Quantum Two
Angular Momentum and Rotations
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Eigenstates and Eigenvalues of Angular Momentum Operators
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As we will see, the process for obtaining this information is very similar to that used to determine the spectrum of the eigenstates of the harmonic oscillator.

We consider, therefore, an arbitrary angular momentum operator $\vec{J}$ whose components satisfy the relations

$$[J_i, J_j] = i \sum_k \varepsilon_{i j k} J_k$$

$$[J_i, J^2] = 0$$
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We then note, as we did for the orbital angular momentum $\vec{ℓ}$, that, since the components $J_i$ do not commute with one another, $\vec{J}$ cannot possess an ONB of eigenstates, i.e., states which are simultaneous eigenstates of all three of its operator components.
In fact, one can show that the only possible eigenstates of $\vec{J}$ are those for which the angular momentum is identically zero (an s-state, in the language of spectroscopy).
In fact, one can show that the only possible eigenstates of $\vec{J}^2$ are those for which the angular momentum is identically zero (an $s$-state, in the language of spectroscopy).

Nonetheless, since, each component of $\vec{J}$ commutes with $J^2$, it is possible to find an ONB of eigenstates common to $J^2$ and to the component of $\vec{J}$ along any single chosen direction.
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Usually the component of $\vec{J}$ along the $z$-axis is chosen, because of the simple form taken by the differential operator representing that component of orbital angular momentum in spherical coordinates.
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Note, however, that due to the cyclical nature of the commutation relations, anything deduced about the spectrum and eigenstates of $J^2$ and $J_z$ must also apply to the eigenstates common to $J^2$ and to any other component of $\vec{J}$. 
We note also, that, as with $\ell^2$, the operator $J^2 = \sum_i J_i^2$ is Hermitian and positive definite, and thus its eigenvalues must be greater than or equal to zero.
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For the moment, we will ignore other quantum numbers and simply denote a common eigenstate of $J^2$ and $J_z$ as $|j, m\rangle$. 
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\begin{align*}
J_z |j, m\rangle &= m |j, m\rangle \\
J^2 |j, m\rangle &= j(j + 1) |j, m\rangle
\end{align*}
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To proceed further, it is convenient to trade in the two components of $\vec{J}$ along the $x$ and $y$ axes for the non-Hermitian operator

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Of course when needed, we can always get back to the original operators

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J_x = \frac{1}{2} (J_+ + J_-) \quad \text{and} \quad J_y = \frac{i}{2} (J_- - J_+)
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Thus, in determining the spectrum and common eigenstates of $J^2$ and $J_z$, we will find it convenient to work with the set of operators

$$\{J_+, J_-, J_z, J^2\} \quad \text{rather than the set} \quad \{J_x, J_y, J_z, J^2\}$$
To solve this problem, we will need commutation relations for the operators in this new set.
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The commutator of $J_{\pm}$ with $J_z$ is also readily established; we find that

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[J_z, J_-] &= [J_z, J_x] - i [J_z, J_y] = iJ_y - J_x = -J_- 
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[J_z, J_-] = [J_z, J_x] - i [J_z, J_y] = iJ_y - J_x = -J_-
\]

or

\[
[J_z, J_{\pm}] = \pm J_{\pm}
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Note that these can be written in the following useful form:

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J_z J_- = J_- J_z - J_-
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J_z J_+ = J_+ J_z + J_+ = J_+ (J_z + 1)
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J_z J_- = J_- J_z - J_- = J_- (J_z - 1)
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Similarly, the commutator of \( J_+ \) and \( J_- \) is

\[
[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]
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\[
= [J_x, J_x] + i [J_y, J_x] - i [J_x, J_y] - i [J_y, iJ_y]
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\begin{align*}
[J_+, J_-] &= [J_x + iJ_y, J_x - iJ_y] \\
&= [J_x, J_x] + i [J_y, J_x] - i [J_x, J_y] - i [J_y, iJ_y] \\
&= 2J_z.
\end{align*}
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&= 2J_z.
\end{align*}

Thus the commutation relations of interest take the form

\[ [J_z, J_\pm] = \pm J_\pm \quad [J_+, J_-] = 2J_z \quad [J^2, J_\pm] = 0 = [J^2, J_z] \]
It will also be necessary in what follows to express the operator $J^2$ in terms of the new "components" $\{J_+, J_-, J_z\}$ rather than the old components $\{J_x, J_y, J_z\}$. Adding these last two results, dividing by two and adding gives the relation
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$$= J_x^2 + J_y^2 + J_z = J^2 - J_z (J_z - 1).$$

Adding these last two results, dividing by two and adding $J_z^2$ gives the relation

$$J^2 = \frac{1}{2} [J_+ J_- + J_- J_+] + J_z^2.$$
With these relations we can now proceed to deduce allowed values in the spectrum of $J^2$ and $J_z$. 

