Modal Analysis of a Tight String

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See: http://web.mst.edu/~stutts/ME_Classes.html

Basic Theory

The string under tension is the simplest example of a continuous structure, but the concepts presented here are readily extensible to more complicated structures such as beams, plates, and shells. That is, to all structures which are relatively thin in at least one dimension with respect to the others. Some very important and powerful concepts will be introduced which apply to many other physical problems, such as conduction in heat transfer, diffusion problems, and problems in electromagnetism, to name a few.

Key Concepts

1. Separation of variables in partial differential equations
2. Eigenvalue problems in continuous systems.
3. Orthogonality of modes of vibration.
4. Expansion of the forced solution in terms of the homogeneous solution.

Tight String Model Derivation

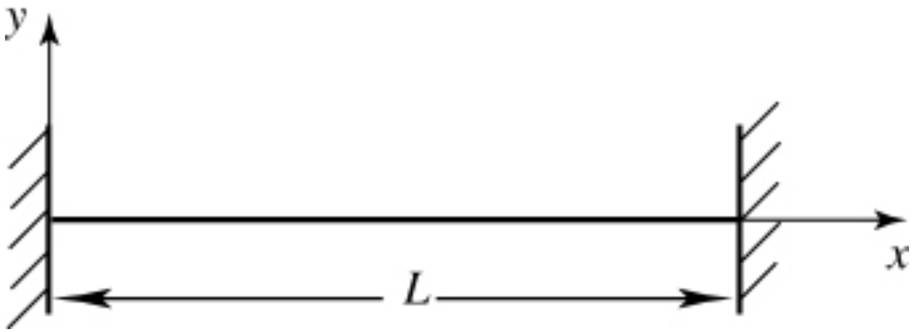

Figure 1. Geometry of tight string with fixed ends.

We will focus our attention on a string of length, L, with fixed ends as shown in Figure 1.

Free Body Diagram of String Element

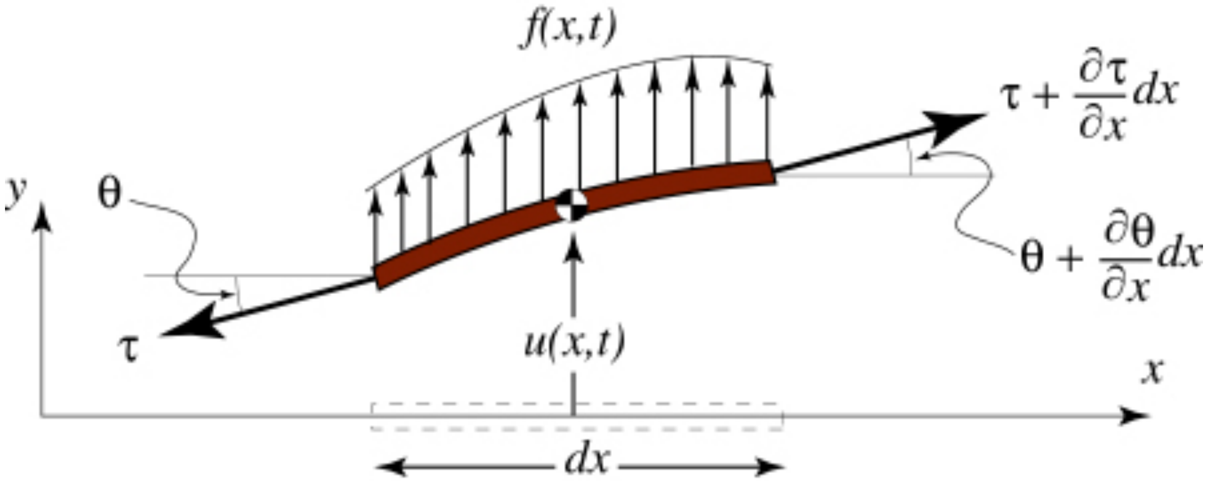

Figure 2. Free body diagram of string element.

