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Dynamic Stability of Long Cylindrical Sandwich Shells and Panels Subject to Periodic-in-time Lateral Pressure

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ABSTRACT: The paper presents an analysis of dynamic stability of long cylindrical sandwich shells and shallow panels subject to a uniform periodic lateral pressure. The solution is obtained using the Sanders shell theory by assumption that the shell or panel remains in the state of plane strain during both steady-state and perturbed motion. The steady-state motion of a shell is axisymmetric, while perturbed vibrations superimposed on the steady response are asymmetric. The analysis of perturbed motion is reduced to specifying the conditions of stability of the Mathieu equation. Subsequently, the criteria of dynamic stability and the boundaries of the regions of unstable motion in the pressure amplitude–pressure frequency plane are immediately available. A shallow panel subjected to hydrodynamic pressure experiences forced vibrations. However, these vibrations can become unstable. Dynamic stability of such vibrations is investigated through the solution of the linearized equations for perturbed motion. It is shown that these equations can be reduced to a system of Mathieu equations.

KEY WORDS: dynamic stability, cylindrical shell, cylindrical panel, sandwich.

INTRODUCTION

The problems of dynamic stability of structures subjected to periodic-in-time loads have been investigated since the first work published by Beliaev in 1924 [1]. Originally,
the studies were limited to columns, but eventually, dynamic stability of thin-walled structures, i.e. plates and shells, received significant attention. The monograph of Bolotin [2] illustrated the solution of a typical problem for isotropic structures. The review papers on this subject published by Hsu [3] and Simitses [4] contain additional material on relevant problems, including dynamic stability of shell structures.

Dynamic stability of isotropic cylindrical shells was first investigated by Vol’mir [5], Bolotin [2] and Roth and Klosner [6]. The first paper on dynamic stability of anisotropic shells was published by Markov [7]. The studies of dynamic stability of composite cylindrical shells were published by Cederbaum [8], Schokker et al. [9], and Ng et al. [10].

The study of dynamic stability of cylindrical sandwich shells and panels has to be conducted accounting for transverse shear effects. This implies that the classical shell theory has to be replaced with a first-order or higher-order shear deformation version. In the present paper, the solution of dynamic stability problem is obtained for a long cylindrical sandwich shell or panel subject to a periodic-in-time uniform lateral dynamic pressure. In this case, the cross sections can be assumed to deform without warping, i.e. the shell remains in the state of plane strain.

The steady-state motion of the shell is axisymmetric, but the superimposed unstable motion is asymmetric. The solution obtained in the paper results in closed-form relationship between the frequency and amplitude of the applied pressure that correspond to the boundaries of the regions of dynamic instability.

In addition to the analysis of dynamic stability of a long cylindrical sandwich shell, the problem of stability of forced vibrations of a long shallow cylindrical sandwich panel is formulated. As shown in the paper, this problem can be reduced to the analysis of stability of solutions of a system of Mathieu equations.

ANALYSIS OF DYNAMIC STABILITY OF A LONG CYLINDRICAL SANDWICH SHELL

Consider a long cylindrical sandwich shell subject to a lateral periodic-in-time uniform pressure. The solution of the dynamic stability problem is obtained by the following assumptions:

1. The shell is sufficiently long to assume that it remains in the state of plane strain both during the steady-state motion and during the perturbed motion.
2. The facings of the shell are in the state of plane stress, while transverse shear stresses are carried by the core.
3. The amplitudes of the perturbed motion that is superimposed on the steady-state vibrations are small. Accordingly, the equations describing this motion can be obtained by a linearization of the corresponding equations where the total motion is compiled of the steady-state axisymmetric and perturbed asymmetric contributions. This means that the solution establishes the boundaries of the instability regions, while the amplitude of unstable vibrations is not determined. This amplitude could be found from the solution of nonlinear equations of perturbed motion.
4. The shell theory employed in the analysis represents a first-order shear-deformable version of the “improved first approximation” Sanders shell theory [11].

