1. Introduction.

The fundamental theorems of mathematical physics applied to classical electromagnetism are

- principle of duality;
- image principle;
- surface equivalence theorem (Love/Schelkunoff’s formulation);
- Lorentz lemma and reciprocity principle;
- radiation (Sommerfeld) condition; principle of absorption at the infinity;
- uniqueness theorems for the solutions of Maxwell equations;
- Huygens principle and Kirchhoff’s integral;
- reaction theorem;
- volume equivalence;
- induction equivalent theorem;
- physical equivalent and physical optics equivalent;
- approximate boundary conditions.

2. Lorentz Lemma.

Lorentz lemma is an auxiliary mathematical relation used for

- formulating the problem of electromagnetic field excitation in space with arbitrarily placed inhomogeneities;
- giving a proof of some basic concepts of electrodynamics.

Let us consider Lorentz lemma in differential form.

Let us consider isotropic medium, generally inhomogeneous medium with continuously varying complex constitutive parameters \( \varepsilon_a, \mu_a \). Let the distribution of external (impressed) currents \( \mathbf{J}^{e,m} \) (at frequency \( \omega \)) be known and producing the field \( \mathbf{E}_1, \mathbf{H}_1 \), satisfying the Maxwell equations,

\[
\nabla \times \mathbf{H}_1 = j \omega \varepsilon_0 \mathbf{E}_1 + \mathbf{J}_1^e; \quad (1a) \\
\nabla \times \mathbf{E}_1 = -j \omega \mu_0 \mathbf{H}_1 - \mathbf{J}_1^m. \quad (1b)
\]

Moreover, there is a system of impressed currents \( \mathbf{J}^{e,m}_2 \) (at the same frequency \( \omega \)), exciting the field \( \mathbf{E}_2, \mathbf{H}_2 \), satisfying the equations

\[
\nabla \times \mathbf{H}_2 = j \omega \varepsilon_0 \mathbf{E}_2 + \mathbf{J}_2^e; \quad (2a) \\
\nabla \times \mathbf{E}_2 = -j \omega \mu_0 \mathbf{H}_2 - \mathbf{J}_2^m. \quad (2b)
\]
Let us find the relation between the fields and impressed currents. To do this, let us multiply in scalar manner and subtract:

\[(1a) \cdot \bar{E}_2 - (2b) \cdot \bar{H}_1;\]
\[(1b) \cdot \bar{H}_2 - (2a) \cdot \bar{E}_1.\]

Remember that \(\nabla \cdot (\bar{a} \times \bar{b}) = \bar{b} \cdot \nabla \times \bar{a} - \bar{a} \cdot \nabla \times \bar{b}.\) Then, we can get

\[-\nabla \cdot \left[\bar{E}_2 \times \bar{H}_1\right] = j\omega \epsilon \bar{E}_2 \cdot \bar{E}_1 + j\omega \mu \bar{H}_2 \cdot \bar{H}_1 + \bar{J}^e_1 \cdot \bar{E}_2 + \bar{J}^m_1 \cdot \bar{H}_1;\]
\[\nabla \cdot \left[\bar{E}_1 \times \bar{H}_2\right] = j\omega \epsilon \bar{E}_1 \cdot \bar{E}_2 + j\omega \mu \bar{H}_1 \cdot \bar{H}_2 + \bar{J}^e_1 \cdot \bar{E}_1 + \bar{J}^m_1 \cdot \bar{H}_2.\]

Adding (3a) and (3b), we will get

\[\nabla \cdot \left[\bar{E}_1 \times \bar{H}_2\right] - \nabla \cdot \left[\bar{E}_2 \times \bar{H}_1\right] = \bar{J}^e_1 \cdot \bar{E}_2 - \bar{J}^m_1 \cdot \bar{H}_2 + \bar{J}^e_2 \cdot \bar{E}_1 - \bar{J}^m_2 \cdot \bar{H}_1.\]

This is the Lorentz lemma in the differential form.