Applying Newton' 2nd Law

For a string of density ρ(*x*) per unit length, we have by summing the forces in the *y*-direction:

$$\rho(x)dx\frac{\partial^2 u}{\partial t^2} = f(x,t)dx + \left(\tau(x) + \frac{\partial \tau}{\partial x}dx\right)\sin\left(\theta + \frac{\partial \theta}{\partial x}dx\right) - \tau(x)\sin\theta$$

$$= f(x,t)dx + \left(\tau(x) + \frac{\partial \tau}{\partial x}dx\right)\left(\sin\theta\cos\left(\frac{\partial \theta}{\partial x}dx\right) + \sin\left(\frac{\partial \theta}{\partial x}dx\right)\cos\theta\right) - \tau(x)\sin\theta \qquad (1)$$

For small θ and $\frac{\partial \theta}{\partial x} dx$,

$$\sin\theta \approx \theta \approx \tan\theta = \frac{\partial u}{\partial x}$$

$$\sin\left(\frac{\partial \theta}{\partial x} dx\right) \approx \frac{\partial \theta}{\partial x} dx = \frac{\partial^2 u}{\partial x^2} dx$$

and

$$\cos\theta \approx \cos\left(\frac{\partial \theta}{\partial x} dx\right) \approx 1$$

Hence, neglecting higher order terms (containing dx^2) the equation of motion becomes

$$\frac{\partial}{\partial x}\left(\tau(x)\frac{\partial u}{\partial x}\right) + f(x,t) = \rho(x)\frac{\partial^2 u}{\partial t^2} \qquad (2)$$

The equation of motion (the domain equation) plus the initial values and the boundary values constitutes what is known as the well-posed IBVP (initial boundary value problem). The boundary values for a fixed string are:

$$u(0,t) = u(L,t) = 0 \tag{3}$$

The initial values are determined by the initial shape and velocity distribution of the string:

$$u(x,0) = g(x) \tag{4}$$

and

$$\frac{\partial u(x,0)}{\partial t} = h(x) \tag{5}$$

Separation of Variables and the Free-Vibration Problem

The free-vibration, or eigenvalue problem, given by

$$\frac{\partial}{\partial x}\left(\tau(x)\frac{\partial u}{\partial x}\right) = \rho(x)\frac{\partial^2 u}{\partial t^2} \tag{6}$$

may be readily solved by the method of separation of variables. Letting

$$u(x,t) = U(x)T(t), \tag{7}$$

we have

$$T(t)\frac{d}{dx}\left(\tau(x)\frac{dU}{dx}\right)=\rho(x)U(x)\frac{d^2T}{dt^2} \qquad (8)$$

Dividing by $\rho(x)U(x)T(t)$ we obtain

$$\frac{1}{\rho(x)U(x)}\frac{d}{dx}\left(\tau(x)\frac{dU}{dx}\right)=\frac{1}{T(t)}\frac{d^2T}{dt^2}=-\omega^2 \qquad (9)$$

Where $-\omega^2$ is a constant. The reason for the negative sign will become obvious shortly.

The only way that a function of one variable, say x, can be equal to a function of another variable, in this case t, if for both functions be be equal to a constant. This being the case, Equation (9) may be recast as two equations:

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \qquad (10)$$

$$\frac{d}{dx}\left(\tau(x)\frac{dU}{dx}\right) + \rho(x)\omega^2 U = 0, \ \text{for } 0 \le x \le L \qquad (11)$$

Equations (10) and (11) + the initial and boundary values form the IBVP.

Equation (10) implies the expected result – that the solution will be harmonic in time. Had we chosen a positive constant in Equation (9), the temporal solution would be exponential in nature – this is clearly non-physical. Hence, the temporal solution may be written

$$T(t) = A\cos\omega t + B\sin\omega t \tag{12}$$

For simplicity, we consider the case of a tight string under constant tension (τ = const.), and with constant mass density per unit length (ρ = const.). Equation (11) becomes

$$\frac{d^2U}{dx^2} + \frac{\omega^2}{c^2}U = 0, \text{ for } 0 \le x \le L \tag{13}$$