Let $\mathbf{v}$ be an arbitrary nonzero eigenvector of $J^2$ and $J_z$, where the eigenvalues of $J^2$ satisfy the inequalities

Using this, and the commutation relations, we now prove a few theorems.

1. For a given value of $J^2$, the eigenvalue must lie in the range $J^2$. To show this, consider the vectors $\mathbf{v}$ and $\mathbf{w}$ whose squared norms are $57$. 


With these relations we can now proceed to deduce allowed values in the spectrum of $J^2$ and $J_z$.

Let $|j, m\rangle$ be an arbitrary nonzero eigenvector of $J^2$ and $J_z$ with angular momentum $(j, m)$, where the eigenvalues of $J^2$ satisfy the inequalities

$$j(j + 1) \geq 0 \quad \text{with} \quad j \geq 0$$
With these relations we can now proceed to deduce allowed values in the spectrum of \( \mathbf{J}^2 \) and \( \mathbf{J}_z \).

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\[
j(j + 1) \geq 0 \quad \text{with} \quad j \geq 0
\]

Using this, and the commutation relations, we now prove a few theorems.
With these relations we can now proceed to deduce allowed values in the spectrum of $J^2$ and $J_z$.

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Using this, and the commutation relations, we now prove a few theorems.

1. For a given value of $j$, the eigenvalue $m$ must lie in the range

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To show this, consider the vectors $J_+ |j, m\rangle$ and $J_- |j, m\rangle$.
With these relations we can now proceed to deduce allowed values in the spectrum of $J^2$ and $J_z$.

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\[ ||J_+ |j, m\rangle||^2 = \langle j, m | J^- J_+ |j, m\rangle \]
With these relations we can now proceed to deduce allowed values in the spectrum of $J^2$ and $J_z$.

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Using this, and the commutation relations, we now prove a few theorems.

1. For a given value of $j$, the eigenvalue $m$ must lie in the range

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To show this, consider the vectors $J_+|j, m\rangle$ and $J_-|j, m\rangle$ whose squared norms are

\[ ||J_+|j, m\rangle||^2 = \langle j, m|J_-J_+|j, m\rangle \quad \text{and} \quad ||J_-|j, m\rangle||^2 = \langle j, m|J_+J_-|j, m\rangle \]
But we have already shown that

\[ J_- J_+ = J^2 - J_z (J_z + 1) \]
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which clearly requires \( j(j + 1) \geq m(m + 1) \).

With \( j \geq 0 \) this implies that for positive \( m \), we must have \( j \geq m \).
But we have already shown that

$$J_-J_+ = J^2 - J_z(J_z + 1)$$

so the statement

$$\|J_+j, m\| ^2 = \langle j, m| J_-J_+|j, m \rangle \geq 0$$

can be written

$$\langle j, m| J^2 - J_z(J_z + 1)|j, m \rangle = [j(j + 1) - m(m + 1)] \|j, m\|^2 \geq 0$$

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\[ \langle j, m|J^2 - J_z (J_z + 1)|j, m \rangle = [j(j + 1) - m(m + 1)] ||j, m||^2 \geq 0 \]

which clearly requires \[ j(j + 1) \geq m(m + 1) \].

With \( j \geq 0 \) this implies that for positive \( m \), we must have \( j \geq m \).

This is also clearly satisfied for negative \( m \). Thus, for any such state \(|j, m \rangle\) we have the upper bound

\[ j \geq m \]
Similarly, we have shown that

\[ J_+ J_- = J^2 - J_z (J_z - 1) \]
Similarly, we have shown that

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so the statement \( |J\_m| j, m \rangle^2 = \langle j, m |J_+ J_- | j, m \rangle \geq 0 \) can be written.
Similarly, we have shown that
\[ J_+ J_- = J^2 - J_z (J_z - 1) \]
so the statement
\[ \|J_-|j, m\| \|^2 = \langle j, m|J_+ J_-|j, m\rangle \geq 0 \]
can be written
\[ \langle j, m|J^2 - J_z (J_z - 1)|j, m\rangle \geq 0 \]
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With \( j \geq 0 \) and negative \( m \), introduce the positive quantity \( M = -m \) so that
Similarly, we have shown that

$$J_+ J_- = J^2 - J_z (J_z - 1)$$

so the statement

$$\|J_- | j, m\rangle\|^2 = \langle j, m | J_+ J_- | j, m\rangle \geq 0$$

can be written

$$\langle j, m | J^2 - J_z (J_z - 1) | j, m\rangle = [j(j + 1) - m(m - 1)] \|j, m\|^2 \geq 0$$