Let us obtain the **Lorentz lemma in the integral form.**

Consider the region of space \(V_0\) limited by the closed surface \(S\), which can be either a border of two media, or some auxiliary surface, embracing the impressed currents totally or partially. Let the impressed currents \(\bar{J}^{e,m}_1\) (at frequency \(\omega\)) be distributed in some volume \(V_1\) (inside the surface \(S\)), these currents producing the field \(\bar{E}_1, \bar{H}_1\). Also, let the impressed currents \(\bar{J}^{e,m}_2\) (at frequency \(\omega\)) be distributed in the volume \(V_2\) (inside the surface \(S\)), these currents producing the field \(\bar{E}_2, \bar{H}_2\). Let us integrate (4) over the volume \(V_0\). Applying the Gauss Theorem to the left-hand side of (4) and taking into
account that the impressed currents are non-zero only in the volumes \( V_{1,2} \), we will get the Lorentz lemma in the integral form,

\[
\int_S \left[ \vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1 \right] d\vec{S}_q = \int_{V_1} \left[ \vec{J}_{1}^s(q) \cdot \vec{E}_2(q) - \vec{J}_{1}^m(q) \cdot \vec{H}_2(q) \right] dV_q - \int_{V_2} \left[ \vec{J}_{2}^s(q) \cdot \vec{E}_1(q) - \vec{J}_{2}^m(q) \cdot \vec{H}_1(q) \right] dV_q,
\]

(5)

where \( q \) is the point of integration (on the surface or inside the volume).

This equation is the relation between the fields and the impressed currents producing these fields, so it takes into account the geometry of the surface \( S \) and position of the distributed currents. This equation is widely used in electrodynamics to compose integral equations.

2. Integral relations for the fields.

It is reasonable to suppose that if there is a boundary between two media, the following processes might take place.

Impressed currents excite the electromagnetic field in all the space; this field is propagating in all the directions; when it reaches the surface, limiting the given volume, it is reflected (scattered) by this surface. Then, there are waves, propagating in the directions, opposite to the initial waves, coming from the sources. Due to re-reflections from the boundaries, there is sustained some resulting field.

Physical representation is the following. The primary field from the impressed sources produces the secondary surface currents, which, in their turn, excite secondary field in the volume. These secondary surface currents depend on

- impressed sources;
- constitutive parameters of media;
- geometry of the boundary surface;
- boundary conditions on the surface.

Let us determine the integral relations for the fields.

Consider the region of space \( V_0 \) limited by the closed surface \( S \). In the region \( V \) inside \( V_0 \) the impressed currents \( \vec{J}_{e,m} \) (at frequency \( \omega \)) are distributed. These currents excite the electromagnetic field \( \vec{E}(p), \vec{H}(p), \ p \in V_0 \), satisfying the boundary conditions at \( S \).
Let us apply the Lorentz lemma, so that $\vec{J}^m_1 = \vec{J}^e_1; \vec{E}_1 = \vec{E}(p), \vec{H}_1 = \vec{H}(p)$. 

First, let us introduce the auxiliary elementary electric dipole with the current density $\vec{J}^e_2 = \vec{a}^0 \delta(q - p)$, while $\vec{J}^m_2 = 0$; $q$ is an arbitrary point, $q \in V_0$. Then, $\vec{E}_2 = \vec{E}^e(q, p)$, and $\vec{H}_2 = \vec{H}^e(q, p)$ are the intensities of electric and magnetic fields excited in the point $q$ (point of observation) by the auxiliary dipole placed in the point $p$ (source point). $\vec{a}^0$ is a unit vector along the elementary dipole.

