BUCKLING AND BENDING OF BEAMS SUBJECT TO A NONUNIFORM THERMAL FIELD

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Introduction

The effect of temperature on structural response is usually studied by assumption that material properties remain unaffected by variations of temperature. However, such a simplification is unacceptable for composite materials and numerous alloys subject to variations of temperature of several hundred degrees or more. Therefore, the properties of materials should be adjusted according to the actual environment.

The situation is even more confusing if temperature is nonuniformly distributed throughout the structure and the gradients are very large. Then the material properties appear to vary throughout the structure according to the thermal field. An attempt to incorporate an analytical material property-temperature relationship into analysis can be found in the recent paper of Artemyan[1] where the modulus of elasticity was taken as a linear function of temperature. However, in general, analytical relationships between material properties and temperature are not readily available. Therefore an approximation is needed to analyze the structure, either analytically or numerically.

In this paper an analytical approach to the analysis of nonuniformly heated structures is proposed. This approach is based on the representation of the actual thermal field by a combination of regions with a constant temperature within each region. Alternatively, one can assume the material properties to remain constant within certain ranges of temperature. Then the analysis within each region with constant material properties can be carried out using standard solutions. The junction conditions at the boundaries of the regions must be satisfied; this enables us to obtain a continuous solution throughout the structure. Note that the problem becomes similar to those of response of structures with piecewise distribution of stiffnesses [2,3]. A closed form solution for numerous linear elastic structures can be obtained using this approach. Here the solutions of buckling and bending problems of a prismatic beam are obtained to illustrate the application of the method.
Buckling Problem

Consider an elastic prismatic beam subject to the action of a compressive force $P$ and temperature $T(x)$, $x$ being the axial coordinate. Equation of equilibrium of the beam with the ends restrained against axial movements is

$$\frac{d^2}{dx^2} \left[ (EI) \frac{d^2w}{dx^2} \right] + \frac{d}{dx} \left[ (P + E\alpha T) \frac{dw}{dx} \right] = 0 \quad (1)$$

where $E$ is the modulus of elasticity of the material, $I$ an $A$ are the moment of inertia and the cross-sectional area respectively, $\alpha$ is the coefficient of thermal expansion and $w$ is transverse deflection. If the gradient $dT(x)/dx$ is large, the values of $E$ and $\alpha$ significantly vary along the beam. The dimensions of the beam cross-section also change due to elevated temperatures. Therefore, $I = I(x)$ and $A = A(x)$, although these effects cannot be very considerable.

Replacing the actual curve $T(x)$ by a piece-wise function

$$T = T(1), \quad T(1) = T_i \quad \text{if} \quad x_i < x \leq x_{i+1} \quad (2)$$

one obtains the equation of equilibrium for each region $(x_i, x_{i+1})$:

$$(EI) \frac{d^4w_i}{dx^4} + (P + E\alpha T)_i \frac{d^2w_i}{dx^2} = 0 \quad (3)$$

where $(\ldots)_i$ denotes the value of $(\ldots)$ at $T = T_i$ and $w_i = w(x_i < x \leq x_{i+1})$.

The solution of (3) is

$$w_i = B_{0i} + B_{1i} k_i x + B_{2i} \cos k_i x + B_{3i} \sin k_i x \quad (4)$$

where

$$k_i^2 = \frac{P + (E\alpha T)_i}{(EI)_i} \quad (5)$$

and $B_{fi}$ are constants of integration. There are $4n$ unknown constants of integration for a beam divided into $n$ regions. These constants are related to each other through 4 continuity conditions at each junction of the regions and two boundary conditions at each end of the beam. Substituting (4) into these conditions one obtains a set of $4n$ homogeneous algebraic equations with respect to $B_{fi}$. The nonzero requirement yields the buckling equation from which the relations between the compressive load and temperature resulting in buckling can
be evaluated. The accuracy of such closed form solution is limited only by an error resulting from representation of the actual thermal field \( T(x) \) by (2). Sometimes an acceptable estimate of buckling conditions can be obtained using the mode shape of buckling corresponding to uniform temperature throughout the beam. For example, if the ends are simply supported one can assume the buckling mode being

\[
    w = W \sin \frac{\pi x}{\ell}
\]

The substitution of (6) into (3) and Galerkin procedure yield

\[
    P_{cr} = \left( \frac{E I}{\ell} \right)_{mod} - (E A \alpha T)_{mod}
\]