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$$j(j + 1) \geq m(m - 1) \ .$$

With \( j \geq 0 \) and negative \( m \), introduce the positive quantity \( M = -m \) so that

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so the statement \( ||J_-|j, m||^2 = \langle j, m|J_+J_-|j, m \rangle \geq 0 \) can be written
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which then requires \( j \geq M = -m \).
Similarly, we have shown that

$$J_+ J_- = J^2 - J_z (J_z - 1)$$

so the statement $||J_-|j, m\rangle||^2 = \langle j, m|J_+ J_-|j, m\rangle \geq 0$ can be written

$$\langle j, m|J^2 - J_z (J_z - 1)|j, m\rangle = [j(j + 1) - m(m - 1)]||j, m\rangle||^2 \geq 0$$

which clearly requires $j(j + 1) \geq m(m - 1)$.

With $j \geq 0$ and negative $m$, introduce the positive quantity $M = -m$ so that

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which then requires $j \geq M = -m$, Multiplying this last inequality by -1 reverses that inequality and gives the lower bound
Similarly, we have shown that
\[ J^+_J^- = J^2 - J_z(J_z - 1) \]
so the statement \[ ||J^-|j, m||^2 = \langle j, m|J^+_J^-|j, m \rangle \geq 0 \] can be written
\[ \langle j, m|J^2 - J_z(J_z - 1)|j, m \rangle = [j(j + 1) - m(m - 1)]||j, m||^2 \geq 0 \]
which clearly requires \[ j(j + 1) \geq m(m - 1) \].
With \[ j \geq 0 \] and negative \[ m \], introduce the positive quantity \[ M = -m \] so that
\[ j(j + 1) \geq -M(-M - 1) = M(M + 1) \]
which then requires \[ j \geq M = -m \]. Multiplying this last inequality by \(-1\) reverses that inequality and gives the lower bound
\[ m \geq -j \]
Similarly, we have shown that
\[ J_+ J_- = J^2 - J_z (J_z - 1) \]
so the statement \[ ||J_-|j, m\rangle||^2 = \langle j, m|J_+J_-|j, m\rangle \geq 0 \] can be written
\[ \langle j, m|J^2 - J_z (J_z - 1)|j, m\rangle = [j(j + 1) - m(m - 1)] |||j, m\rangle||^2 \geq 0 \]
which clearly requires \( j(j + 1) \geq m(m - 1) \).

With \( j \geq 0 \) and negative \( m \), introduce the positive quantity \( M = -m \) so that
\[ j(j + 1) \geq -M(-M - 1) = M(M + 1) \]
which then requires \( j \geq M = -m \). Multiplying this last inequality by -1 reverses that inequality and gives the lower bound
\[ m \geq -j \]
which is also obviously satisfied for any positive value of \( m \).
Combining the upper and lower bounds obtained in this way, we verify that for a given value of \( j \), any state \(|j, m\rangle\) of angular momentum \((j, m)\) has a value of \( m \) satisfying

\[
j \geq m \geq -j
\]
Combining the upper and lower bounds obtained in this way, we verify that for a given value of $j$, any state $|j, m\rangle$ of angular momentum $(j, m)$ has a value of $m$ satisfying

$$j \geq m \geq -j$$

Having narrowed the range for the eigenvalues of $J_z$, we now prove a second theorem.
Combining the upper and lower bounds obtained in this way, we verify that for a given value of $j$, any state $|j, m\rangle$ of angular momentum $(j, m)$ has a value of $m$ satisfying

$$j \geq m \geq -j$$

Having narrowed the range for the eigenvalues of $J_z$, we now prove a second theorem.

2. The vector $J_+ |j, m\rangle$ vanishes if and only if $m = j$. Otherwise, $J_+ |j, m\rangle$ is an eigenvector of $J^2$ and $J_z$ with angular momentum $(j, m + 1)$, i.e.,
Combining the upper and lower bounds obtained in this way, we verify that for a given value of \( j \), any state \( |j, m\rangle \) of angular momentum \((j, m)\) has a value of \( m \) satisfying

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\[
\text{it is an eigenvector of } J^2 \text{ with the same eigenvalue } j(j + 1), \text{ but }
\]
Combining the upper and lower bounds obtained in this way, we verify that for a given value of \( j \), any state \(|j, m\rangle\) of angular momentum \((j, m)\) has a value of \( m \) satisfying

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Having narrowed the range for the eigenvalues of \( J_z \), we now prove a second theorem.

