Quantum One
Solving the Eigenvalue Equation in Infinite Dimensional Spaces
In the last segment, we saw how to solve the eigenvalue problem in a finite dimensional state space, in which each linear operator is represented by a finite dimensional square matrix.

For a normal operator, i.e., one that commutes with its adjoint, the solution to the eigenvalue problem in such a space involves a two step process of

1. Finding the eigenvalues and their degeneracies by finding the roots and multiplicities of the characteristic equation, and

2. Solving the linear equations associated with the eigenvalue problem to obtain the expansion coefficients of the eigenstates in some representation.
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2. Solving the linear equations associated with the eigenvalue problem to obtain the expansion coefficients of the eigenstates in some representation.

In this segment, we consider what happens in infinite dimensional spaces, in which there are both discrete and continuous representations.
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Indeed, it is not all that unusual to find Hermitian operators that have no eigenvectors at all in the space itself.

Hopefully, an example will help to make this clear.
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that are not square integrable, because ...
they are of infinite norm. Thus, the position operator contains no eigenstates in the space of square-integrable functions. It is for this reason that we have chosen the space of Fourier transformable functions (which includes the delta functions and plane waves). This does not entirely dispose of the problem, but it does allow us to confront it less frequently.

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2. **A differential or integral equation approach**, in which one expresses the eigenvalue equation as a differential equation, an integral equation, or an integro-differential equation in some continuous representation, that is then solved subject to appropriate boundary conditions, as we have done in Schrödinger's mechanics.
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Later in this course we will use the **algebraic approach** to solve the eigenvalue problem for the **Harmonic Oscillator** Hamiltonian, and for the components of the **angular momentum operator**.
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We can choose either the position or the wavevector representation.
In the position representation, the eigenvector

\[ |\phi\rangle = \int d^3 r \, \phi(r) |r\rangle \]

is represented by the eigenfunction \( \phi(r) \).
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In this representation, the kinetic energy is a differential operator, while the potential energy is a multiplicative operator.
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We “look at” the eigenvalue equation

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\[ \langle \vec{r}| H |\phi\rangle = \langle \vec{r}| E |\phi\rangle \]

which then becomes

\[ -\frac{\hbar^2}{2m} \nabla^2 \phi(\vec{r}) + V(\vec{r})\phi(\vec{r}) = E\phi(\vec{r}) \]
Thus in the position representation the energy eigenvalue equation becomes a partial differential equation

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The eigenvalues of $H$ are then identified with those solutions which lie within the relevant space, which in our case, is the space of Fourier transformable functions (which defines more precisely what we mean by "an acceptable solution", i.e., it includes square-integrable functions, but also includes plane waves, delta functions, and other functions that remain bounded as $|\vec{r}| \rightarrow \infty$. 56
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Thus, for example, when \(V = 0\) this leads us to the plane waves, but excludes exponentials of the form \(\phi(\vec{r}) = \exp(\vec{\alpha} \cdot \vec{r})\) for real vectors \(\vec{\alpha}\).
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Thus, for example, when $V = 0$ this leads us to the plane waves, but excludes exponentials of the form $\phi(\vec{r}) = \exp(\vec{\alpha} \cdot \vec{r})$ for real vectors $\vec{\alpha}$, which diverge along the direction of $\vec{\alpha}$ and are not Fourier transformable.
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\[ |\phi\rangle = \int d^3 k \phi(\vec{k}) \ |\vec{k}\rangle \]

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We "look at" the eigenvalue equation in the momentum representation by multiplying on the left by the basis vector $\langle \vec{k} |$

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On the left, we have

\[ \langle \vec{k} | H |\phi\rangle = \langle \vec{k} | \frac{P^2}{2m} |\phi\rangle + \langle \vec{k} | V |\phi\rangle \]
But in this representation, the kinetic energy terms becomes

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while the potential energy term can be written

$$\langle \vec{k} | V | \phi \rangle = \int d^3 k' \langle \vec{k} | V | \vec{k}' \rangle \langle \vec{k}' | \phi \rangle$$

$$= \int d^3 k' \ V(\vec{k}, \vec{k}') \phi(\vec{k}')$$
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combining these with the expression

\[ \langle \vec{k} | E | \phi \rangle = E \phi(k) \]

which appears on the right hand side of the eigenvalue equation ...
we obtain an integral equation

\[ \frac{\hbar^2 k^2}{2m} \phi(\vec{k}) + \int d^3k' V(\vec{k}, \vec{k'}) \phi(\vec{k'}) = E \phi(\vec{k}) \]

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To put it in a more standard integral equation form, we move the integral to the right and combine the terms proportional to \( \phi(\vec{k}) \) on the left to
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To put it in a more standard integral equation form, we move the integral to the right and combine the terms proportional to \( \phi(\vec{k}) \) on the left to obtain

\[ (E - E_k) \phi(\vec{k}) = \int d^3k' V(\vec{k}, \vec{k}') \phi(\vec{k}') \quad \text{where} \quad E_k = \frac{\hbar^2 k^2}{2m} \]
we obtain an **integral equation**

\[
\frac{\hbar^2 k^2}{2m} \phi(\vec{k}) + \int d^3 k' \, V(\vec{k}, \vec{k}') \phi(\vec{k}') = E \phi(\vec{k})
\]

which represents the eigenvalue equation in the momentum representation.

To put it in a more standard integral equation form, we move the integral to the right and combine the terms proportional to \( \phi(\vec{k}) \) on the left to obtain

\[
(E - E_k) \, \phi(\vec{k}) = \int d^3 k' \, V(\vec{k}, \vec{k}') \phi(\vec{k}') \quad \text{where} \quad E_k = \frac{\hbar^2 k^2}{2m}
\]

or

\[
\phi(\vec{k}) = \frac{1}{(E - E_k)} \int d^3 k' \, V(\vec{k}, \vec{k}') \phi(\vec{k}')
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where $\tilde{V}(\vec{k})$ is “essentially” the Fourier transform of $V(\vec{r})$. 

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the negative gradient of which generates the constant force.
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whose solutions, after considerable work, can be expressed in terms of the Airy functions \( \text{Ai}(x) \).
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and integrated

\[
\int_{\phi(0)}^{\phi(x)} \frac{d\phi}{\phi} = i \int_{0}^{k} (\varepsilon - \alpha k^2) dk
\]
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To normalize these solutions, we see that, since the spectrum is continuous, **Dirac normalization** is the appropriate condition to impose.
Thus we must choose the value of $\phi(0)$ so that
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Thus, for any fixed value of $\alpha$, and for all $\varepsilon \in \mathbb{R}$, the functions

$$\phi_{\varepsilon} (k) = \frac{1}{\sqrt{2\pi}} \exp \left(-i \left[ \varepsilon k + \frac{1}{3} \alpha k^3 \right] \right)$$

represent a complete set of states $|\varepsilon\rangle$ for a particle moving in $1D$. 103
Note that these states, represented by momentum space wavefunctions

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Just as the plane waves form a perfectly good representation, even when the particle is not free.
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Sometimes, Hermitian operators do not have eigenstates that lie within the state space.

So in an infinite dimensional space, not all Hermitian operators are observables for that space.
In the next segment, we finish up our discussion of the properties of linear operators, by considering under what circumstances two different observables can have

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Sets of observables satisfying condition (3), are therefore called compatible observables.