)\, sn^2(v)}. \tag{3.5.4}$$

3.6. Periodicity

Since $u = \displaystyle\int_0^x \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\, dt$, means that $sn(u\,|\,m) = x$, if we let $K = \displaystyle\int_0^1 \frac{1}{\sqrt{(1-t^2)\,(1-mt^2)}}\, dt$, then $sn(K\,|\,m) = 1$, $cn(K\,|\,m) = 0$, and $dn(K\,|\,m) = \sqrt{1-m} = k'$. By the addition theorem for sn, we get

$$sn(u+K) = \frac{sn(u)\, cn(K)\, dn(K) + sn(K)\, cn(u)\, dn(u)}{1 - m\, sn^2(u)\, sn^2(K)} = \frac{cn(u)}{dn(u)} = cd(u).$$

Similarly, the addition theorems for cn and dn give

$$cn(u+K) = \frac{cn(u)\, cn(K) - sn(u)\, sn(K)\, dn(u)\, dn(K)}{1 - m\, sn^2(u)\, sn^2(K)} = -\sqrt{1-m}\, sd(u)$$

$$dn(u+K) = \frac{dn(u)\, dn(K) - m\, sn(u)\, sn(K)\, cn(u)\, cn(K)}{1 - m\, sn^2(u)\, sn^2(K)} = \sqrt{1-m}\, nd(u).$$

These results can be used to get

$$sn(u+2K) = -sn(u) \qquad cn(u+2K) = -cn(u) \qquad dn(u+2K) = dn(u)$$

and

$$sn(u+4K) = sn(u) \qquad cn(u+4K) = cn(u).$$

Thus, sn and cn have period $4K$, and dn has period $2K$, where $K = K(m)$ is the elliptic integral of the first kind. Note that we say "a period" instead of "the period" in the following theorem.

Theorem 3.6.1 (First Period Theorem). *If* $K = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-mt^2)}}\,dt = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-m\sin^2\theta}}$, *then* $4K$ *is a period of* sn *and* cn, *and* $2K$ *is a period of* dn.

Exercise 3.6.1. *Fill in the details of the proof of Theorem 3.6.1.*

To study other possible periods, let $K' = \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-m_1t^2)}}\,dt$. K' is the same function of m_1 as K is of m. Suppose $0 < m < 1$ so that both k and k' are also strictly between 0 and 1.

Exercise 3.6.2. *In the integral for* K', *substitute* $s^2 = \frac{1}{1-m_1t^2}$ *to get* $K' = \int_1^{\frac{1}{k}} \frac{1}{\sqrt{(s^2-1)(1-k^2t^2)}}\,ds$.

Now consider the integral $\int_0^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt$. Since $\frac{1}{k} > 1$, the integrand has a singularity at $t = 1$, and the integral has complex values for $1 < t < \frac{1}{k}$. To deal with these problems, we consider t a complex variable and look at the following path from the origin to $(\frac{1}{k}, 0)$ in the complex plane:

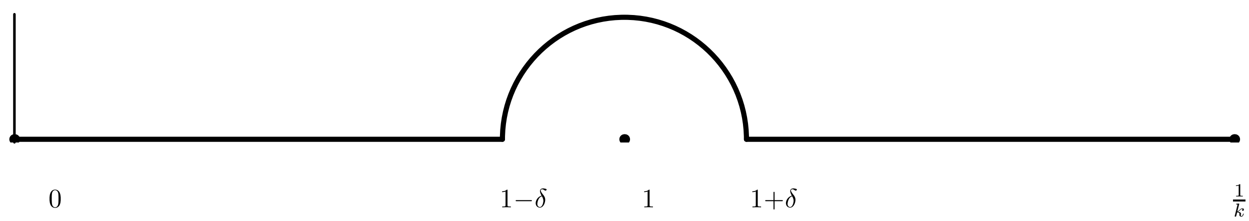

Figure 3.6.1

When t is on the semicircle near $1+\delta$, $t = 1+\delta e^{i\epsilon}$ for some $\epsilon > 0$. This makes $1-t^2 = \delta(2+\delta e^{i\epsilon})e^{i(\pi+\epsilon)}$. In order to get the principal value of the square root, we must replace $e^{i(\pi+\epsilon)}$ by $e^{-i(\pi-\epsilon)}$. Then, as $\epsilon \to 0+$, so that $t \to 1+\delta$ clockwise on the semicircle, we get

$$\begin{aligned}\sqrt{1-t^2} &= \lim_{\epsilon\to 0+} \sqrt{\delta(2+\delta e^{i\epsilon})}e^{i(\pi+\epsilon)/2} \\ &= \sqrt{\delta(2+\delta)}e^{-i\pi/2} \\ &= -i\sqrt{t^2-1}.\end{aligned}$$