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To show the first half of the statement, we note from our previous expression that
\[\|J_+ |j, m\rangle\|^2 = [j(j + 1) - m(m + 1)] \|j, m\|^2\]
To show the first half of the statement, we note from our previous expression that
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||J_+|j, m||^2 = [j(j + 1) - m(m + 1)]||j, m||^2
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Since \(|j, m\rangle \neq 0\), given the bounds on \(m\), it follows that \(J_+|j, m\rangle\) vanishes if and only if \(m = j\).
To show the first half of the statement, we note from our previous expression that

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Since $|j, m\rangle \neq 0$, given the bounds on $m$, it follows that $J_+ |j, m\rangle$ vanishes if and only if $m = j$.

To prove the second part, we first use the commutation relation $[J_z, J_+] = J_+$ in the form $J_z J_+ = J_+(J_z + 1)$ to write
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$$J_zJ_+|j, m\rangle = J_+(J_z + 1)|j, m\rangle = (m + 1)J_+|j, m\rangle$$
To show the first half of the statement, we note from our previous expression that
\[ \|J_+|j, m\|^2 = [j(j + 1) - m(m + 1)] \|j, m\|^2 \]

Since \(|j, m\rangle \neq 0\), given the bounds on \(m\), it follows that \(J_+|j, m\rangle\) vanishes if and only if \(m = j\).

To prove the second part, we first use the commutation relation \([J_z, J_+] = J_+\) in the form \(J_zJ_+ = J_+(J_z + 1)\) to write
\[
J_zJ_+|j, m\rangle = J_+(J_z + 1)|j, m\rangle = (m + 1)J_+|j, m\rangle
\]
showing that \(J_+|j, m\rangle\) is an eigenvector of \(J_z\) with eigenvalue \(m + 1\).
To show the first half of the statement, we note from our previous expression that

\[ ||J_+|j, m\rangle||^2 = \left[ j(j + 1) - m(m + 1) \right] ||j, m\rangle||^2 \]

Since \(|j, m\rangle \neq 0\), given the bounds on \(m\), it follows that \(J_+|j, m\rangle\) vanishes if and only if \(m = j\).

To prove the second part, we first use the commutation relation \([J_z, J_+] = J_+\) in the form \(J_z J_+ = J_+ (J_z + 1)\) to write

\[ J_z J_+|j, m\rangle = J_+ (J_z + 1)|j, m\rangle = (m + 1) J_+|j, m\rangle \]

showing that \(J_+|j, m\rangle\) is an eigenvector of \(J_z\) with eigenvalue \(m + 1\), and then observe that, because \([J^2, J_+] = 0\),

\[ 0 = [J^2, J_+]|j, m\rangle = J_+ J^2|j, m\rangle - J^2 J_+|j, m\rangle = J_+(m + 1)|j, m\rangle - (m + 1) J_+|j, m\rangle = J_+|j, m\rangle, \]

which confirms the second part of the statement.
To show the first half of the statement, we note from our previous expression that
\[ ||J_+|j, m||^2 = [j(j + 1) - m(m + 1)] ||j, m||^2 \]

Since \(|j, m\rangle \neq 0\), given the bounds on \(m\), it follows that \(J_+|j, m\rangle\) vanishes if and only if \(m = j\).

To prove the second part, we first use the commutation relation \([J_z, J_+] = J_+\) in the form \(J_zJ_+ = J_+(J_z + 1)\) to write
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To prove the second part, we first use the commutation relation \([J_z, J_+] = J_+\) in the form \(J_zJ_+ = J_+(J_z + 1)\) to write

\[ J_zJ_+|j, m\rangle = J_+(J_z + 1)|j, m\rangle = (m + 1)J_+|j, m\rangle \]

showing that \(J_+|j, m\rangle\) is an eigenvector of \(J_z\) with eigenvalue \(m + 1\), and then observe that, because \([J^2, J_+] = 0\),

\[ J^2J_+|j, m\rangle = J_+J^2|j, m\rangle = j(j + 1)J_+|j, m\rangle \]
To show the first half of the statement, we note from our previous expression that

$$||J_+ |j, m\rangle||^2 = [j(j + 1) - m(m + 1)] ||j, m\rangle||^2$$

Since $|j, m\rangle \neq 0$, given the bounds on $m$, it follows that $J_+ |j, m\rangle$ vanishes if and only if $m = j$.