The functions

\[
    (E I)_{mod} = \frac{2}{\ell} \sum_{i=1}^{n} (E I)_i \sin^2 \frac{\pi x_i}{\ell} \, dx
\]

\[
    (E A \alpha T)_{mod} = \frac{2}{\ell} \sum_{i=1}^{n} (E A \alpha T)_i \sin^2 \frac{\pi x_i}{\ell} \, dx
\]

can be easily evaluated in the form

\[
    (E I)_{mod} = \sum_{i=1}^{n} [(E I)_i f(i)]
\]

\[
    (E A \alpha T)_{mod} = \sum_{i=1}^{n} [(E A \alpha T)_i f(i)]
\]

\[
    f(i) = \frac{x_{i+1} - x_i}{\ell} - \frac{1}{2\pi} (\sin \frac{2\pi x_i}{\ell} - \sin \frac{2\pi x_{i+1}}{\ell})
\]

**Bending Problems**

The problem of bending of prismatic beams subject to a nonuniform temperature \( T(x) \) and lateral load \( q(x) \) can be treated similar to the previous problem. The equilibrium equation (3) is replaced by

\[
    (E I)_i \frac{d^4 w_i}{dx^4} + (E A \alpha T)_i \frac{d^2 w_i}{dx^2} = q_i
\]
where \( q_i = q(x) \) at \( x_i < x \leq x_{i+1} \)

The solution of (10) depends on the functions \( q_i(x) \). For example if

\[
q_i(x) = q_{i0} + q_{i1}x,
\]

\[
w_i = C_{o1} + C_{i1}k_1x + C_{21} \cos k_1x + C_{31} \sin k_1x +
\]

\[
\frac{q_{i0}x^2}{2(E\alpha T)_i} + \frac{q_{i1}x^3}{6(E\alpha T)_i}
\]

(11)

Using the boundary and continuity conditions one can evaluate constants of integration and determine the elastic curve of the beam.

If the beam is simply supported the solution can be sought in the form

\[
w = \sum W_j \sin \frac{j\pi x}{\ell}
\]

(12)

which upon the substitution into (10) and application of Galerkin procedure yields the set of equations with respect to \( W_j \). The \( n \)-th equation of this set is

\[
\sum [\sum (\frac{j\pi}{\ell})^4 w_j f(i,j,n)(EI)_i] - \sum [\sum (\frac{j\pi}{\ell})^2 w_j f(i,j,n)(E\alpha T)_i] =
\]

\[
\frac{2}{n\pi} \sum q_i \cos \frac{n\pi x_{i+1}}{\ell} - \cos \frac{n\pi x_{i}}{\ell}
\]

(13)

where

\[
f(i,j,n) = \frac{\ell}{(j-n)\pi} \left[ \sin \frac{(j-n)\pi x_{i+1}}{\ell} - \sin \frac{(j-n)\pi x_{i}}{\ell} \right] -
\]

\[
\frac{\ell}{(j+n)\pi} \left[ \sin \frac{(j+n)\pi x_{i+1}}{\ell} - \sin \frac{(j+n)\pi x_{i}}{\ell} \right]
\]

(14)

Numerical Example

Consider buckling load of a prismatic simply supported beam subject to a nonuniform temperature which can be schematically represented by two steps: 900°F in the left half-span and 500°F in the right half-span. The ratio of the moduli of elasticity at 70°F, 500°F and 900°F is 1:0.92:0.84. Other properties are unaffected by temperature.
The buckling load calculated using (7,9) is

\[ P_{cr} = 0.880 \left( \frac{\pi}{L} \right)^2 EI - 1.216(\text{EA}_T) \quad (15) \]

where \( E = E(70^\circ F) \) and \( T = 500^\circ F \).

The load calculated by (7,9) without corrections of material properties (modulus of elasticity in this example), i.e. assuming \((EI)_i = EI\), \((\text{EA}_T)_i = \text{EA}_T\) would be

\[ P_{cr} = \left( \frac{\pi}{L} \right)^2 EI - 1.4(\text{EA}_T) \quad (16) \]

where the values of \( E \) and \( T \) are the same as in (15).

The comparison of (15) and (16) illustrates that the failure to include effect of temperature on material properties can result in significant errors even if the distribution of temperature is satisfactory modelled.

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References