We now can see that

$$\begin{aligned}\int_0^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt &= \int_0^1 \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt + \int_1^{\frac{1}{k}} \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}}\,dt \\ &= K(m) - \frac{1}{i}\int_1^{\frac{1}{k}} \frac{1}{\sqrt{(t^2-1)(1-k^2t^2)}}\,dt \\ &= K(m) + i\,K'(m).\end{aligned}$$

Therefore,

$$sn(K+i\,K') = 1/k, \text{ from which we get } dn(K+i\,K') = 0$$

and

$$cn(K+i\,K') = \lim_{x\to 1/k} \sqrt{1-x^2} = \lim_{x\to 1/k}\left[-i\sqrt{x^2-1}\right] = -i\,\frac{k'}{k},$$

where the limits are taken along the path shown in Figure 3.6.1.

Exercise 3.6.3. *Show that*

$$sn(u+K+i\,K') = \frac{1}{k}\,dc(u), \qquad cn(u+K+i\,K') = -\frac{i\,k'}{k}\,nc(u), \qquad dn(u+K+i\,K') = i\,k'\,sc(u).$$

Exercise 3.6.4. *Show that*

$$sn(u+2K+2i\,K') = -sn(u), \qquad cn(u+2K+2i\,K') = cn(u), \qquad dn(u+2K+2i\,K') = -dn(u).$$

Exercise 3.6.5. *Show that*

$$sn(u+4K+4i\,K') = sn(u), \qquad dn(u+4K+4i\,K') = dn(u).$$

So, the Jacobian elliptic functions also have a complex period.

Theorem 3.6.2 (Second Period Theorem). *If* $K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-mt^2)}}$, *and* $K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-m_1t^2)}}$, *then* $4K+4i\,K'$ *is a period of* sn *and* dn, *and* $2K+2i\,K'$ *is a period of* cn.

Theorem 3.6.3 (Third Period Theorem). *If* $K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)\,(1-m_1t^2)}}$, *then* $4i\,K'$ *is a period of* cn *and* dn, *and* $2i\,K'$ *is a period of* sn.

Exercise 3.6.6. *Prove Theorem 3.6.3. [Hint:* $i\,K' = -K + K + i\,K'$*.]*

The numbers K and iK' are called the real and imaginary *quarter periods* of the Jacobian elliptic functions.

3.7. Zeros, Poles, and Period Parallelograms

Series expansions of sn, cn, and dn around $u=0$ are

$$\begin{aligned}
sn(u\,|\,m) &= u - \frac{1+m}{6}u^3 + \frac{1+14m+m^2}{120}u^5 + O(u^7)\\
cn(u\,|\,m) &= 1 - \frac{1}{2}u^2 + \frac{1+4m}{24}u^4 + O(u^6)\\
dn(u\,|\,m) &= 1 - \frac{m}{2}u^2 + \frac{4m+m^2}{24}u^4 + O(u^6)
\end{aligned}$$

Series expansions of sn, cn, and dn around $u = i\,K'$ are

$$\begin{aligned}
sn(u+i\,K'\,|\,m) &= \frac{1}{k\,sn(u\,|\,m)} = \frac{1}{k}\left[\frac{1}{u} + \frac{1+m}{6}u + \frac{7-22m+7m^2}{360}u^3 + O(u^4)\right]\\
cn(u+i\,K'\,|\,m) &= \frac{-i}{k}ds(u\,|\,m) = \frac{-i}{k}\left[\frac{1}{u} + \frac{1-2m}{6}u + \frac{7+8m-8m^2}{360}u^3 + O(u^4)\right]\\
dn(u+i\,K'\,|\,m) &= -i\,cs(u\,|\,m) = -i\left[\frac{1}{u} + \frac{-2+m}{6}u + \frac{-8+8m+7m^2}{360}u^3 + O(u^4)\right]
\end{aligned}$$

From these expansions we can identify the poles and residues of the Jacobian elliptic functions.