where

$$c = \sqrt{\frac{\tau}{\rho}} \tag{14}$$

The constant c is the speed of sound in the string. Equation (13) has a general solution of the form

$$U(x) = C\cos\frac{\omega}{c}x + D\sin\frac{\omega}{c}x \tag{15}$$

To determine specific mode shape, we must apply the boundary conditions:

$$u(0,t) = U(0)T(t) = 0 \Rightarrow U(0) = 0 \tag{16}$$

$$u(L,t) = U(L)T(t) = 0 \Rightarrow U(L) = 0 \tag{17}$$

Equation (16) implies that

$$U(0) = C = 0 \Rightarrow U(x) = D\sin\frac{\omega}{c}x \tag{18}$$

From Equations (18) and (17), we have

$$U(L) = D\sin\frac{\omega}{c}L = 0 \tag{19}$$

$$\Rightarrow \sin\frac{\omega}{c}L = 0 \tag{20}$$

Hence, we must have

$$\frac{\omega L}{c} = n\pi, \text{ for } n = 1,2,3\cdots \tag{21}$$

Thus, we have that the natural frequencies occur in discrete values given by

$$\omega_n = \frac{n\pi c}{L} = \frac{n\pi}{L}\sqrt{\frac{\tau}{\rho}} \qquad (22)$$

Furthermore, the overall motion is composed of a sum of discrete modes of the form

$$U_n(x) = \sin\frac{n\pi}{L}x \qquad (23)$$

where the the unknown constant, D, has been set to unity because it is arbitrary.

Hence, the total general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right) \sin \frac{n\pi}{L} x \qquad (24)$$

The unknown constants also occur discretely, and must be determined from the initial conditions by applying the principal of orthogonality of modes.

Orthogonality Makes the World Go Around!

From the initial conditions, we have

$$u(x,0) = g(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi}{L} x \qquad (25)$$

Multiplying (25) by $\sin \frac{m\pi}{L} x$ and integrating over the

Domain yields the unknown constant, A_n, because of the following relationship:

$$\int_0^L \sin\frac{n\pi}{L}x\sin\frac{m\pi}{L}x\,dx = \begin{cases} \frac{L}{2} \text{ for } m = n \\ 0 \text{ for } m \neq n \end{cases} = \frac{L}{2}\delta_{mn} \qquad (26)$$

where δ_{mn} is the Kroneker delta. Thus, we obtain

$$A_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi}{L}x\,dx \qquad (27)$$

Similarly, from Equation (5) we obtain

$$B_n = \frac{2}{L\omega_n}\int_0^L h(x)\sin\frac{n\pi}{L}x\,dx \qquad (28)$$

Example: the Plucked Guitar String

Consider the following example of a tight string with initial conditions given by

$$u(x,0)=\begin{cases}\dfrac{2Hx}{L},\ 0\le x\le\dfrac{L}{2}\\ 2H\left(1-\dfrac{x}{L}\right),\ \dfrac{L}{2}<x\le L\end{cases}\tag{29}$$

and

$$\dot{u}(x,0)=0\tag{30}$$

The situation looks like the following:

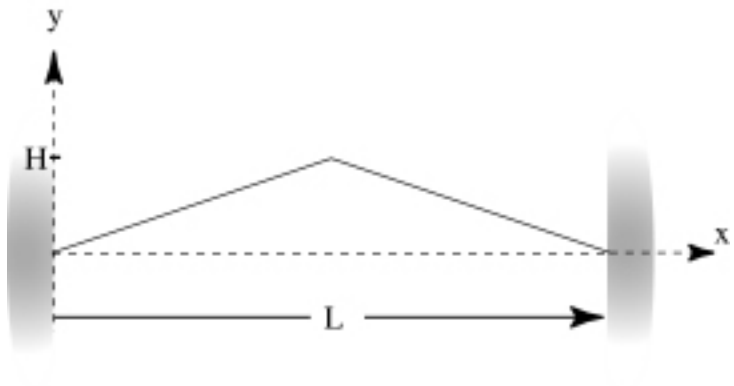

Figure 3. Initial shape of a plucked guitar string.