Equations of motion based on the Sanders theory are obtained in this paper from the equations developed in the paper of Birman and Simitses [12]. The equations of motion for
a long shear-deformable shell subject to periodic external pressure of frequency $\omega$ are:

\[
N_{y,yy} - \frac{N_y}{R} \left( \frac{v}{R} - w_{,y} \right) + \frac{M_{y,yy}}{R} + q_0 \cos \omega t = m\ddot{v}
\]

\[
Q_y = M_{y,y} - I\ddot{\psi}_y
\]

\[
M_{y,yy} = \frac{N_y}{R} - \left[ N_y \left( \frac{v}{R} - w_{,y} \right) \right], y + q \cos \omega t = m\ddot{w}
\]

where $N_y$ and $M_y$ are circumferential stress resultant and stress couple, respectively, $Q_y$ is a transverse shear stress resultant acting in the planes parallel to the shell axis, $y$ is the circumferential coordinate, $v$ and $w$ are circumferential and radial displacements of the shell, $\psi_y$ is the angle of rotation of an infinitesimal element about the longitudinal $x$-axis, and $R$ is the shell radius. The terms $q$ and $q_0$ denote pressure components perpendicular to the underformed shell surface and the projection of pressure in the circumferential direction of the undeformed shell, respectively. Note that the loading case considered in this paper corresponds to hydrodynamic pressure, i.e. pressure is directed perpendicular to the shell surface at each time instant. The mass of the shell per unit surface area is denoted by $m$, while the rotational inertia coefficient is

\[
I = \int_{-h/2}^{h/2} \rho z^2 \, dz
\]

where $h$ is the thickness of the shell, $z$ is the coordinate counted from the shell’s middle surface in the thickness direction, and $\rho = \rho(z)$ is the mass density of the material.

Note that Equation (1) describes the motion of a shell that is sufficiently long to justify the assumption that it deforms into a cylindrical surface, i.e. the shell remains in the state of plane strain. Accordingly, the derivatives with respect to the axial ($x$) coordinate as well as the displacements and inertia terms in the corresponding direction are omitted.

The strains corresponding to a nonlinear version of the Sanders theory [13] are

\[
\varepsilon_y^0 = v_{,y} + w \ddot{R} + \frac{1}{2} \left( w_{,y} - \frac{v}{R} \right)^2
\]

\[
\kappa_y = \psi_{y,yy}
\]

\[
\gamma_{yz} = \psi_{y} + w_{,y} - \frac{v}{R}
\]

where $\varepsilon_y^0$, $\kappa_y$, and $\gamma_{yz}$ are the middle surface extensional strain in the circumferential direction, the curvature change in the plane $x = \text{constant}$, and the transverse shear strain in the same plane, respectively.

The constitutive relations are presented here for the case of a shell that is symmetrically laminated with respect to its middle surface:

\[
N_y = A\varepsilon_y^0
\]

\[
M_y(v, w) = D\kappa_y
\]

\[
Q_y = A_{44}\gamma_{yz}
\]

where $A$ and $D$ are the extensional and bending stiffnesses of the shell in the circumferential direction determined from standard equations. The transverse shear
extensional stiffness is obtained by assumption that the facings are in the state of plane stress, i.e. the core is the only element contributing to this stiffness:

\[
A_{44} = k \int_{-h_c/2}^{h_c/2} Q_{44} \, dz
\]  

(5)

In (5), \(h_c\) is the thickness of the core, \(k\) is the shear correction factor, and \(Q_{44} = Q_{44}(z)\) is the transverse shear stiffnesses of the core in the circumferential direction.

As discussed above, dynamically unstable asymmetric motion is superimposed on axisymmetric steady-state vibrations. If all derivatives with respect to the \(y\)-coordinate are omitted in (1) and the circumferential displacements as well as rotations \(\psi_y\) are set equal to zero, the equation of the axisymmetric motion becomes

\[
m \ddot{w}_0 + \frac{N_y^0}{R} = q \cos \omega t
\]

(6)

where \(w_0\) denotes the axisymmetric radial deflection (breathing mode) and the circumferential stress resultant is \(N_y = A(w_0/R)\).

The solution of (6) corresponding to the steady-state motion is, of course,

\[
w_0 = \frac{q}{(A/R^2) - m\omega^2}\cos \omega t
\]

(7)

The natural frequency of free axisymmetric vibration is

\[
\omega_0 = \sqrt{\frac{A}{mR^2}}
\]

(8)

It is convenient to introduce the nondimensional coefficient

\[
f = \frac{1}{1 - (\omega/\omega_0)^2}
\]

(9)

The dynamic circumferential stress resultant generated during the axisymmetric motion can be written in the form

\[
N_y^0 = qRf \cos \omega t
\]

(10)

Now consider dynamic instability that is characterized by the motion with a number of half-waves in the circumferential direction. The state of plane strain that is assumed in the shell implies that it is possible to use Equation (1) where

\[
N_y = N_y^0 + N'_y
\]

\[
M_y = M'_y
\]

\[
w = w_0 + w'
\]

\[
v = v'
\]

\[
\psi_y = \psi'_y
\]

(11)
In these equations, the primed quantities denote the perturbed asymmetric motion associated with dynamic instability.