Theorem 3.7.1 (First Pole Theorem). *At the point $u = i\,K'$ the functions sn, cn, and dn have simple poles with residues $1/k$, $-i/k$, and $-i$ respectively.*

Theorem 3.7.2 (Second Pole Theorem). *At the point $u = 2K + i\,K'$ the functions sn and cn have simple poles with residues $-1/k$ and i/k. At the point $u = 3i\,K'$ the function dn has a simple pole with residue i.*

Exercise 3.7.1. *Prove both Pole Theorems. If you want to verify the series expansions, use Mathematica or Maple.*

From the period theorems we see that each Jacobian elliptic function has a smallest *period parallelogram* in the complex plane. It is customary to translate period parallelograms so that no zeros or poles are on the boundary. When this is done, we see that the period parallelogram for each Jacobian elliptic function contains exactly two zeros and two poles in its interior. Unless stated otherwise, we shall assume that any period parallelogram has been translated in this manner. See Figure 3.7.1, in which zeros are indicated by a *o* and poles by a $*$.

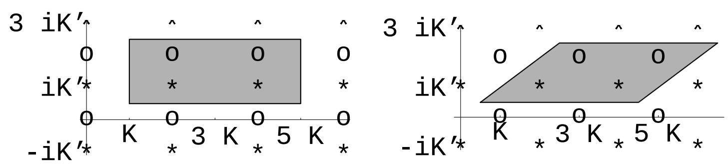

Figure 3.7.1. Period parallelograms for *sn* and *cn*.

Exercise 3.7.2. *Sketch period parallelograms for dn, sc, and cd with no zeros or poles on the boundaries.*

If C denotes the counterclockwise boundary of a period parallelogram for some Jacobian elliptic function, the Residue Theorem from complex analysis says that the the integral around C of the Jacobian elliptic function is $1/2\pi i$ times the sum of the residues at the two poles inside C. In the next section we will see that for elliptic functions in general this integral is zero.

Exercise 3.7.3. *If C denotes the boundary of a period parallelogram, oriented counterclockwise, compute $\int_C sn(u)\,du$, $\int_C cn(u)\,du$, and $\int_C dn(u)\,du$.*

A useful device for dealing with the Jacobian elliptic functions is the doubly infinite array, or lattice, consisting of the letters s, c, d, and n shown in Figure 3.7.2. Think of this lattice in the complex plane and denote one of the points labelled s by K_s. Then denote the point to the east labelled c by K_c, the point to the north labelled n by K_n, and the point to the southwest labelled d by K_d. If we put the origin at K_s, then the sum (of complex numbers, or of vectors) $K_s + K_c + K_d + K_n = 0$. Assume the scale on the lattice is such that $K_c = K$, $K_n = iK'$, and $K_d = -K - iK'$, where K and iK' are the real and imaginary quarter periods.

s	c	s	c	s	c	s
n	d	n	d	n	d	n
s	c	s	c	s	c	s
n	d	n	d	n	d	n

Figure 3.7.2. This pattern is repeated indefinitely on all sides.

If the letters p, q, r, and t are any permutation of s, c, d, and n, then the Jacobian elliptic function *pq* has the following properties. See also A&S, p.569.

(1) *pq* is doubly periodic with a simple zero at K_p and a simple pole at K_q.

(2) The step $K_q - K_p$ from the zero to the pole is a half-period; the numbers K_c, K_n, and K_d not equal to $K_q - K_p$ are quarter-periods.

(3) In the series expansion of *pq* around $u = 0$ the coefficient of the leading term is 1.

Here are plots of the *modular surfaces* over one period parallelogram of the functions $w = sn(u \,|\, \frac{1}{2})$ and $w = dn(u \,|\, \frac{1}{2})$, where $u = x + i\,y$. Complex functions of a complex variable require four dimensions for a complete graph, but a useful compromise is to plot the modulus, or absolute value of a complex function, which is essentially a real function of the real and imaginary parts of the complex variable. The surfaces in Figure 3.7.3 are plots of $w = |\, sn(x + i\,y \,|\, \frac{1}{2}) \,|$ and $w = |\, dn(x + i\,y \,|\, \frac{1}{2}) \,|$. The depressions correspond to zeros and the towers correspond to poles.