Substitute these currents and fields in the Lorentz lemma (5).

\[
\int_S \left[ \vec{E}(q) \times \vec{H}^e(q, p) \right] dS_q = \int_{V_1} \left[ \vec{J}^e(q, p) - \vec{E}^e(q) \cdot \vec{H}(q) \right] dV_q - \int_{V_2} \vec{J}^e(q) \cdot \vec{E}(q) dV_q \tag{6}
\]

Since the dipole is elementary, integration over $V_2$ is substituted by multiplication. Then (6) can be re-written as

\[
\vec{a}^0 \cdot \vec{E}(p) = \int_S \left[ \vec{E}^e(q, p) \times \vec{H}(q) \right] dS_q + \int_{V_1} \left[ \vec{J}^e(q) \vec{E}^e(q, p) - \vec{J}^m(q) \vec{H}^e(q, p) \right] dV_q. \tag{7}
\]

This is the integral relation for the electric field intensity in any point $p$.

To determine the magnetic field intensity, let us introduce an auxiliary elementary magnetic dipole $\vec{J}^m_2 = \vec{b}^0 \delta(q - p)$, oriented along the unit vector $\vec{b}^0$. The auxiliary electric current density is zero, $\vec{J}^e_2 = 0$. Then, from the Lorentz lemma we will get
\[
\vec{b}^0 \cdot \vec{H}(p) = -\int_S \left[ \vec{E}^m(q, p) \times \vec{H}(q) - [\vec{E}(q) \times \vec{H}^m(q, p)] \right] dS_q = \\
\int_{V_0} \left[ \vec{j}^e(q) \vec{E}^m(q, p) - \vec{j}^m(q) \vec{H}^m(q, p) \right] dV_q.
\]

(8)

The scalar products \(a^0 \cdot \vec{E}(p)\) and \(b^0 \cdot \vec{H}(p)\) allow to determine any component of \(\vec{E}\) and \(\vec{H}\) fields, in any coordinate system.

In (7) and (8) integration over the volume can always be fulfilled, since the impressed currents are given functions of coordinates, and the auxiliary dipoles are also known (let us suppose that these are fields of elementary dipoles in unbounded space). As a result of integration, some functions of the coordinates of \(p\) can be obtained.

Let us multiply the surface integrals in (7) and (8) in a scalar manner by the normal to the surface vector \(\vec{n}\), then, interchanging the order in the mixed (triple) product, we can get

\[
\int_S \left[ \vec{E}^{e,m}(q, p) \times \vec{H}(q) - [\vec{E}(q) \times \vec{H}^{e,m}(q, p)] \right] dS_q = \\
\int_S \left[ \vec{H}(q) \times \vec{n} \right] \left[ \vec{E}^{e,m}(p, q) - [\vec{n} \times \vec{E}(q)] \vec{H}^{e,m}(q, p) \right] dS_q.
\]

(9)

Herein, index “e” is taken when calculating the field \(\vec{E}\), and index “m” is taken when calculating the field \(\vec{H}\).

Since the vector products \([\vec{H} \times \vec{n}]\) and \([\vec{n} \times \vec{E}]\) determine the tangential components of the fields on the surfaces, and they are in the integrands of the surface integrals, then, to find the field in an arbitrary point \(p \in V_0\), one should know the tangential components of the same field on the surrounding surface \(S\). In some important problems of electromagnetics the tangential components of the fields on the surfaces \(S\) may be given using some approximate representations: using measured (experimental) data; or calculated using approximate solutions of the problem. As a result of integration over the surface \(S\), some function of \(p\) can be obtained, and the fields \(\vec{E}(p)\) and \(\vec{H}(p)\) are then found.

It is sufficient to find only one vector – either \(\vec{E}(p)\) or \(\vec{H}(p)\), the second is found from the Maxwell’s equations.

If the point of observation \(p\) is properly chosen on the surface \(S\), and the tangential component of either \(\vec{E}(p)\) or \(\vec{H}(p)\) is somehow found, then, it can be seen in (7) and (8) that this tangential component is also in the integrand of the right-hand side. This way the integral equation in respect of the unknown tangential component of the field can be obtained.
3. **Theorem of the equivalent surface currents.**

The surface equivalence was introduced by Schelkunoff in 1936. It is based on the uniqueness theorem. By the surface equivalence theorem, the fields outside an imaginary closed surface are obtained by placing over the closed surface suitable electric and magnetic current densities that satisfy boundary conditions. The current densities are selected so that the fields inside the closed surface are zero, and outside are equal to the radiation produced by actual sources. Thus, this technique can be used to get the fields radiated outside a closed surface by sources enclosed within it.