Upon linearization, the equations of motion (1) become

\[
N'_{y,y} - q f \left( \frac{v'}{R} - w'_{,y} \right) \cos \omega t + \frac{M'_{y,y}}{R} + \delta q \beta \cos \omega t = m \ddot{v}'
\]

\[
Q'_y = M'_{y,y} - I \ddot{\psi}'_y
\]

\[
M'_{y,y} - N'_{y} \frac{v'}{R} - q f R \left( \frac{v'}{R} - w'_{,y} \right) \cos \omega t + \delta q \cos \omega t = m \ddot{w}'
\]

where \( N'_y, M'_y, \) and \( Q'_y \) are given in terms of buckling deformation by (4) and the linear version of (3) accounting for the perturbed asymmetric motion:

\[
\varepsilon^0_y = \psi_{y,x} + \frac{w'}{R}
\]

\[
\kappa'_y = \psi'_{y,y}
\]

\[
\gamma'_{y,z} = \psi'_{y} + w'_{,y} - \frac{v'}{R}
\]

The terms \( \delta q \) and \( \delta q \beta \) represent the differences between the projections of the dynamic pressure acting perpendicular to the surface of the vibrating shell on the radial and tangential axes of the static configuration and the hydrodynamic pressure state (\( q = \) applied radial pressure, \( q \beta = 0 \)). It can be shown that subject to the assumption of small deformations, the corresponding terms are [14]:

\[
\delta q = 0
\]

\[
\delta q \beta = -q \left( w_{,y} - \frac{v}{R} \right)
\]

Now the equations of motion become

\[
N'_{y,y} - q (f - 1) \left( \frac{v'}{R} - w'_{,y} \right) \cos \omega t + \frac{M'_{y,y}}{R} = m \ddot{v}'
\]

\[
Q'_y = M'_{y,y} + I \ddot{\psi}'_y
\]

\[
M'_{y,y} - N'_{y} \frac{v'}{R} - q f R \left( \frac{v'}{R} - w'_{,y} \right) \cos \omega t = m \ddot{w}'
\]

Let

\[
w' = W(t) \sin \frac{ny}{R}
\]

\[
v' = V(t) \cos \frac{ny}{R}
\]

\[
\psi'_y = F(t) \cos \frac{ny}{R}
\]
where \( n \) is an integer representing the number of half-waves in the dynamic instability mode shape.

The substitution of (16) into (4), (13), and (12) yields the equations of motion in terms of displacement functions of time:

\[
\mathbf{C} \ddot{\mathbf{f}}, t + (\mathbf{E} - \alpha \mathbf{A} - \beta \mathbf{B} \cos \omega t) \mathbf{f} = 0
\]  

(17)

where the matrices and vectors of the third order are

\[
\mathbf{C} = \begin{bmatrix}
  m & 0 & 0 \\
  0 & I & 0 \\
  0 & 0 & m
\end{bmatrix}
\]

\[
\mathbf{E} - \alpha \mathbf{A} = \begin{bmatrix}
  \left(\frac{n}{R}\right)^2 A & \left(\frac{n}{R}\right)^2 D & -\frac{n}{R^2} A \\
  -\frac{A_{44}}{R} & A_{44} + \left(\frac{n}{R}\right)^2 D & \frac{n}{R} A_{44} \\
  -\frac{n}{R^2} A & -\left(\frac{n}{R}\right)^3 D & \frac{A}{R^2}
\end{bmatrix}
\]

(18)

\[
-\beta \mathbf{B} = \begin{bmatrix}
  \frac{q}{R} (f - 1) & 0 & -\frac{qn}{R} (f - 1) \\
  0 & 0 & 0 \\
  -\frac{qnf}{R} & 0 & \frac{qfn^2}{R}
\end{bmatrix}
\]

\[
\mathbf{f} = \begin{bmatrix}
  V \\
  F \\
  W
\end{bmatrix}
\]

Note that the notation for the matrices in (18) is chosen to comply with standard notation in literature on dynamic instability [2].