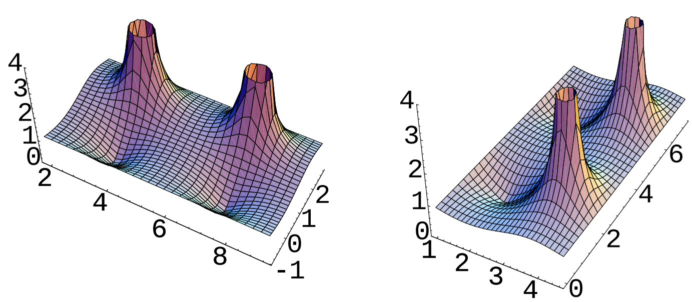

Figure 3.7.3. Modular surfaces of *sn* and *dn*.

Exercise 3.7.4. *Use the lattice in Figure 3.7.2 to determine the periods, zeros, and poles of cd and ds. Use Mathematica or Maple to plot the modular surfaces over one period parallelogram.*

Exercise 3.7.5. *Why are the functions sn, cn, and dn called "the Copolar Trio" in A&S, 16.3?*

3.8. General Elliptic Functions

Definition 3.8.1. *An elliptic function of a complex variable is a doubly periodic function which is meromorphic, i.e., analytic except for poles, in the finite complex plane.*

If the smallest periods of an elliptic function are $2\omega_1$ and $2\omega_2$, then the parallelogram with vertices 0, $2\omega_1$, $2\omega_1 + 2\omega_2$, and $2\omega_2$ is the *fundamental period parallelogram.* A *period parallelogram* is any translation of a fundamental period parallelogram by integer multiples of $2\omega_1$ and/or $2\omega_2$. A *cell* is a translation of a period parallelogram so that no poles are on the boundary. Parts of the proofs of the next few theorems depend on the theory of functions of a complex variable. If you have not studied that subject (or if you have forgotten it), learn what the theorems say, and come back to the proofs after you have studied the theory of functions of a complex variable. In fact, this material should be an incentive to take a complex variables course!

Theorem 3.8.1. *An elliptic function has a finite number of poles in any cell.*

Outline of Proof. A cell is a bounded set in the complex plane. If the number of poles in a cell is infinite, then by the two-dimensional Bolzano-Weierstrass Theorem, the poles would have a limit point in the cell. This limit point would be an essential singularity of the elliptic function. It could not be a pole, because a pole is an isolated singularity. This is a contradiction of the definition of an elliptic function. ♠

Theorem 3.8.2. *An elliptic function has a finite number of zeros in any cell.*

Proof. If not, then the reciprocal function would have an infinite number of poles in a cell, and as in the proof of Theorem 3.8.1, would have an essential singularity in the cell. This point would also be an essential singularity of the original function, contradicting the definition of an elliptic function. ♠

Theorem 3.8.3. *In a cell, the sum of the residues at the poles of an elliptic function is zero.*

Proof. Let C be the boundary of the cell, oriented counterclockwise, and let the vertices be given by t, $t + 2\omega_1$, $t + 2\omega_1 + 2\omega_2$, and $t + 2\omega_2$, where $2\omega_1$ and $2\omega_2$ are the periods. Call the elliptic function f. The sum of the residues is

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\left[\int_t^{t+2\omega_1} f(z)\,dz + \int_{t+2\omega_1}^{t+2\omega_1+2\omega_2} f(z)\,dz + \int_{t+2\omega_1+2\omega_2}^{t+2\omega_2} f(z)\,dz + \int_{t+2\omega_2}^{t} f(z)\,dz\right].$$

In the second integral, replace z by $z + 2\omega_1$, and in the third integral, replace z by $z + 2\omega_2$. We then get

$$\frac{1}{2\pi i}\int_C f(z)\,dz = \frac{1}{2\pi i}\int_t^{t+2\omega_1} (f(z) - f(z + 2\omega_2))\,dz - \frac{1}{2\pi i}\int_t^{t+2\omega_2} (f(z) - f(z + 2\omega_1))\,dz$$

By the periodicity of f, each of these integrals is zero. ♠

Theorem 3.8.4 (Liouville's Theorem for Elliptic Functions). *An elliptic function having no poles in a cell is a constant.*

Proof. If f is elliptic having no poles in a cell, then f is analytic both in the cell and on the boundary of the cell. Thus, f is bounded on the closed cell, and so there is an M such that for z in the closed cell, $|f(z)| < M$. By periodicity, we then have that $|f(z)| < M$ for all z in the complex plane, and so by Liouville's (other, more famous) Theorem, f is constant. ♠

Definition 3.8.2. *The order of an elliptic function f is equal to the number of poles, counted according to multiplicity, in a cell.*

The following lemma is useful in determining the order of elliptic functions.