Let us consider the equivalent surface currents. According to the boundary conditions, the tangential components of the field vectors on the surface $S$ have the dimensions of the surface densities for electric and magnetic current, respectively:

$$\left[\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1)\right] = \mathbf{J}_S^e \quad \text{and} \quad \left[\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1)\right] = -\mathbf{J}_S^m.$$

Since the fields in the second medium are not of interest, they can be set as zero (*Love’s equivalence principle* – a null field within the imaginary surface $S$, or replaced by a PEC), then

$$\left[\mathbf{H} \times \mathbf{n}\right] = \mathbf{J}_S^e; \quad \left[\mathbf{n} \times \mathbf{E}\right] = \mathbf{J}_S^m.$$

Then, from (7) and (8), one can get for any $\mathbf{p} \in V_0$

$$\frac{\mathbf{a}^0 \cdot \mathbf{E}(\mathbf{p})}{\mathbf{b}^0 \cdot \mathbf{H}(\mathbf{p})} = \pm \int \left[\mathbf{J}_S^e \cdot \mathbf{E}^e,m - \mathbf{J}_S^m \cdot \mathbf{H}^e,m\right] dV_q \pm \int \left[\mathbf{J}_S^e \cdot \mathbf{E}^e,m - \mathbf{J}_S^m \cdot \mathbf{H}^e,m\right] dS_q. \quad (10)$$

It is seen that the equivalent surface currents play the same part in excitation as the impressed currents. Important that the impressed currents are given functions, while the equivalent surface currents are secondary, since they appear due to the existence of the fields of the impressed currents.

The **Theorem of Equivalent Surface Currents** is often used for many important problems of electrodynamics.

The field in the region free of sources can be produced by the electric and magnetic currents, distributed over the limiting surface. Thus, the actual sources can be substituted by the equivalent surface currents.
Suppose that we have a problem where the isolated closed surfaces $S_{0,1,2}$ are given (there may be a number of other surfaces). The field sources are given in the volumes $V'$ and $V''$, generally placed at different distances from the coordinate origin (let the latter be inside the volume $V_0$. In the formula (10), the integrals are taken over the surfaces $S_{0,1,2}$, which might have a complex shape. To simplify computations, let us assume that there are some fictitious surfaces $S'$ and $S''$ around the sources, such, that in the volume $\hat{V}$, limited by the surfaces $S'$ and $S''$, there would be no impressed sources. These surfaces $S'$ and $S''$ might partially intersect with $S_{0,1,2}$; choosing of their geometry is the matter of simplifying the solution of the problem.

Let the point of observation be inside the volume $\hat{V}$, $p \in \hat{V}$, the region free of impressed sources. Then, Maxwell’s equations in this region are homogeneous (uniform), then in (10) the volume integral is zero. The surface integral is taken over $S'$ and $S''$. Then,

$$\begin{align}
\vec{a}^0 \cdot \vec{E}(p) &= \pm \int_{S' \cup S''} \left[ \vec{j}^e \cdot \vec{E}^e - \vec{j}^m \cdot \vec{H}^e \right] dS_q, \quad p \in \hat{V}.
\end{align}$$

As follows from (11), the electromagnetic field in the source-free region is excited by the equivalent surface currents distributed over the surfaces limiting this volume. If the electromagnetic field outside the region $\hat{V}$ is somehow found, then, the tangential components of the field vectors at these surfaces are determined. Knowing these tangential components, one can find the equivalent surface currents.

4. **Uniqueness theorem.**

The uniqueness theorem shows how to formulate a boundary problem, for it would have a unique solution. The proof of uniqueness can be obtained directly from the energy balance of electromagnetic field.
Let us have a volume $V_0$ limited by the closed surface $S$. Let the medium inside the volume be isotropic and generally inhomogeneous with complex parameters $\varepsilon_a, \mu_a$. In the region $V \in V_0$, the impressed currents are given $\vec{J}^{e,m}$. The boundary conditions on the surface $S$ are given: on the part of the surface $S_1$ only the tangential component of the electric field is given, and on the rest of the surface $S_2$ only the tangential component of the magnetic field is given.