From Equations (27) and (29) we have

$$A_n = \frac{2}{L}\left[\int_0^{\frac{L}{2}} \frac{2Hx}{L}\sin\frac{n\pi}{L}x\,dx + \int_{\frac{L}{2}}^{L} 2H\left(1-\frac{x}{L}\right)\sin\frac{n\pi}{L}x\,dx\right] \quad (31)$$

Evaluating the integrals yields

$$A_n = \frac{8H}{n^2\pi^2}\sin\frac{n\pi}{2} = \begin{cases} 0 \text{ for } n \text{ even} \\ \dfrac{8H}{n^2\pi^2}(-1)^{\frac{n-1}{2}} \text{ for } n \text{ odd} \end{cases} \tag{32}$$

Thus,

$$u(x,t) = \frac{8H}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi}{L}x\cos\omega_n t$$

or

$$u(x,t) = \frac{8H}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}(-1)^{\frac{n-1}{2}}\sin\frac{n\pi}{L}x\cos\omega_n t \tag{33}$$

Alternatively, the dummy index, n, may be shifted to avoid the odd indicial notation:

$$u(x,t) = \frac{8H}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\frac{(2m+1)\pi}{L}x\cos\left(\frac{2m+1}{L}\right)\pi ct \qquad (34)$$

The Effect of Damping on the Response of the Plucked String

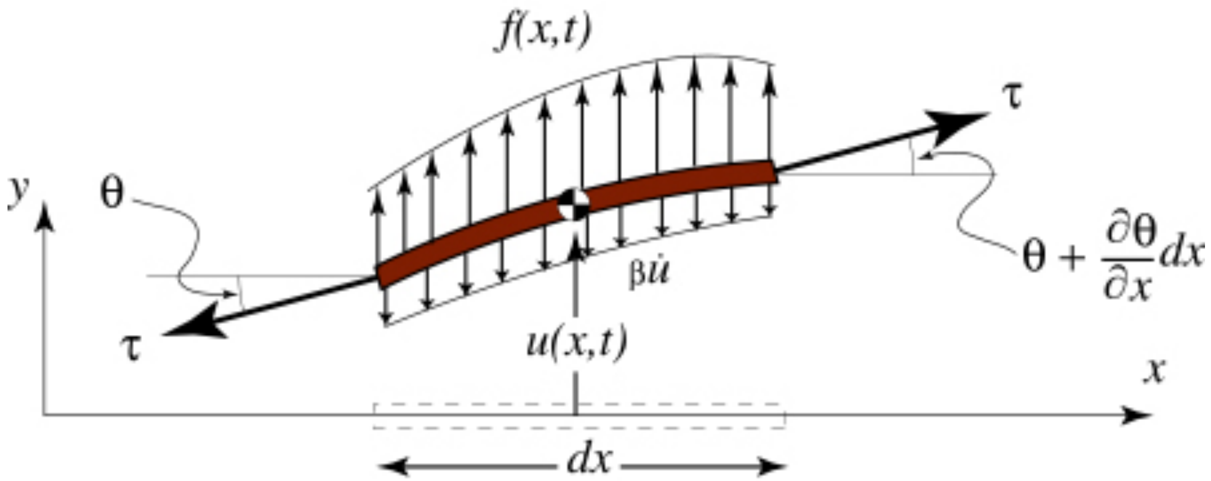

Figure 4. FBD of string with damping and constant tension.

The equation of motion tight string with constant tension and density, but including distributed damping as shown in Figure 4 may be shown to be

$$\rho \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} - \tau \frac{\partial^2 u}{\partial x^2} = f(x,t) \tag{35}$$

where β is the distributed viscous damping constant with units Newton-seconds/meter^2. We note that in Figure 4, the damping force opposes the motion of the string, so results in a negative applied force in Newton's 2nd law. We will seek a solution in terms of the previously determined eigen function of the form