Lemma 3.8.1. *If f is an elliptic function and z_0 is any complex number, the number of roots of the equation $f(z) = z_0$ in any cell depends only on f and not on z_0.*

Proof. Let C be the boundary, oriented counterlockwise, of a cell, and let Z and P be the respective numbers of zeros and poles, counted according to multiplicity, of $f(z) - z_0$ in the cell. Then by the Argument Principle,

$$Z - P = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - z_0}\, dz.$$

Breaking the integral into four parts and substituting as in the proof of Theorem 3.8.3, we get $Z - P = 0$. Thus, $f(z) - z_0$ has the same number of zeros as poles; but the number of poles is the same as the number of poles of f, which is independent of z_0. ♠

Definition 3.8.3. *The order of an elliptic function f is equal to the number of zeros, counted according to multiplicity, of f in any cell.*

Exercise 3.8.1. *Prove that if f is a nonconstant elliptic function, then the order of f is at least two.*

Thus, in terms of the number of poles, the simplest elliptic functions are those of order two, of which there are two kinds: (1) those having a single pole of order two whose residue is zero, and (2) those having two simple poles whose residues are negatives of one another. The Jacobian elliptic functions are of the second kind. An example of an elliptic function of the first kind will be given in the next section.

3.9. Weierstrass' P-function

Let ω_1 and ω_2 be two complex numbers such that the quotient ω_1/ω_2 is not a real number, and for integers m and n let $\Omega_{m,n} = 2m\,\omega_1 + 2n\,\omega_2$. Then Weierstrass' P-function is defined by

$$P(z) = \frac{1}{z^2} + {\sum_{m,n}}' \left[\frac{1}{(z - \Omega_{m,n})^2} - \frac{1}{\Omega_{m,n}^2} \right], \tag{3.9.1}$$

where the sum is over all integer values of m and n, and the prime notation indicates that m and n simultaneously zero is not included. The series for $P(z)$ can be shown to converge absolutely and uniformly except at the points $\Omega_{m,n}$, which are poles.

Exercise 3.9.1. *Show that $P'(z) = -2 \sum_{m,n} \frac{1}{(z - \Omega_{m,n})^3}$. Note the absence of the prime on the sum.*

Exercise 3.9.2. *Show that P is an even function and that P' is an odd function.*

Theorem 3.9.1. *$P'(z)$ is doubly periodic with periods $2\omega_1$ and $2\omega_2$, and therefore P' is an elliptic function.*

Proof. The sets $\{\Omega_{m,n}\}$, $\{\Omega_{m,n} - 2\omega_1\}$, and $\{\Omega_{m,n} - 2\omega_2\}$ are all the same. ♠

Theorem 3.9.2. *P is an elliptic function with periods $2\omega_1$ and $2\omega_2$.*

Proof. Since $P'(z+2\omega_1) = P'(z)$, we have $P(z+2\omega_1) = P(z) + A$, where A is a constant. If $z = -\omega_1$, we have $P(\omega_1) = P(-\omega_1) + A$, and since P is an even function, $A = 0$. Similarly, $P(z+2\omega_2) = P(z)$. ♠

$\tilde{P}(z) = P(z) - z^{-2}$ is analytic in a neighborhood of the origin and is an even function, so $\tilde{P}$ has a series expansion around $z = 0$:

$$\tilde{P}(z) = a_2 z^2 + a_4 z^4 + O(z^6)$$

$$a_2 = \frac{6}{2!} \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^4} = \frac{1}{20} g_2$$

$$g_2 = 60 \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^4}$$

$$a_4 = \frac{120}{4!} \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^6} = \frac{1}{28} g_3$$

$$g_3 = 140 \sum_{m,n}{}' \frac{1}{\Omega_{m,n}^6}$$

Thus we get the following series representations.