It is necessary to show that the solution of the Maxwell’s equations

$$\nabla \times \vec{H} = j \omega \varepsilon_a \vec{E} + \vec{J}^e; \quad \nabla \times \vec{E} = -j \omega \mu_a \vec{H} - \vec{J}^m, \quad (26a)$$

$$\nabla \times \vec{H} = j \omega \varepsilon_a \vec{E} + \vec{J}^e; \quad \nabla \times \vec{E} = -j \omega \mu_a \vec{H} - \vec{J}^m, \quad (26b)$$

satisfying the boundary conditions is unique.

Let us consider the internal electrodynamic problem, when the volume $V_0$ is finite. Suppose that there are TWO solutions: $\vec{E}_1, \vec{H}_1$ and $\vec{E}_2, \vec{H}_2$. The differences of the solutions are

$$\vec{E} = \vec{E}_1 - \vec{E}_2 \quad \text{and} \quad \vec{H} = \vec{H}_1 - \vec{H}_2.$$ 

Since $\vec{E}_1, \vec{H}_1$ and $\vec{E}_2, \vec{H}_2$ satisfy inhomogeneous Maxwell’s equations (26), the differences also satisfy the homogeneous Maxwell’s equations,

$$\nabla \times \vec{H} = (j \omega \varepsilon_a + \sigma) \vec{E}; \quad \text{where} \quad j \omega \varepsilon_a = j \omega \varepsilon_a + \sigma; \quad (27a)$$

$$\nabla \times \vec{E} = -j \omega \mu_a \vec{H}, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (27b)$$

and the homogeneous boundary conditions,
\[ E_i = 0 \big|_{S_1} \quad \text{and} \quad H_i = 0 \big|_{S_2}. \quad (28) \]

From the energy balance we will have,
\[ 0 = j\omega \int_{V} \left( \mu_a \frac{\vec{H} \cdot \vec{H}^*}{2} - \varepsilon_a \frac{\vec{E} \cdot \vec{E}^*}{2} \right) dV + \int_{s} \sigma \frac{\vec{E} \cdot \vec{E}^*}{2} dV + \oint_{S} \left[ \vec{E} \times \vec{H}^* \right] \tilde{n} dS. \quad (29) \]

The left-hand side is zero because there are no impressed currents for the difference fields.

From the boundary conditions (28) it follows that the Poynting vector flux through the closed surface \( S \) is zero, because
\[ \left[ \vec{E} \times \vec{H}^* \right] \cdot \vec{n} = \left[ \vec{n} \times \vec{E} \right] \cdot \vec{H}^* = \vec{E}_i \cdot \vec{H}^* = 0 \big|_{S_1}; \]
\[ \left[ \vec{E} \times \vec{H}^* \right] \cdot \vec{n} = \left[ \vec{H}^* \times \vec{n} \right] \cdot \vec{E} = \vec{H}_i \cdot \vec{E} = 0 \big|_{S_2}. \quad (30) \]

This means that there is not input of energy for the difference field in the volume \( V_0 \), and there are no impressed sources, so loss of energy for the difference field is also zero. Hence,
\[ \int_{V} \sigma \frac{\vec{E} \cdot \vec{E}^*}{2} dV = 0. \quad (31) \]

As follows from (29),
\[ \int_{V} \left( \mu_a \frac{\vec{H} \cdot \vec{H}^*}{2} - \varepsilon_a \frac{\vec{E} \cdot \vec{E}^*}{2} \right) dV = 0. \quad (32) \]

At the analysis of (31) and (32) there can be two cases.

Case 1. Conductivity of the medium is non-zero. Then, it follows from (31) that \( \vec{E} = 0 \) at \( p \in V_0 \), and from (32) follows that \( \vec{H} = 0 \) at \( p \in V_0 \). Then, the solution is unique, \( \vec{E}_1 = \vec{E}_2 \) and \( \vec{H}_1 = \vec{H}_2 \).

