Quantum One
Postulate Three
The Measurement Process
Postulate Three: Measurement of Quantum Observables

In the last lecture, we investigated the conditions under which two or more observables can have none, some, or a complete set of common eigenstates.

We saw that to have any common eigenstates at all, it is necessary that the commutator of the two observables have eigenvectors with eigenvalue zero.

For two operators that commute, this is automatically satisfied, and in that case we can always construct an ONB of simultaneous eigenstates common to them, or to any set of mutually commuting observables.

When we have enough commuting observables, so that the resulting basis vectors are uniquely labeled, we have found a complete set of commuting observables.

In this lecture, we begin discussing the third postulate, which describes what happens when an arbitrary observable $A$ is measured on the system when it is in an arbitrary state $|\psi\rangle$ at the time of measurement.
Postulate Three: Measurement of Quantum Observables

We first note first that since $A$ is an observable it has, by definition, a set of eigenvalues $\{a\}$ and an ONB of eigenvectors

$$\{|a, \tau\} \mid \tau = 1, \ldots, n_a$$

which satisfy orthonormality and completeness relations

$$\langle a, \tau \mid a', \tau' \rangle = \delta_{a,a'} \delta_{\tau,\tau'} \quad \sum_a \sum_{\tau=1}^{n_a} |a, \tau\rangle \langle a, \tau| = 1.$$ 

in terms of which we can expand the state on which a measurement will be made as

$$|\psi\rangle = \sum_a \sum_{\tau=1}^{n_a} |a, \tau\rangle \langle a, \tau|\psi\rangle = \sum_a \sum_{\tau=1}^{n_a} \psi_{a,\tau} |a, \tau\rangle.$$
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where the expansion coefficients are just given by the inner products

$$\psi_{a,\tau} = \langle a, \tau | \psi \rangle.$$ 

Now, the completeness relation can be written in the form

$$\sum_{a} \sum_{\tau=1}^{n_a} |a, \tau \rangle \langle a, \tau | = \sum_{a} P_a = 1$$

where, for fixed $a$,

$$P_a = \sum_{\tau=1}^{n_a} |a, \tau \rangle \langle a, \tau |$$

is the **sum of the orthogonal projectors** onto the basis vectors that span the eigenspace $S_a$ of $A$ with eigenvalue $a$.

The operator $P_a$ is thus the **total projector on to that subspace**.
Postulate Three: Measurement of Quantum Observables

We can therefore write

\[ |\psi\rangle = \sum_a P_a |\psi\rangle = \sum_a |\psi_a\rangle \]

where

\[ |\psi_a\rangle = P_a |\psi\rangle = \sum_{\tau=1}^{n_a} \psi_{a,\tau} |a, \tau\rangle \]

is just the part of the state \(|\psi\rangle\) lying in the eigenspace \(S_a\).

With these definitions, we can now state the first part of the third postulate as it applies to an observable with a discrete spectrum.
Postulate Three: Measurement of Quantum Observables

Postulate Three (a) - (Values obtained during measurement)

The only value which can be obtained as a result of measuring an observable $A$ of a quantum mechanical system at a time when the system is in the normalized state $|\psi\rangle$, is one of the eigenvalues $a$ in the spectrum of $A$.

Exactly which eigenvalue will be obtained cannot generally be predicted.

It is possible, however, to predict the probability of obtaining each eigenvalue of $A$.

The probability $P(a)$ that a measurement will yield the discrete eigenvalue $a$ of $A$ is the expectation value

$$P(a) = \langle \psi | P_a | \psi \rangle$$

taken with respect to the state $|\psi\rangle$ of the projector $P_a$ onto the eigenspace of $A$ associated with that eigenvalue.
Postulate Three: Measurement of Quantum Observables

A geometric interpretation of this postulate arises from the observation that

$$P_a = \sum_{\tau=1}^{n_a} |a, \tau\rangle \langle a, \tau| = P_a^+$$

is obviously Hermitian, and that it is a projector, so $P_a = P_a^2$. Thus, we can write

$$\langle \psi | P_a | \psi \rangle = \langle \psi | P_a P_a | \psi \rangle = \| P_a | \psi \rangle \|^2 = \| \psi_a \|^2$$

which implies that the probability to obtain the eigenvalue $a$, is just the squared length

$$P(a) = \| \psi_a \|^2$$

of that part of $|\psi\rangle$ that lies inside the eigenspace $S_a$. 
Postulate Three: Measurement of Quantum Observables

Note that, in this discrete case, if the eigenvalue $a$ is nondegenerate, so that there is only one linearly independent basis vector $|a\rangle$, then the projector

$$P_a = |a\rangle\langle a|$$

onto the subspace $S_a$ is just the projector onto this one state.