$$P(z) = z^{-2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6)$$

$$P'(z) = -2z^{-3} + \frac{1}{10} g_2 z + \frac{1}{7} g_3 z^3 + O(z^5)$$

$$P(z)^3 = z^{-6} + \frac{3}{20} g_2 z^{-2} + \frac{3}{28} g_3 + O(z^2)$$

$$(P'(z))^2 = 4z^{-6} - \frac{2}{5} g_2 z^{-2} - \frac{4}{7} g_3 + O(z^2)$$

Combining these leads to

$$(P'(z))^2 - 4\,P^3(z) + g_2 P(z) + g_3 = O(z^2). \tag{3.9.1}$$

Exercise 3.9.3. *Verify the details in the derivation of equation (3.9.1).*

The left side of (3.9.1) is an elliptic function with periods the same as P and is analytic at $z = 0$. By periodicity, then, it is analytic at each of the points $\Omega_{m,n}$. But the points $\Omega_{m,n}$ are the only points where the left side of (3.9.1) can have poles, so it is an elliptic function with no poles, and hence a constant. Let $z \to 0$ to see that the constant is 0.

Exercise 3.9.4. *Verify the statements in the last paragraph.*

The numbers g_2 and g_3 are called the *invariants* of P. P satisfies the differential equation

$$(P'(z))^2 = 4\,P^3(z) - g_2 P(z) - g_3. \tag{3.9.2}$$

3.10. Elliptic Functions in Terms of P and P'

Suppose f is an elliptic function and let P be the Weierstrass elliptic function (WEF) with the same periods as f. Then

$$\begin{aligned} f(z) &= \frac{1}{2}\left[f(z) + f(-z)\right] + \frac{1}{2}\left[(f(z) - f(-z))\,(P'(z))^{-1}\right] P'(z) \\ &= (\text{even elliptic function}) + (\text{even elliptic function}) P'(z) \end{aligned}$$

So, if we can express any *even* elliptic function in terms of P and P', then we can so express any elliptic function.

Suppose ϕ is an even elliptic function. The zeros and poles of ϕ in a cell can each be arranged in two sets:

zeros: $\{a_1, a_2, \ldots, a_n\}$ and additional points in the cell congruent to $\{-a_1, -a_2, \ldots, -a_n\}$.

poles: $\{b_1, b_2, \ldots, b_n\}$ and additional points in the cell congruent to $\{-b_1, -b_2, \ldots, -b_n\}$.

Consider the function

$$G(z) = \frac{1}{\phi(z)} \prod_{j=1}^{n} \frac{(P(z) - P(a_j))^?}{(P(z) - P(b_j))^?}.$$

Exercise 3.10.1. *Prove that G is a constant function when the question marks are suitably replaced. (For a specific case, see Example 3.10.1 below.)*

Thus, $\phi(z) = A \prod_{j=1}^{n} \frac{(P(z) - P(a_j))^?}{(P(z) - P(b_j))^?}$, and we have the following theorem.

Theorem 3.10.1. *Any elliptic function can be expressed in terms of the WEFs P and P' with the same periods. The expression will be rational in P and linear in P'.*

A related theorem is the following.

Theorem 3.10.2. *An algebraic (polynomial, I believe - LMH) relation exists between any two elliptic functions with the same periods.*

Outline of proof: Let f and ϕ be elliptic with the same periods. By Theorem 3.10.1, each can be expressed as a rational function of the WEFs P and P' having the same periods, say $f(z) = R_1(P(z), P'(z))$, $\phi(z) = R_2(P(z), P'(z))$. We can get an algebraic relation between f and ϕ by eliminating P and P' from these equations plus (3.9.2). ♠

Corollary 3.10.1. *Every elliptic function is related to its derivative by an algebraic relation.*

Proof: Clear, and left to the reader.

Example 3.10.1. *Express $cn\, z$ in terms of P and P'. Since cn is even, has periods $4K$ and $2K + 2iK'$, has zeros at K and $3K$, and has poles at iK' and $2K + iK'$, we can let $a_1 = K$ and $b_1 = iK'$. Thus,*

$$A = \frac{1}{cn\, z} \frac{P(z) - P(K)}{P(z) - P(iK')}.$$

As $z \to 0$ we see that $A = 1$, so

$$cn\, z = \frac{P(z) - P(K)}{P(z) - P(iK')}.$$

Exercise 3.10.2. *Let $m = .5$ and verify all statements in Example 3.10.1. Compare modular surface plots of cn and its representation in terms of P.*

The algebraic relation between two equiperiodic elliptic functions depends on the orders of the elliptic functions. Recalling Lemma 3.8.1, if f has order m and ϕ has order n, then corresponding to any value of $f(z)$, there are m values of z. Corresponding to each of these m values of z there are m values of $\phi(z)$. Similarly, to each value of $\phi(z)$ there correspond n values of $f(z)$. Thus, the algebraic relation between f and ϕ will be of degree m or lower in ϕ and degree n or lower in f.