Case 2. Conductivity of the medium is zero. Then, (31) is satisfied at any non-zero field. It is necessary only that average electric energy equals average magnetic energy. This can be at free oscillations inside the volume.
Hence, for the lossless media, the uniqueness of the solution can be at any frequencies, except for the resonance frequencies. On the boundary surface it is necessary to determine only the tangential electric field or tangential magnetic field.

The uniqueness can be also proved for the external electromagnetic problems, also using the energy balance principle. For the external problem it is necessary to demand that the solution satisfies electromagnetic radiation conditions.

It is important to note that the integral expressions for fields (7) and (8) agree with the uniqueness theorem. Actually, using (7) and (8) it is possible to find fields in any point \( p \) of the region of consideration, if besides the impressed sources the tangential components of the electric and magnetic fields are given on the surface \( S \) limiting the region. As for auxiliary fields, any arbitrary boundary conditions can be applied. However, we supposed that the fields of auxiliary dipoles satisfy the radiation conditions at infinity. Thus, at the surface \( S \) the zero tangential conditions for the auxiliary fields \( \vec{H}^{c,m} = 0 \) or \( \vec{E}^{c,m} = 0 \) can be also applied. Then, in (7) and (8) either first or second terms in the integrals over the surface are zero. Hence, according to the uniqueness, boundary conditions for either \( E \) or \( H \) fields are sufficient and necessary.

5. Huygens principle and Kirchhoff’s integral.

The Kirchhoff’s integral is the mathematical representation of the Huygens principle. According to the Huygens principle, the field of the wave process in any point is a superposition of spherical waves radiated by the elementary sources, distributed over the given surface, which is the wave front.

To obtain the Kirchhoff’s integral, let us consider region \( \hat{V} \) free from the impressed sources, and limited by the surfaces \( S' \) and \( S'' \). Let us find integral expressions for the vectors \( \vec{E} \) and \( \vec{H} \) in \( \hat{V} \).

The Maxwell’s equations are homogeneous in \( \hat{V} \), hence, the Helmholtz equations are also homogeneous. In the Cartesian coordinates (for simplicity), for any component of electromagnetic field we have

\[
\nabla^2 E_i + \gamma^2 E_i = 0; \quad (33a)
\]
\[
\nabla^2 H_i + \gamma^2 H_i = 0, \quad (33b)
\]

\((i = x, y, z; \ p \in \hat{V}).\)

Let in the point \( q \in \hat{V} \) be an auxiliary point source. Its scalar field is the Green’s function, satisfying the inhomogeneous Helmholtz equation

\[
\nabla^2 G(p, q) + \gamma^2 G(p, q) = -\delta(p - q). \quad (34)
\]
Let us multiply (34) by $E_i$ and (33a) by $G = G(p,q)$, and subtract the first result from the second. Integrating by the volume $\hat{V}$ gives

$$\int_{\hat{V}} (G \nabla^2 E_i - E_i \nabla^2 G) dV = \begin{cases} E_i(q), & q \in \hat{V} \\ 0, & q \notin \hat{V} \end{cases}.$$  

(35)

According to Green’s theorem known from Vector Calculus, we have

$$\int_{\hat{V}} (G \nabla^2 E_i - E_i \nabla^2 G) dV = \int_{S''} \left( G \frac{\partial E_i}{\partial n} - E_i \frac{\partial G}{\partial n} \right) dS.$$  

(36)

Interchanging the points $p$ and $q$, we have

$$E_i(p) = \int_{S''} \left( G(p,q) \frac{\partial E_i(p)}{\partial n} - E_i(q) \frac{\partial G(p,q)}{\partial n} \right) dS_q, \quad p \in \hat{V}.$$  

(37)

If the surface $S''$ is shifted to infinity, then, using the radiation condition, we will get

$$E_i(p) = \int_{S'} \left( G(p,q) \frac{\partial E_i(p)}{\partial n} - E_i(q) \frac{\partial G(p,q)}{\partial n} \right) dS_q, \quad p \in \hat{V}.$$  

(38)

Here $G(p,q)$ is the Green’s function for unlimited 3D space. This integral allows determining the function $E_i$ everywhere in the volume, if the values of $E_i$ and $\frac{\partial E_i}{\partial n}$ are known on the boundary surface. This is the Kirchhoff’s integral.