In this limit, the probability $P(a)$ reduces to

$$P(a) = \langle \psi | a \rangle \langle a | \psi \rangle = \psi_a^* \psi_a = |\psi_a|^2$$

where $\psi_a = \langle a | \psi \rangle$ is the associated expansion coefficient for the state $|\psi\rangle$ in the basis of eigenstates of $A$.

Thus, the probability reduces to the squared magnitude of the associated amplitude, exactly as we asserted in Schrödinger’s mechanics.
Postulate Three: Measurement of Quantum Observables

For degenerate eigenvalues, we don’t just get a single squared amplitude,

\[ P(a) = \langle \psi | P_a | \psi \rangle = \sum_{\tau=1}^{n_a} \langle \psi | a, \tau \rangle \langle a, \tau | \psi \rangle = \sum_{\tau=1}^{n_a} |\psi_{a,\tau}|^2 \]

we get a sum of squared amplitudes for the different basis vector associated with that eigenvalue.

In practice, this last expression is usually the way we actually calculate the probability for degenerate eigenvalues.

Find the amplitudes for each basis vector with the same eigenvalue, take their squared magnitudes, and add them up.
Postulate Three: Measurement of Quantum Observables

An important part of the interpretation of the measurement process is that the value of an observable $A$ is really not defined unless the system is in an eigenstate associated with that observable.

If it is in such an eigenstate, let us call it $|\psi_a\rangle$, then it lies entirely within the associated eigenspace $S_a$.

The act of a projector $P_a'$ on such a state will be to annihilate the state if $a \neq a'$, and to leave it alone if $a = a'$. Thus, under such circumstances, it is clear that

$$P(a') = \begin{cases} 1 & \text{if } a = a' \\ 0 & \text{if } a \neq a' \end{cases}$$

Hence the probability of obtaining the eigenvalue $a$ associated with an eigenstate having that eigenvalue is equal to unity, which is the only time that the value of the associated observable $A$ is well defined.
Postulate Three: Measurement of Quantum Observables

We now need to address the second part of the measurement postulate which describes what happens to a quantum mechanical system itself as a result of the measurement process performed upon it.

We assert that in an ideal measurement, which is one which perturbs the system as little as possible, the state of the system immediately after the measurement is one that:

1. is consistent with the particular eigenvalue obtained as a result of the measurement process, and

2. retains as much information about the state of the system immediately before the measurement as is consistent with (1).

These ideas form the basis for the following:
Postulate Three: Measurement of Quantum Observables

Postulate Three (b) (Collapse of the State Vector)

Immediately after a measurement of an observable $A$ performed on a system in the state $|\psi\rangle$ that yields the value $a$, the state of the system is left in the normalized projection of $|\psi\rangle$ onto the eigenspace $S_a$ associated with the eigenvalue measured, i.e., it is left in that part of $|\psi\rangle$ lying within $S_a$.

We schematically indicate this as follows:

$$
|\psi\rangle \quad \overset{A}{\longrightarrow} \quad \frac{P_a|\psi\rangle}{\|P_a|\psi\rangle}\n$$

Thus, nature just seems to throw away those parts of the state vector which are not consistent with the actual value obtained.
Postulate Three: Measurement of Quantum Observables

Note that this only indicates one possible branch of the change in the system during the course of the measurement process, that which occurs when the particular eigenvalue $a$ is obtained.

As we have seen, it is not possible to predict which of these branches will actually be followed by any single quantum mechanical system.

Thus, this change in the state vector during measurement is inherently non-deterministic.

\[ |\psi\rangle \xrightarrow{A} \begin{cases} P_a |\psi\rangle & \text{with probability } P(a) \\ P_{a'} |\psi\rangle & \text{with probability } P(a') \\ P_{a''} |\psi\rangle & \text{with probability } P(a'') \\ \vdots \\ \vdots \\ \vdots \\ \end{cases} \]

Presumably, his collapse of the state vector occurs as the result of an unspecified interaction with the (classical) measuring device used to measure the observable.
Postulate Three: Measurement of Quantum Observables

Extension to Continuous Eigenvalues - As with Schrödinger’s mechanics, we have initially stated this postulate in a form which assumes that the spectrum of the observable of interest is discrete.