In the case where we are concerned with static buckling, \( f = 1 \), and the buckling equation is available from (18) in the form

\[
\begin{vmatrix}
  \left(\frac{n}{R}\right)^2 A & \left(\frac{n}{R}\right)^2 D & -\frac{nA}{R^2} \\
  -\frac{A_{44}}{R} & A_{44} + \left(\frac{n}{R}\right)^2 D & \frac{nA_{44}}{R} \\
  -\frac{n}{R} \left(\frac{A}{R} + q\right) & -\left(\frac{n}{R}\right)^3 D & \frac{1}{R} \left(\frac{A}{R} + qn^2\right)
\end{vmatrix} = 0
\]  

(19)
The buckling pressure for a shear-deformable long shell is available from (19):

\[ q_{cr} = -\frac{D(n^2 - 1)}{R^3(1 + (D/A_{44})(n/R)^2)} \]

Note that the buckling pressure given by (20) is similar to the result obtained by Smith and Simitses [14] for a circular ring subject to a hydrostatic pressure. The mode shape of instability corresponds to \( n = 2 \), as is also the case for a ring. If the shear stiffness of the structure is infinite (thin or classical shell theory), \( A_{44} = 1 \), and Equation (20) yields the classical ring solution.

The squared natural frequency of the shell is also available from (18). In particular, if the circumferential and rotational inertias are neglected, the frequency equation is

\[
\begin{aligned}
\begin{vmatrix}
(n/R)^2 A & (n/R)^2 D & -nA \\
-A_{44} & A_{44} + (n/R)^2 D & nA_{44} \\
-n/R^2 A & -(n/R)^3 D & A/R^2 - m\omega^2
\end{vmatrix}
&= 0 \\
\end{aligned}
\]

The solution of (21) yields

\[ \omega_n^2 = \frac{(n^2 - 1)^2 DA A_{44}}{mR^4[(n/R)^2 DA + AA_{44} + DA_{44}/R^2]} \]

If \( n^2 D \ll A_{44} R^2 \) and \( D \ll A R^2 \), the frequency given by (22) is simplified:

\[ \omega_n = \frac{(n^2 - 1)}{R^2} \sqrt{\frac{D}{m}} \]

Note that this frequency coincides with the solution available from the thin-shell version of the theory where according to the Sanders and Love shell theories \( \psi_j = -w, \alpha = v/R \).

A comprehensive treatment of stability of the solutions of (17) has been considered by a number of investigators. The classical treatise on the subject was published by Bolotin [2]. According to the approach outlined in this monograph, the solution can be sought in the form

\[ \bar{f} = e^{st} \left[ \frac{\bar{b}_0}{2} + \sum_{k=1}^{\infty} (\bar{a}_k \sin k\omega t + \bar{b}_k \cos k\omega t) \right] \]

where \( s \) is a complex number and \( \bar{b}_0, \bar{a}_k, \bar{b}_k \) are constant vectors.

The substitution of (24) into (17) and the requirement of a nontrivial solution yields

\[
\begin{vmatrix}
(s^2 - \omega^2) \bar{C} + \bar{E} - \alpha \bar{A} & -\frac{\beta \bar{B}}{2} & 2s \bar{C} \omega \\
-\beta \bar{B} & s^2 \bar{C} + \bar{E} - \alpha \bar{A} & 0 \\
-2s \bar{C} \omega & 0 & (s^2 - \omega^2) \bar{C} + \bar{E} - \alpha \bar{A}
\end{vmatrix}
= 0
\]
According to Bolotin [2], the solutions of (25) are stable, if the numbers $s$ found from (25) are such that for all of them $\text{Re}(s)$ are nonpositive.

In addition to the regions of dynamic instability determined according to the Bolotin’s procedure outlined above, additional combination resonances can be found as was shown, for example, by Yakubovich and Starzhinskii [15]. However, considering a relatively high damping in sandwich structures and the well-known effect of damping on the instability regions, it is possible to concentrate on the principal instability region ($n = 2$). This is due to the fact that higher instability regions experience a relatively larger effect of damping. Therefore, it is necessary to apply a larger parametric load to realize instability in these regions than in the case of the principal region. Sandwich structures being relatively stiff, a large load amplitude that may be needed to realize higher regions of instability is likely to cause the loss of strength prior to the instability phenomenon. Therefore, it is logical to concentrate on the principal region of instability, as is done in this study.