Notice that $E_i$ and $\frac{\partial E_i}{\partial n}$ are tightly related, they are not arbitrary.

Comparison with the expressions (11) which take into account the vector character of electromagnetic field, can lead to the following conclusion:

The component of the electromagnetic field and its normal derivative on the boundary play a part of equivalent surface currents; auxiliary fields $\tilde{E}^{e,m}$ and $\tilde{H}^{e,m}$ play parts of the Green’s function and its normal derivative. The surface integrals in (11) and (38) allow for summatting radiation of elementary sources, distributed on the surface limiting the volume. Green’s function describing the radiation of the elementary sources, qualitatively characterizes field of the radiators distributed over the surface.

Green’s function with applied boundary conditions can satisfy Dirichlet ($G=0$) and Neumann ($\frac{\partial G}{\partial n} = 0$) boundary problems at $S'$, at $S'' \to \infty$. 

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If in (38) use the Green’s function \( G_i \) for the Dirichlet problem or \( G_2 \) for the Neumann problem, then the Kirchhoff’s integral is

\[
E_i(p) = -\int_{\partial V} E_i(q) \frac{\partial G_i(p,q)}{\partial n} dS_q, \quad p \in \hat{V} \quad (39a)
\]

or

\[
E_i(p) = \int_{\partial V} \left(G_2(p,q) \frac{\partial E_i(p)}{\partial n}\right) dS_q, \quad p \in \hat{V} \quad (39b)
\]

The equations (11) are the formulation of the theorem of equivalent surface currents, and they express Huygens principle for vector fields, while Kirchhoff’s integral does not take into account the vector character of electromagnetic field.

If in \( V \in V_0 \) \((V_0 \) is limited by \( S \) \) there are given the impressed currents, then, to calculate \( \vec{E} \) and \( \vec{H} \) it is convenient to use not (7) and (8), but some modified equations, based on Maxwell’s equations in terms of vector potentials.

\[
\vec{E}^e = \frac{1}{j\omega \varepsilon} \left( j^2 \vec{a}_0 G + \nabla (\vec{a} \cdot \vec{a}_0 G) \right); \quad \vec{H}^e = \nabla \times \vec{a}_0 G; \quad (39a)
\]

\[
\vec{E}^m = \frac{1}{j\omega \mu} \left( j^2 \vec{b}_0 G + \nabla (\vec{b} \cdot \vec{b}_0 G) \right); \quad \vec{H}^m = -\nabla \times \vec{b}_0 G, \quad (39b)
\]

where \( G \) is the Green’s function for the unbounded space; \( \vec{a}_0 \) and \( \vec{b}_0 \) are unit vectors in the point \( p \).

Substituting in (7) and (8) (integration is over the points \( q \), where the currents are non-zero) gives

\[
\vec{E}(p) = \frac{1}{j\omega \varepsilon} \left( j^2 \vec{F} + \nabla (\vec{a} \cdot \vec{F}) \right) - \nabla \times \vec{A} ; \quad (40a)
\]

\[
\vec{H}(p) = \frac{1}{j\omega \mu} \left( j^2 \vec{F} + \nabla (\vec{A} \cdot \vec{F}) \right) + \nabla \times \vec{F} , \quad (40b)
\]

where the vector potentials are

\[
\vec{A} = \oint_{V} \vec{J}^e GdV_q + \oint_{S} \vec{J}^e GdS_q ; \quad (41a)
\]

\[
\vec{F} = \oint_{V} \vec{J}^m GdV_q + \oint_{S} \vec{J}^m GdS_q . \quad (41b)
\]