Example 3.10.2. *The functions $f(z) = P(z)$ and $\phi(z) = P^2(z)$ have orders 2 and 4, so their relation will be of degree at most 2 in ϕ and at most 4 in f. The relation between them is obviously $\phi = f^2$, of less than maximum degrees.*

Example 3.10.3. *Let $f(z) = P(z)$ (order 2) and $\phi(z) = P'(z)$ (order 3). Their relation is given by equation (3.9.2), and is of degree 2 in ϕ and degree 3 in f.*

3.11. Elliptic Wheels - An Application

The material in this section is taken from: Leon Hall and Stan Wagon, Roads and wheels, *Mathematics Magazine* 65, (1992), 283-301. See this article for more details.

Suppose we are given a wheel in the form of a function defined by $r = g(\theta)$ in polar coordinates with the axle of the wheel at the origin, or pole. The problem is, what road is required for this wheel to roll on so that the axle remains level? The axle may or may not coincide with the wheel's geometric center, and the road is assumed to provide enough friction so the wheel never slips. Assume the road (to be found) has equation $y = f(x)$.

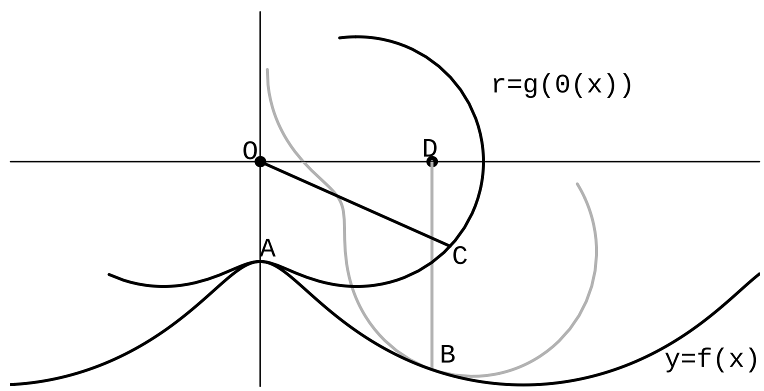

Figure 3.11.1. Wheel - road relationships

Three conditions will guarantee that the axle of the wheel moves horizontally on the x-axis as the wheel rolls on the road. These conditions are illustrated in Figure 3.11.1. First, the initial point of contact must be directly below the origin, which means that when $x = 0$, $\theta = -\pi/2$. Second, corresponding arc lengths along the road and on the wheel must be equal. In Figure 3.11.1, this means that the road length from A to B must equal the wheel length from A to C. Third, the radius of the wheel must match the depth of the road at the corresponding point, which means that $OC = DB$, or $g(\theta(x)) = -f(x)$. The arc length condition gives

$$\int_0^x \sqrt{1 + f'(u)^2}\, du = \int_{-\pi/2}^{\theta} \sqrt{g(\phi)^2 + g'(\phi)^2}\, d\phi.$$

Differentiation with respect to x and simplification leads to the initial value problem

$$\frac{d\theta}{dx} = \frac{1}{g(\theta)}, \qquad \theta(0) = -\pi/2,$$

whose solution expresses θ as a function of x. The road is then given by $y = -g(\theta(x))$.

Exercise 3.11.1. *Fill in the details of the derivation sketched above.*

Example 3.11.1. *Consider the ellipse with polar equation* $r = \dfrac{k\,e}{1 - e\,\sin\theta}$*, where* e *is the eccentricity of the ellipse and* k *is the distance from the origin to the corresponding directrix. The axle for this wheel is the focus which is initially at the origin. The details get a bit messy (guess who gets to do them!) but no special functions are required. The solution of the IVP turns out to be*

$$\frac{a\,x}{2\,k\,e} = \arctan\left(\frac{tan(\theta/2) - e}{a}\right) + \arctan\left(\frac{1+e}{a}\right),$$

where $a = \sqrt{1-e^2}$*. Now take the tangent of both sides and do some trig to get*

$$\frac{(1+e)(1-\cos^2(cx))}{(1-e)(1+\cos(cx))^2} = \frac{1+\sin\theta}{1-\sin\theta},$$

where $c = a/(ke)$*. Finally, solve for* $\sin\theta$ *and substitute into* $y(x) = \dfrac{k\,e}{1 - e\,\sin\theta(x)}$ *to get the road*

$$y = -\frac{k\,e}{a^2}(1 - e\,\cos(cx).$$

Thus, the road for an elliptic wheel with axle at a focus is essentially a cosine curve. Figure 3.11.2 illustrates the case $k = 1$ *and* $e = 1/\sqrt{2}$*.*