If the analysis is limited to the principal instability region, the boundaries of the corresponding region can approximately be determined from [2]

$$\left| \frac{E - \alpha A + \beta B}{2} - \frac{C \omega^2}{4} \right| = 0$$

The substitution of the matrices given by (18) into (26) yields a relationship between the amplitude of the applied pressure and the frequency corresponding to the boundaries of the principal instability region for the prescribed mode shape of instability ($n$). Note that the inertia terms associated with the circumferential and rotational modes of motion are often neglected in the analysis of transverse (radial) motion. Accordingly, let the matrix

$$\overline{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m \end{bmatrix}$$

Then the following equation can be used to specify the boundaries of the principal instability region for the case where the excitation frequency is of the same order as the fundamental frequency of the shell vibrating forming a cylindrical surface (plane strain state in the cross sections):

$$\left| \begin{bmatrix} \left( \frac{n}{R} \right)^2 A + \frac{q(f - 1)}{2} \\ -A_{44} \\ - A R + \frac{q f}{2} \end{bmatrix} - \begin{bmatrix} \left( \frac{n}{R} \right)^2 D \\ A_{44} + \left( \frac{n}{R} \right)^2 D \\ (A \pm \frac{q f n^2}{2}) \end{bmatrix} \right| = 0$$

The boundary frequencies defining the principal instability regions can now be expressed in terms of the amplitude of the applied pressure as

$$\frac{m \omega^2}{4} = \pm k \frac{q}{l} + k_0$$

$$\pm r \frac{q}{l} + r_0$$
The terms at the second power of pressure $q$ in the equation obtained from (28) cancel each other. The coefficients in (29) are given by

$$k_1 = \frac{1}{2R^2} (n^2 - 1) \left[ f(n^2 - 1) + 1 \right] A \left[ \left( \frac{n}{R} \right)^2 D + A_{44} \right]$$

$$k_0 = (\frac{n}{R})^2 (n^2 - 1)^2 \frac{DA_{44}}{R^4}$$

$$r_1 = \frac{(f - 1)}{2R} \left[ \left( \frac{n}{R} \right)^2 D + A_{44} \right]$$

$$r_0 = \left( \frac{n}{R} \right)^2 \left[ \left( \frac{n}{R} \right)^2 DA + AA_{44} + \frac{DA_{44}}{R^2} \right]$$

(30)

Note that if the amplitude of the applied dynamic pressure approaches zero, the frequency corresponding to the “origin” of the instability region is equal to twice the natural frequency of free vibrations with the same mode shape. This immediately follows from the comparison of (26) where $\beta \bar{B} = 0$ and the free frequency equation

$$| \bar{E} - \alpha \bar{A} - \bar{C} \omega^2 | = 0$$

(31)

From (29), the free vibration frequency corresponding to the mode shape described by (16) is

$$m \omega_n^2 = \frac{k_0}{r_0}$$

(32)

Neglecting the terms $(n/R)^2 DA$ and $DA_{44}/R^2$, as compared to the term $AA_{44}$ in the denominator of this equation yields the frequency for a thin sandwich shell given by (23). It is noted that such simplification is justified in most practical configurations if the number of half-waves in the circumferential direction, $n$, is small. As discussed above, the most dangerous principal dynamic instability region is anticipated in the vicinity of excitation frequencies equal to twice the fundamental frequency of the shell $(n = 2)$. Accordingly, in this case, we can use simplified expressions for the corresponding coefficients in (30).

In the static case, $f = 1$. The stability equation is

$$| \bar{E} - \alpha \bar{A} - \bar{B} | = 0$$

(33)

This equation coincides with (19) and yields the same result for the critical pressure that can also be presented as

$$q_{cr} = q_{cr}(n) = -\frac{k_0}{2k_1'}$$

(34)

where $k_1' = k_1(f = 1)$.

Accordingly, using a nondimensional pressure defined by

$$\bar{q}_n = \frac{q}{q_{cr}(n)}$$

(35)
one obtains the expression for the boundaries of the principal instability region

\[
\left( \frac{\omega}{\omega_n} \right)^2 = 4 \left( \frac{1 \pm (k_1/2k'_1) \tilde{q}_n}{1 \pm (r_1/r_0)(k_0/2k'_1) \tilde{q}_n} \right)
\] (36)

Note that (36) cannot be immediately used to evaluate the relationships between the amplitude of the applied pressure and its frequency corresponding to the boundaries of the instability regions. This is because the coefficients \(k_1\) and \(r_1\) depend on \(f\) that in turn, depends on the frequency of motion. Therefore, it is more convenient to rewrite (36) defining the pressure as a function of the driving frequency:

\[
\tilde{q}_n = \frac{4 - (\omega/\omega_n)^2}{2(k_1(\omega)/k'_1) - (r_1(\omega)/r_0)(k_0/2k'_1)(\omega/\omega_n)^2}
\] (37)