We now discuss how the third postulate needs to be modified for the case of observables with a continuous spectrum of eigenvalues \( \{ \alpha \} \).

In this circumstance, \( A \) still has an ONB of eigenvectors \( \{ |\alpha, \tau\rangle \} \) which satisfy the Dirac orthonormality condition and the completeness relation

\[
\langle \alpha, \tau | \alpha', \tau' \rangle = \delta(\alpha - \alpha') \delta_{\tau, \tau'} \quad \int d\alpha \sum_{\tau=1}^{n_{\alpha}} |\alpha, \tau\rangle \langle \alpha, \tau| = \mathbf{1},
\]

and in terms of which we can expand the state

\[
|\psi\rangle = \int d\alpha \sum_{\tau=1}^{n_{\alpha}} |\alpha, \tau\rangle \langle \alpha, \tau| \psi\rangle = \int d\alpha \sum_{\tau=1}^{n_{\alpha}} \psi_{\tau}(\alpha)|\alpha, \tau\rangle.
\]
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where the expansion coefficients are just given by the inner products

$$\psi_i(\alpha) = \langle \alpha, \tau | \psi \rangle.$$  

Now, the completeness relation can be written in the form

$$\int d\alpha \sum_{\tau=1}^{n_\alpha} |\alpha, \tau \rangle \langle \alpha, \tau | = \int d\alpha \ \rho_\alpha = \mathbf{1},$$

where for fixed $\alpha$, the operator

$$\rho(\alpha) = \sum_{\tau=1}^{n_\alpha} |\alpha, \tau \rangle \langle \alpha, \tau |$$

is the sum of the orthogonal projector densities onto the basis vectors that span the eigenspace $S_\alpha$ of $A$ with eigenvalue $\alpha$.

It is thus the total projector density associated with that eigenvalue.
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We can therefore write

$$|\psi\rangle = \int d\alpha \rho_\alpha |\psi\rangle = \int d\alpha |\psi_\alpha\rangle$$

where

$$|\psi_\alpha\rangle = \rho_\alpha |\psi\rangle = \sum_{\tau=1}^{n_\alpha} \psi_\tau(\alpha)|\alpha, \tau\rangle$$

is just the part of the state $|\psi\rangle$ lying in the eigenspace $S_\alpha$.

Then, in describing measurements of an observable $A$ with a continuous spectrum, rather than discuss the probability of obtaining a particular eigenvalue (which is zero), we talk of the probability density $\rho(\alpha)$. 
Postulate Three: Measurement of Quantum Observables

Note that, if the eigenvalue $\alpha$ is nondegenerate, so that there is only one linearly independent basis vector $|\alpha\rangle$ then

$$\rho_\alpha = |\alpha\rangle\langle\alpha|.$$  

In this limit,

$$\rho(\alpha) = \langle\psi|\alpha\rangle\langle\alpha|\psi\rangle = |\psi(\alpha)|^2.$$  

Thus, the probability density in this case reduces to the squared magnitude of the associated wave function, exactly as we saw in Schrödinger’s mechanics.

Thus, for a single particle, we have

$$\rho(\vec{r}) = \langle\psi|\vec{r}\rangle\langle\vec{r}|\psi\rangle = |\psi(\vec{r})|^2$$ and  

$$\rho(\vec{k}) = \langle\psi|\vec{k}\rangle\langle\vec{k}|\psi\rangle = |\psi(\vec{k})|^2.$$
Postulate Three: Measurement of Quantum Observables

For degenerate eigenvalues, we don’t just get a single squared amplitude

\[ \rho(\alpha) = \langle \psi | \rho_\alpha | \psi \rangle = \sum_{\tau=1}^{n_\alpha} \langle \psi | \alpha, \tau \rangle \langle \alpha, \tau | \psi \rangle = \sum_{\tau=1}^{n_\alpha} |\psi_\tau(\alpha)|^2, \]

we get a sum of squared amplitudes for the different basis vector associated with that eigenvalue.

Note in the expressions that we have given above the index \( \tau \), which we have written as though it were discrete, can sometimes be continuous if the associated eigenvalue is infinitely degenerate.