$$u(x,t) = \sum_{n=1}^{\infty} \eta_n(t) \sin \frac{n\pi}{L} x \tag{36}$$

Substitution of (36) into (35) yields

$$\sum_{n=1}^{\infty}\left[\ddot{\eta}_n + \frac{\beta}{\rho}\dot{\eta}_n + c^2\left(\frac{n\pi}{L}\right)^2 \eta_n\right]\sin\frac{n\pi}{L}x = \frac{1}{\rho}f(x,t) \qquad (37)$$

where use has been made of Equation (14) and the dot notation for derivatives with respect to time.
Multiplication of (37) by

$$\sin\frac{m\pi}{L}x$$

and integration with respect to x over $(0,L)$ yields for $m = n$

$$\ddot{\eta}_n + \frac{\beta}{\rho}\dot{\eta}_n + c^2\left(\frac{n\pi}{L}\right)^2 \eta_n = \frac{2}{\rho L}\int_0^L f(x,t)\sin\frac{n\pi}{L}x\,dx \qquad (38)$$

because the left hand side of Equation (37) vanishes identically for $m \neq n$. In canonical form, (37) becomes

$$\ddot{\eta}_n + 2\zeta_n \omega_n \dot{\eta}_n + \omega_n^2 \eta_n = \frac{2}{\rho L}\int_0^L f(x,t)\sin\frac{n\pi}{L}x\,dx \tag{39}$$

where

$$\omega_n = \frac{n\pi}{l}c = \frac{n\pi}{l}\sqrt{\frac{\tau}{\rho}} \tag{40}$$

and

$$\zeta_n = \frac{\beta}{2\rho\omega_n} = \frac{\beta L}{2n\pi\sqrt{\tau\rho}} \tag{41}$$

Plucked Guitar String with Viscous Damping

As in the previous example which neglected damping, $f(x,t) = 0$, and the initial conditions are given by Equations (29) and (30). The equation for the so-called modal participation factor, Equation (39), becomes

$$\ddot{\eta}_n + 2\zeta_n\omega_n\dot{\eta}_n + \omega_n^2\eta_n = 0 \tag{42}$$

The solution may easily be shown to be

$$\eta_n(t) = e^{-\zeta_n\omega_n t}\left(A_n \cos\omega_d t + B_n \sin\omega_d t\right) \tag{43}$$

where the coefficients A_n and B_n must be determined from the initial conditions.

The frequency of oscillation is effected by damping, and is referred to as the *damped natural frequency of oscillation*, and given by

$$\omega_d = \omega_n \sqrt{1 - \zeta_n^2} \tag{44}$$

Hence, the general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} \left(e^{-\zeta_n \omega_n t} \left(A_n \cos \omega_d t + B_n \sin \omega_d t \right) \right) \sin \frac{n\pi}{L} x \tag{45}$$

Application of the initial conditions yields the same value for A_n as given in Equation (32), but the initial velocity equation yields

$$B_n = \frac{\zeta_n}{\sqrt{1 - \zeta_n^2}} A_n \tag{46}$$

Thus, the total solution may be written

$$u(x,t) = \sum_{n=0}^{\infty}\left\{\frac{8H}{\pi^2}\frac{(-1)^n}{(2n+1)^2}e^{-\zeta_n\omega_n t}\left(\cos\omega_d t + \frac{\zeta_n}{\sqrt{1-\zeta_n^2}}\sin\omega_d t\right)\right. \quad (47)$$

$$\left.\sin\frac{(2n+1)\pi}{L}x\right\}$$

where the index has again been shifted to account for the odd-only indices. In terms of the physical parameters, we note that

$$\frac{\zeta_n}{\sqrt{1-\zeta_n^2}} = \frac{\beta L}{\sqrt{4(2n+1)^2\pi^2\tau\rho - \beta^2 L^2}} \quad (48)$$

The Notes and Frequencies on a Classical Guitar

Table 1. Notes and frequencies on the classical guitar listed as: frequency (Hz)/string/fret.