If \(f \approx 1\), as is the case for a thin shell, \(k'_1 = k_1\) and \(r_1 = 0\). Then, as follows from (36),

\[
\left( \frac{\omega}{\omega_n} \right)^2 = 4 \left( 1 \pm \frac{1}{2} \tilde{q}_n \right)
\] (38)

Equation (38) corresponds to straight lines representing the boundaries of the corresponding principal region of instability on the squared driving frequency–amplitude of applied pressure plane. In thin sandwich configurations, the natural frequency and the critical pressure are given by Equation (23) and \(q_{cr} = -D(n^2 - 1)/R^3\), respectively. If the simplifying assumptions are unacceptable, i.e. all terms have to be accounted for in the corresponding solutions (37), the equations for the natural frequency and critical pressure are replaced with (32) and (20).

ANALYSIS OF A LONG CYLINDRICAL SANDWICH PANEL SUBJECTED TO HYDRODYNAMIC PERIODIC-IN-TIME PRESSURE

In this section we illustrate the solution of the dynamic stability problem for a long cylindrical sandwich panel subjected to a periodic pressure \(q \cos \omega t\). The pressure is assumed to remain perpendicular to the surface of the vibrating panel. The width of the panel is equal to \(b\); the circumferential coordinate is \(y\).

The solution consists of two phases. In the first phase, the linear forced vibrations of the panel are analyzed. This analysis is based on the linear version of Equation (1) where the tangential component of the applied pressure is omitted in the first equation by assumption that the panel is shallow:

\[
N_{y,y} + \frac{M_{y,y}}{R} = m\ddot{y}
\]

\[
Q_y = M_{y,y} - \frac{N_{y}}{R}
\]

\[
M_{y,yy} - \frac{N_y}{R} + q \cos \omega t = m\ddot{w}
\] (39)

Note that the assumption of a shallow panel introduced above is equivalent to the assumption that the load remains parallel to its original direction during vibrations [16].
If this assumption is violated, the first Equation (39) would include the terms dependent on deformations multiplied by the applied pressure, i.e. the motion would be generated by a combination of forced and parametric excitations.

The linear expressions for the stress resultants and stress couple are

\[ N_y = A \left( v_y + \frac{w}{R} \right) \]
\[ M_y = D \psi_y, y \]
\[ Q_y = A_{44} \left( \psi_y + w_y - \frac{v}{R} \right) \]

The solution of the forced vibration problem is sought in the form of series that satisfy the boundary conditions of simple support along the edges \( y = 0 \) and \( y = b \):

\[ v = \sum_n V_n \cos \omega t \cos \frac{n\pi y}{b} \]
\[ \psi_y = \sum_n F_n \cos \omega t \cos \frac{n\pi y}{b} \]
\[ w = \sum_n W_n \cos \omega t \sin \frac{n\pi y}{b} \]

The substitution of (41) into (40) and (39) yields a system of three algebraic equations for each combination of harmonics amplitudes \( \{V_n, F_n, W_n\} \):

\[
\begin{bmatrix}
\left( \frac{n\pi}{b} \right)^2 A - m\omega^2 \\
- \frac{A_{44}}{R} & A_{44} + \left( \frac{n\pi}{b} \right) D - i\omega^2 & - \frac{n\pi A}{b} R \\
- \frac{n\pi A}{b} & - \left( \frac{n\pi}{b} \right)^3 D & A - m\omega^2
\end{bmatrix}
\begin{bmatrix}
V_n \\
F_n \\
W_n
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
q_n
\end{bmatrix}
\]

where \( q = \sum_n q_n \sin n\pi/b \).

The stability of steady-state forced vibration can be evaluated by considering the system of nonlinear equations of perturbed motion. This system can be obtained if we consider the motion of the panel including steady-state and perturbation components: \( \{v + v'(t), \psi_y + \psi'_y(t), w + w'(t)\} \) where \( v'(t), \psi'_y(t), \) and \( w'(t) \) are functions of time. If these functions increase with time, the motion is unstable. The motion is assumed to remain independent of the longitudinal coordinate, i.e. the state of plane strain in the central part of the panel, away from the curved edges is preserved.