In that case, all of the expressions above involving a sum over \( \tau \) can be modified by re-writing them in terms of an integral, e.g.,

\[ \sum_{\tau=1}^{n_\alpha} |\alpha, \tau \rangle \langle \alpha, \tau | \quad \rightarrow \quad \int d\tau \ |\alpha, \tau \rangle \langle \alpha, \tau |. \]

Postulate Three: Preparation of a Quantum Ensemble

Clearly, in order to actually test the third postulate, it is necessary to generate an ensemble of quantum mechanical systems that are all in the same physical state, since the content of that postulate involves statistical predictions that apply collectively to such an ensemble.

For internal consistency, therefore, it is necessary to specify how such an ensemble could, in principle, be generated, assuming the postulate to be valid.

To do this, assume that one starts with an arbitrary ensemble $\{|\psi\rangle\}$ of physically similar quantum systems, in which the individual elements are in arbitrary initial states, $|\psi\rangle$ which may or may not be known at the outset.

According to the postulates, it is then possible to measure compatible observables, or commuting observables, either simultaneously or sequentially over a brief period of time on each of the members of this ensemble.
Postulate Three: Preparation of a Quantum Ensemble

Suppose, therefore, the each member of the ensemble is subjected to the measurement of a complete set of commuting observables \( \{A, B, C\} \), say.

For such a set, there exists, by assumption, a complete ONB of common eigenstates \( \{|a, b, c\}\rangle \), each element of which is uniquely labeled by its eigenvalues \( a, b, c \).

For a member of our ensemble initially in a quantum state

\[
|\psi\rangle = \sum_{a', b', c'} \psi_{a', b', c'} |a', b', c'\rangle
\]

a measurement of these three compatible observables in a very short interval of time will bring about the collapse of the state vector onto just one of these basis vectors.
Postulate Three: Preparation of a Quantum Ensemble

This is schematically indicated in the following diagram

\[ |\psi\rangle \xrightarrow{A} \ldots \xrightarrow{a} \sum_{b',c'} \psi_{a,b',c'} |a, b', c'\rangle \]

\[ \xrightarrow{B} \ldots \xrightarrow{b} \sum_{c'} \psi_{a,b,c'} |a, b, c'\rangle \]

\[ \xrightarrow{C} \ldots \xrightarrow{c} \frac{\psi_{a,b,c}}{|\psi_{a,b,c}|} |a, b, c\rangle \]

where we have performed the required re-normalization in the last step, leaving the final state determined up to a phase factor

\[ e^{i\phi} = \frac{\psi_{a,b,c}}{|\psi_{a,b,c}|} \].
Postulate Three: Preparation of a Quantum Ensemble

Thus, each member of the ensemble is left, up to a phase factor, in a well-defined physical state, which is now known as a result of the measurement process.

Thus, after performing such a complete series of measurements on an ensemble of arbitrary initial state vectors, we can extract those which end up, within a phase factor, in a particular quantum state $|a, b, c\rangle$ to produce a sub-ensemble of systems upon which to perform further experiments, and thereby test the statistical predictions of the postulate.

Note that predictions of the postulate, which are all stated in terms of expectation values, are insensitive to such a phase factor. That is, if we consider two states

$$|\psi\rangle \quad \text{and} \quad |\psi'\rangle = e^{i\phi}|\psi\rangle$$

then

$$P_\psi(a) = \langle \psi | P_a | \psi \rangle,$$

while

$$P_{\psi'}(a) = \langle \psi' | P_a | \psi' \rangle = e^{-i\phi} \langle \psi | P_a | \psi \rangle e^{i\phi} = \langle \psi | P_a | \psi \rangle = P_\psi(a).$$
Postulate Three: Measurement of Quantum Observables

In this lecture, we stated the third postulate in a form that takes into account the possible degeneracy of the eigenvalues of an observable being measured.

In any measurement of an observable \( A \), the only values that can be obtained are the eigenvalues of that observable.

The third postulate gives an expression for the probability, or probability density to obtain an eigenvalue in the discrete or continuous part of the spectrum.

When a particular eigenvalue is obtained in a measurement, the state vector collapses onto the eigenspace associated with that eigenvalue. The parts associated with different eigenvalues are simply removed.

In the next lecture, we consider some examples, and discuss useful statistical implications of the third postulate.