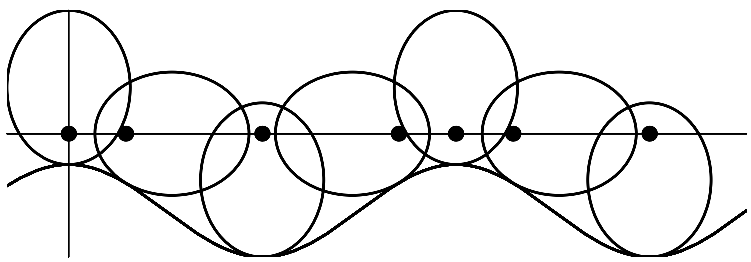

Figure 3.11.2. The axle at a focus yields a cosine road.

Exercise 3.11.2. *Fill in all the details in Example 3.11.1 for the case* $k = 1$ *and* $e = 1/\sqrt{2}$*.*

Example 3.11.2. *We now consider a rolling ellipse where the axle is at the center of the ellipse. The ellipse* $x^2/a^2 + y^2/b^2 = 1$ *has polar representation* $r = b/\sqrt{1 - m\,\cos^2\theta}$*, where* $m = 1 - b^2/a^2$ *(assume* $a > b$*). The IVP in* θ *and* x *is again separable and we get*

$$\int_{-\pi/2}^{\theta(x)} \frac{d\phi}{\sqrt{1 - m\,\cos^2\phi}} = \int_0^x \frac{dt}{b}.$$

The substitution $\psi = \phi + \pi/2$ *yields*

$$\int_0^{\theta(x)+\pi/2} \frac{d\psi}{\sqrt{1 - m\,\sin^2\psi}} = \frac{x}{b},$$

which involves an incomplete elliptic integral of the first kind. In terms of the Jacobian elliptic functions, we get $\sin(\theta + \pi/2) = sn(\frac{x}{b}\,|\,m)$*. The road is then*

$$y = \frac{-b}{dn(\frac{x}{b}\,|\,m)} = -b\;nd(\frac{x}{b}\,|\,\frac{a^2-b^2}{a^2}).$$

See Figure 3.11.3 for the $a = 1$ and $b = 1/2$ case.

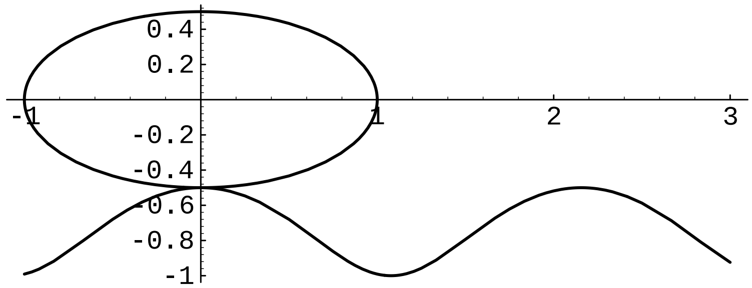

Figure 3.11.3. The axle at the center leads to Jacobian elliptic functions.

Exercise 3.11.3. *Fill in all the details in Example 3.11.2 for the case $a = 1$ and $b = 1/2$.*

Exercise 3.11.4. *Determine the road in terms of a Jacobian elliptic function for a center-axle elliptic wheel, $x^2/a^2 + y^2/b^2 = 1$, when $b > a$.*

3.12. Miscellaneous Integrals

Exercise 3.12.1. *Evaluate* $\displaystyle\int_x^\infty \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}$, *where $a > b$. (See A&S, p. 596.)*

Exercise 3.12.2. *Evaluate* $\displaystyle\int_a^x \frac{dt}{\sqrt{(t^2-a^2)(t^2-b^2)}}$, *where $a > b$. (See A&S, p. 596.)*

Exercises 3.12.3-7. *Do Examples 8-12, A&S, pp.603-04.*