Note	1st	2nd	3rd	4th
A	110/5/0	220/3/2	440/1/5	880/1/17
B	123.75/5/2	247.5/2/0	495/1/7	990/1/19
C	130/5/3	260/2/1	520/1/8	
D	146.667/4/0	293.334/2/3	586.668/1/10	
E	82.5/6/0	165/4/2	330/1/0	660/1/12
F	87.5/6/1	175/4/3	350/1/1	700/1/13
G	97.778/6/3	195.556/3/0	391.11/1/3	782.224/1/15

The guitar provides an important application for the plucked string model. The notes and their corresponding frequencies are listed in Table 1.

Equations (34) and (47) are simulated (see accompanying simulations) using data for in the the 'D' string of an acoustic guitar. The data used are as follows:

$$\tau = 69.39 \text{ N}$$

$$\rho = 0.0019206 \text{ kg/m}^3$$

$$c = 190.0 \text{ m/s}$$

$$H = 0.001 \text{ m}$$

$$L = 0.648 \text{ m}$$

$$\beta = 0.177 \text{ N - s/m}^2$$

The actual value of β for the D string is much less, (about 0.00071 N-s/m^2), but the above value was

was used to keep the size of the resulting simulation down. The D string on one of my classical guitars, although old and rather dirty, rang perceptibly for about 13 seconds. Hence, the resulting animation would be approximately 4 megabytes in size – far too large to download even from a campus computer.

The next topic to address is the last case we will cover in this study of the plucked string. We will examine how to handle a particular case of forcing. The methods shown will easily extend to other cases.

The Plucked Guitar String with Delayed Impulsive Forcing

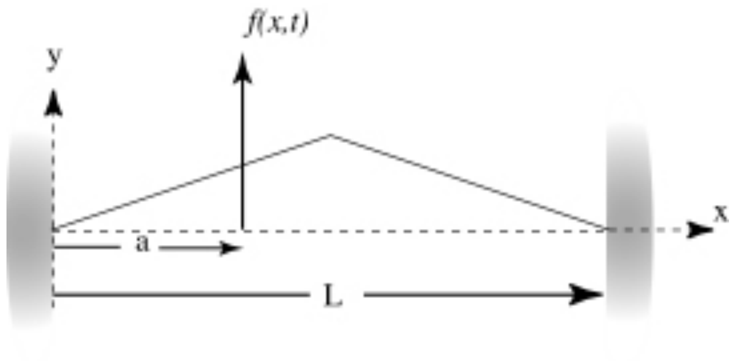

Figure 5. Plucked string with point force located at $x = a$.

The next case we will consider has an applied force within the domain (0,L). The situation is identical to the previous example, so we will start with Equation (39) and assume

$$f(x,t) = F_0\delta(x-a)\delta(t-\alpha) \qquad (49)$$

Equation (49) describes an impulsive force of magnitude, F_0, applied at a point $x = a$. This model is analogous to a piano string struck by the hammer. In our example, it might model the impact of a finger nail on the hand of a flamenco guitar player as he percusssively strums through the D string. The magnitude of the impulse, F_0, has units of N-s. That this must be the case is stems from the fact that the delta function is related to the Heaviside step function as follows:

$$\delta(x-a) = \frac{d}{dx} U(x-a) \qquad (50)$$

Hence, δ(x) has units of 1/m because *U(x)* is dimensionless. A similar relationship exists for δ(t-α):

$$\delta(t-\alpha) = \frac{d}{dt} U(t-\alpha) \qquad (51)$$

Thus, δ(t) has units of 1/t. The 1/m unit gets cancelled out during the integration process of Equation (38). The units on the right and left-hand sides of Equations (38) and (39) are m/s^2– acceleration. Hence, F_0 must have units consistent with an impulse, that is, N-s. Upon integration and the application of orthogonality, Equation (39), the modal participation factor equation becomes

$$\ddot{\eta}_n + 2\zeta_n\omega_n\dot{\eta}_n + \omega_n^2\eta_n = \frac{2}{\rho L}\int_0^L F_0\delta(x-a)\delta(t-\alpha)\sin\frac{n\pi}{L}x\,dx \qquad (52)$$