Our goal being to specify the conditions for dynamic instability, rather than the motion and stresses corresponding to the unstable motion, the equations of perturbed motion can be linearized with respect to small perturbations yielding

\[ N'_y, y - \frac{N'_y}{R} \left( \frac{v}{R} - w_y \right) - \frac{N_y}{R} \left( \frac{v'}{R} - w'_y \right) + \frac{M'_{y, y}}{R} = m\ddot{v} \]
\[ Q'_y = M'_{y, y} - i\dot{\psi}_y \]
\[ M'_{y, y} - \frac{N'_y}{R} - \left[ N_y \left( \frac{v}{R} - w_y \right) + N_y \left( \frac{v'}{R} - w'_y \right) \right] , y = m\ddot{w}' \]
In the above equations, the linearized perturbed motion contributions to the stress resultants and stress couple are

\[ N'_y = A \left( y' + \frac{w'}{R} \right) \]
\[ M'_y = D \psi'_{,y} \]
\[ Q'_y = A_{44} \left( \psi'_y + w'_{,y} - \frac{v'}{R} \right) \]

The solution is obtained by the assumption that the perturbed motion can be represented in the form of the following series that satisfy the boundary conditions:

\[ v' = \sum_i V'_i(t) \cos \frac{n \pi y}{b} \]
\[ \psi'_y = \sum_i F'_i(t) \cos \frac{n \pi y}{b} \]
\[ w' = \sum_i W'_i(t) \sin \frac{n \pi y}{b} \]

The substitution of the steady-state solution (41) and the perturbed motion (45) into the constitutive equations and equation of motion (43) and the application of the Galerkin procedure yield a system of Mathieu equations that can be written in the form

\[ C' f'_{,tt} + \left( E' - \alpha A' - \beta B' \cos \omega t \right) f' = 0 \]

where

\[ f' = \{ V'_1 F'_1 W'_1 \ldots W'_n \}^T \]

is the vector of the corresponding functions of time in the series (45). The expressions for the matrices \( C' \), \( E' - \alpha A' \), and \( \beta B' \) are omitted for brevity. However, it should be noted that the elements of the latter matrix depend on the amplitudes of the steady-state motion.

The stability of the system of equations (46) can be investigated similar to that of the system of equations (17). Note that if the panel is relatively narrow, the series (41) and (45) can be reduced to just one-term series, i.e. \( n = 1 \). In this case, the analytical solution of the problem can easily be obtained.

**NUMERICAL EXAMPLES**

Principal regions of dynamic instability were determined for long cylindrical sandwich shells with quasi-isotropic glass/epoxy facings formed from QC-8700 short-fiber composite (www.quantumcomposites.com). This material chosen for its excellent fatigue resistance consists of chopped 1’’ long glass fibers with the volume fraction equal to 46\% (weight fraction of fibers is equal to 63\%). The elastic modulus and the Poisson ratio...
of this material are equal to 26.9 GPa and 0.386, respectively. The Balsa core has the modulus of elasticity equal to 280.8 GPa and the shear modulus equal to 108.0 MPa. The mass densities of the facings and core are 1870 kg/m³ and 24 kg/m³, respectively.

The boundaries of the principal dynamic instability region obtained by the assumption that the shell is thin are shown in Figures 1 and 2. As follows from these figures, an increase in the thickness of the core results in a significant increase of the excitation frequencies corresponding to the principal region of dynamic instability. The amplitude of the applied dynamic pressure in these figures should be compared to the static buckling pressure to better interpret the results. The latter pressure is given in Table 1 for all cases considered in the figures. As follows from this table, the shell can remain dynamically

![Figure 1](image-url)

**Figure 1.** The boundaries of principal regions of dynamic instability for long cylindrical sandwich shells with the middle surface radius \( R = 1.0 \) m and 5-mm thick facings. The mode shape of instability corresponds to \( n = 2 \).

![Figure 2](image-url)

**Figure 2.** The boundaries of principal regions of dynamic instability for long cylindrical sandwich shells with the middle surface radius \( R = 2.0 \) m and 5-mm thick facings. The mode shape of instability corresponds to \( n = 2 \).
stable, even if the amplitude of dynamic pressure exceeds the static buckling value. On the other hand, dynamic instability is possible, even at the amplitude of pressure that is much smaller than the static buckling pressure. It should be noted, however, that at small pressure amplitudes, dynamic instability may not be realized due to structural damping. The width of the principal instability regions in Figures 1 and 2 depend on the thickness of the core. However, a wider instability region corresponding to a thinner core should not be understood as an indicator of a larger vulnerability of the shell to dynamic instability. As shown in Table 1, the static buckling pressure of the shells with a thinner core is much smaller than that of the counterparts with a thick core. Accordingly, the diapasons of the applied pressure amplitudes in Figures 1 and 2 include the values that are much higher than the static buckling value for 10-mm core shells. In the contrary, the shells with a thicker core, whose instability regions are shown in these figures, have the static buckling pressure that is higher than the largest amplitude values in Figures 1 and 2.