To prove the second part, we first use the commutation relation $[J_z, J_+] = J_+$ in the form $J_z J_+ = J_+ (J_z + 1)$ to write

$$J_z J_+ |j, m\rangle = J_+ (J_z + 1) |j, m\rangle = (m + 1) J_+ |j, m\rangle$$

showing that $J_+ |j, m\rangle$ is an eigenvector of $J_z$ with eigenvalue $m + 1$, and then observe that, because $[J^2, J_+] = 0$,

$$J^2 J_+ |j, m\rangle = J_+ J^2 |j, m\rangle = j(j + 1) J_+ |j, m\rangle$$

showing that $J_+ |j, m\rangle$ is then also an eigenvector of $J^2$ with eigenvalue $j(j + 1)$.
We then prove a third final theorem:
We then prove a third final theorem:

3. The vector $J_- |j, m\rangle$ vanishes if and only if $m = -j$. Otherwise, $J_- |j, m\rangle$ is an eigenvector of $J^2$ and $J_z$ with angular momentum $(j, m - 1)$, i.e.,
We then prove a third final theorem:

3. The vector $J_- |j, m\rangle$ vanishes if and only if $m = -j$.

Otherwise, $J_- |j, m\rangle$ is an eigenvector of $J^2$ and $J_z$, with angular momentum $(j, m - 1)$, i.e.,

it is an eigenvector of $J^2$ with the same eigenvalue $j(j + 1)$, but
We then prove a third final theorem:

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Otherwise, $J_- |j, m\rangle$ is an eigenvector of $J^2$ and $J_z$ with angular momentum $(j, m - 1)$, i.e.,

- it is an eigenvector of $J^2$ with the same eigenvalue $j(j + 1)$, but
- it is an eigenvector of $J_z$ with eigenvalue $m - 1$ that is lower by one relative to the state $|j, m\rangle$. 
We then prove a third final theorem:

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- it is an eigenvector of $J^2$ with the same eigenvalue $j(j + 1)$, but
- it is an eigenvector of $J_z$ with eigenvalue $m - 1$ that is lower by one relative to the state $|j, m\rangle$.

To show the first half of the statement, we note from our previous expression that

$$||J_- |j, m\rangle||^2 = [j(j + 1) - m(m - 1)] ||j, m\rangle||^2$$
We then prove a third final theorem:

3. The vector $J_-|j, m\rangle$ vanishes if and only if $m = -j$.

Otherwise, $J_-|j, m\rangle$ is an eigenvector of $J^2$ and $J_z$ with angular momentum $(j, m - 1)$, i.e.,

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To show the first half of the statement, we note from our previous expression that

$$||J_-|j, m\rangle||^2 = [j(j + 1) - m(m - 1)] ||j, m\rangle||^2$$

Since $|j, m\rangle \neq 0$, given the bounds on $m$ it follows that $J_-|j, m\rangle$ vanishes if and only if $m = -j$, for which
We then prove a third final theorem:

2. The vector $J_- |j, m\rangle$ vanishes if and only if $m = -j$.

Otherwise, $J_- |j, m\rangle$ is an eigenvector of $J^2$ and $J_z$ with angular momentum $(j, m - 1)$, i.e.,

it is an eigenvector of $J^2$ with the same eigenvalue $j(j + 1)$, but it is an eigenvector of $J_z$ with eigenvalue $m - 1$ that is lower by one relative to the state $|j, m\rangle$.

To show the first half of the statement, we note from our previous expression that

$$||J_- |j, m\rangle||^2 = [j(j + 1) - m(m - 1)] ||j, m\rangle||^2$$

Since $|j, m\rangle \neq 0$, given the bounds on $m$ it follows that $J_- |j, m\rangle$ vanishes if and only if $m = -j$, for which

$$j(j + 1) - m(m - 1) = j(j + 1) - (-j)(-j - 1)$$
$$= j(j + 1) - (j)(j + 1) = 0$$
To prove the second part, we first use the commutation relation $[J_z, J_-] = -J_-$ in the form $J_z J_- = J_-(J_z - 1)$ to write

$$J_z J_- |j, m\rangle = J_-(J_z - 1)|j, m\rangle$$
To prove the second part, we first use the commutation relation \([J_z, J_-] = -J_-\) in the form \(J_z J_- = J_- (J_z - 1)\) to write

\[
J_z J_- |j, m\rangle = J_- (J_z - 1) |j, m\rangle = (m - 1) J_- |j, m\rangle
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To prove the second part, we first use the commutation relation $[J_z, J_-] = -J_-$ in the form $J_z J_- = J_-(J_z - 1)$ to write

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showing that $J_- |j, m\rangle$ is an eigenvector of $J_z$ with eigenvalue $m - 1$, and then observe that, because $[J^2, J_-] = 0$,

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showing that $J_- |j, m\rangle$ is also an eigenvector of $J^2$ with eigenvalue $j(j + 1)$.