Carrying out the spatial integration, we have by the filtering property of the Dirac delta function

$$\ddot{\eta}_n + 2\zeta_n\omega_n\dot{\eta}_n + \omega_n^2\eta_n = \frac{2}{\rho L}F_0\sin\frac{n\pi}{L}a\delta(t-\alpha) \tag{53}$$

Taking the Laplace transform of Equation (53), and solving for the $\eta_n(s)$, yields

$$\eta_n(s) = \frac{\frac{2F_0}{\rho L}\sin\frac{n\pi a}{L}e^{-\alpha s}}{\left(s+\zeta_n\omega_n\right)^2+\omega_d^2} + \frac{\left(s+\zeta_n\omega_n\right)\eta_n(0)}{\left(s+\zeta_n\omega_n\right)^2+\omega_d^2}$$

$$+\frac{\dot{\eta}_n(0)+\zeta_n\omega_n\eta_n(0)}{\left(s+\zeta_n\omega_n\right)^2+\omega_d^2} \tag{54}$$

Taking the inverse Laplace transformation yields

$$\eta_n(t) = \frac{2F_0}{\rho L \omega_d} \sin\left(\frac{n\pi}{L} a\right) e^{-\zeta_n \omega_n (t-\alpha)} \sin \omega_d (t-\alpha) U(t-\alpha)$$

$$+ e^{-\zeta_n \omega_n t}\left(\eta_n(0) \cos \omega_d t + \frac{\dot{\eta}_n(0) + \zeta_n \omega_n \eta_n(0)}{\omega_d} \sin \omega_d t\right) \quad (55)$$

Applying the initial conditions, we have

$$u(x,0) = \sum_{n=1}^{\infty} \eta_n(0) \sin \frac{n\pi}{L} x = \begin{cases} \frac{2Hx}{L}, 0 \le x \le \frac{L}{2} \\ 2H\left(1 - \frac{x}{L}\right), \frac{L}{2} < x \le L \end{cases} \quad (56)$$

Multiplying Equation (56) by $\sin\frac{m\pi}{L}x$, we have for $m = n$

$$\eta_n(0) = \frac{8H}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad (57)$$

which is the same result we obtained for A_n in Equation (32). That this is so should not be surprising since the two systems have the same initial shape, which is not effected by either the damping or the forcing in this case. The applied impulse occurs at time $t = \alpha > 0$, so the response due to this forcing does not exist at time $t = 0$.

The initial velocity is zero, so we have

$$\dot{u}(x,0) = \sum_{n=1}^{\infty} \dot{\eta}_n(0) \sin \frac{n\pi}{L} x = 0 \Rightarrow \dot{\eta}_n(0) = 0 \qquad (58)$$

The validity of Equation (58) stems from the fact that the sine terms are all linearly independent – i.e. you can’t get any one of the terms in the series from a linear combination of the others. Therefore, if the sum of all terms vanishes, then each of the coefficients must vanish.

Hence, we obtain the modal participation factor

$$\eta_n(t) = \frac{2F_0}{\rho L\omega_d}\sin\left(\frac{n\pi}{L}a\right)e^{-\zeta_n\omega_n(t-\alpha)}\sin\omega_d(t-\alpha)U(t-\alpha)$$

$$+\frac{8H}{n^2\pi^2}e^{-\zeta_n\omega_n t}\left(\cos\omega_d t+\frac{\zeta_n}{\sqrt{1-\zeta_n^2}}\sin\omega_d t\right) \qquad (59)$$

The total solution is thus

$$u(x,t) = \sum_{n=1}^{\infty}\left\{\frac{2F_0}{\rho L\omega_d}\sin\left(\frac{n\pi}{L}a\right)e^{-\zeta_n\omega_n(t-\alpha)}\sin\omega_d(t-\alpha)U(t-\alpha)\right.$$

$$\left.+\frac{8H\sin\frac{n\pi}{2}}{n^2\pi^2}e^{-\zeta_n\omega_n t}\left(\cos\omega_d t+\frac{\zeta_n}{\sqrt{1-\zeta_n^2}}\sin\omega_d t\right)\right\}\sin\frac{n\pi}{L}x \qquad (60)$$