As follows from the comparison of Figures 1 and 2, more shallow shells with a larger radius have much lower frequencies corresponding to the principal dynamic instability region and these regions become wide at a much lower amplitude of the applied pressure. The latter fact is easily understood, if one observes that the static buckling pressure of shallow shells is much lower than that of the shells with a smaller radius, as reflected in Table 1.

It is interesting to compare results generated using Equations (37) and (38). As indicated above, the latter equation is applicable if \( f \approx 1 \). This condition is realized for principal regions of dynamic instability considered in the paper. For example, the largest deviation from \( f = 1 \) for the cases considered in Figure 1 was in the case of a 30-mm core and it was equal to 1.5% (\( f = 1.015 \) for \( q = 125 \) KPa and \( \omega = 468.21 \) l/s). In Figure 2, the maximum deviation from \( f = 1 \) was also observed for a 30-mm core. This deviation was equal to 0.4% (\( f = 1.004 \) for \( q = 15 \) KPa and \( \omega = 116.61 \) l/s). Accordingly, a difference between the boundaries of the principal instability regions calculated by (37) and (38) is negligible. However, this difference becomes much more pronounced, if the radius of the shell is small or if higher instability regions occurring at much higher excitation frequencies are considered.

The effect of the thickness of the facings on the boundaries of the principal instability regions is considered in Figure 3. As follows from this figure, thicker facings do not significantly increase the frequencies corresponding to dynamic instability. The region of frequencies corresponding to dynamic instability is much wider for the shell with 2-mm

<table>
<thead>
<tr>
<th>Figure</th>
<th>( R ) (m)</th>
<th>( h_f ) (mm)</th>
<th>( h_c ) (mm)</th>
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facings. However, this is explained by a lower static buckling pressure of such shell (see Table 1). Similar to the observation made regarding the cases considered in Figures 1 and 2, a difference between the results obtained by Equations (37) and (38) was negligible in this figure.

CONCLUSIONS

This paper presents the analysis of dynamic stability of long cylindrical sandwich shells subject to a periodic-in-time dynamic lateral pressure. The analysis is conducted by assumption that the shell deforms into a cylindrical surface, i.e. it remains in the state of plane strain. This assumption is acceptable, in case where the shell is sufficiently long (cylindrical tube). The analysis is conducted by the first-order shear-deformable version of the Sanders shell theory. The steady-state motion of the shell is axisymmetric, while the perturbed motion associated with dynamic instability is asymmetric.

The solution is obtained for an arbitrary number of half-waves in the circumferential direction in the mode shape of dynamically unstable motion. Numerical results are generated for the principal region of instability characterized by two half-waves, i.e. the same shape as that of static buckling. Numerical results for higher regions of dynamic instability are not shown in this paper since the realization of these regions require a much larger amplitude of the applied dynamic pressure necessary to overcome damping. Considering a relatively high damping in sandwich structures, it may be beneficial to analyze its effect on dynamic instability identifying the minimum amplitude of the applied pressure making this phenomenon possible.

In addition to the problem of dynamic stability of long cylindrical shells, the stability of steady-state forced vibrations of long shallow cylindrical panels subjected to a periodic hydrodynamic pressure is investigated. The steady-state motion occurs with finite amplitudes and the frequency of the applied pressure. The stability of this motion can be
determined by the analysis of perturbed vibrations. The condition of dynamic instability of the steady-state motion is reduced to the analysis of a system of Mathieu equations whose coefficients depend on the amplitudes of the steady-state vibrations.

The results obtained in the paper confirm that dynamic instability of a long cylindrical sandwich shell is possible at the amplitude of the applied dynamic pressure that can be much lower than the static buckling value. On the other hand, the shell can remain dynamically stable, even if the amplitudes of pressure significantly exceed the static buckling pressure. If the shell is shallow, i.e. it has a large radius, the frequencies corresponding to the principal region of dynamic instability are much lower than in counterparts with a smaller radius. An increase in the thickness of the core results in a significant “shift” of the principal instability region to higher frequencies. Thicker facings produce a similar but much less pronounced effect.

Future work on dynamic stability of cylindrical sandwich shells and panels should address a number of issues. Some of them include the effect of damping on the realization of instability regions at low excitation amplitudes, dynamic wrinkling of facings, inelastic effects, and geometrically nonlinear motion of a dynamically unstable shell or panel (this issue is particularly important to predict whether the structure will collapse or experience fatigue damage).

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