Because of their effects on the states $|j, m\rangle$, the operator $J_+$ is referred to as the raising operator, since it acts to increase the component of angular momentum along the $z$-axis by one unit and $J_-$ is referred to as the lowering operator, since it acts to decrease it by one.
With these three theorems in hand, we now proceed to restrict even further the spectra of $J^2$ and $J_z$. 
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We note, e.g., that, given any state \( |j, m\rangle \) of angular momentum \( (j, m) \) we can produce a sequence of eigenvectors of \( J^2 \) and \( J_z \).
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$$|j, m\rangle$$

with eigenvalue

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$$\begin{align*}
|j, m\rangle & \quad J_+ |j, m\rangle & \quad J^2_+ |j, m\rangle & \quad J^3_+ |j, m\rangle & \quad \cdots \\
\quad m & \quad (m + 1) & \quad (m + 2) & \quad (m + 3) & \quad \cdots
\end{align*}$$

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|j, m\rangle \quad J_+ |j, m\rangle \quad J_+^2 |j, m\rangle \quad J_+^3 |j, m\rangle \quad \cdots
$$

with eigenvalues $m \quad (m + 1) \quad (m + 2) \quad (m + 3) \quad \cdots$

This sequence must terminate, or else produce eigenvectors of $J_z$ with eigenvalues violating the upper bound $j \geq m$. 
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with eigenvalues

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But termination can only occur when $J_+$ acts on the last nonzero vector of the sequence, $J^m_+|j, m\rangle$ with eigenvalue $m' = m + n$ say, and takes it on to the null vector.
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\[
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Similar arguments can be made for the sequence of eigenvectors of $J^2$ and $J_z$:
\[ |j, m\rangle \quad J_- |j, m\rangle \]

with eigenvalues
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\[
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Similar arguments can be made for the sequence of eigenvectors of \( J^2 \) and \( J_z \)

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|j, m\rangle \quad J_- |j, m\rangle \quad J_-^2 |j, m\rangle
\]

with eigenvalues

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To now avoid violating the lower bound $m \geq -j$, the operator $J_-$ must act on the last nonzero vector of the sequence, $J_-^{n'} |j, m\rangle$ with eigenvalue $m' = m - n'$ to take it onto the null vector.
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with eigenvalues

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Similar arguments can be made for the sequence of eigenvectors of \( J^2 \) and \( J_z \)

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Thus, there exists an integer \( n' \) such that

\[ n' = j + m \]
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If \( N \) is an even integer, then in this case, \( j \in \{0, 1, 2, \cdots\} \) and is said to be an integral value of angular momentum.
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For this situation, the results of the proceeding analysis indicate that \( m \)
must also be an integer and take on each of the \( 2j + 1 \) integer values

\[ m = 0, \pm 1, \pm 2, \ldots \pm j \]
If $N$ is an odd integer, then $j$ differs from an integer by $1/2$, i.e., it is in the set $j \in \{1/2, 3/2, 5/2, \ldots \}$ and is then said to be half-integral (short for half-odd-integral).
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For a given half-integral value of \( j \), the values of \( m \) must then take on each of the \( 2j + 1 \) half-odd-integer values \( m = \pm 1/2, \ldots , \pm j \).
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Thus, we have deduced the values of \( j \) and \( m \) that are consistent with the commutation relations.
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In particular, the allowed values of \( j \) that can occur are the non-negative integers and half-odd-integers.
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In particular, the allowed values of \( j \) that can occur are the non-negative integers and half-odd-integers.

For each value of \( j \) the eigenvectors of \( J^2 \) and \( J_z \) always come in sets, or multiplets, of \( 2j + 1 \) fold mutually-orthogonal eigenvectors

\[
\{ |j, m\rangle | m = -j, -j + 1, \cdots, j \}
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Other particles, referred to as bosons, are particles that are empirically found to have integer spins.
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