DYNAMIC STABILITY OF REINFORCED COMPOSITE CYLINDRICAL SHELLS IN THERMAL FIELDS

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The dynamic stability of thin, multi-layered composite shells reinforced by axial and ring stiffeners and subjected to pulsating loads acting in the axial direction is considered. The presence of a thermal field is shown to affect the location of dynamic instability regions in the load amplitude–frequency plane. The boundaries of these regions can be evaluated as functions of static critical loads, natural frequencies, and thermal terms; the latter can be easily calculated. This permits straightforward conclusions to be drawn regarding dynamic stability of shells for which the buckling and free vibration problems have already been solved.

1. INTRODUCTION

Problems of dynamic stability of unreinforced composite cylindrical shells have been intensively studied. Thin shells were first considered by Markov [1]. Further studies of this subject were pursued both in the U.S.S.R. [2–5] as well as in the West [6]. The problem was considered thoroughly in the recent book of Bogdanovich [7]. Viscoelastic action was considered in references [8, 9]. Effects of shear deformability of thick composite shells on their dynamic stability were investigated by Bogdanovich [10] and by Bert and Birman [11].

Temperature can significantly affect dynamic behavior of composite shells [12, 13]. However, the influence of temperature on parametric vibrations of composite cylinders, let alone reinforced shells, has not been considered. In this paper the dynamic stability of composite circular cylindrical shells reinforced in both the axial and circumferential directions is studied. The effect of non-uniform thermal fields on stability is taken into consideration. In the first part of the paper, governing equations for heated composite shells subject to axial pulsating loading are derived. In the second part, the boundaries of the instability regions are obtained as functions of critical static loads, natural frequencies of the shell and a thermal term. The latter term can be found for particular temperature distributions and boundary conditions. Therefore, dynamic stability can be analyzed if buckling and free vibration solutions at normal temperature (i.e., neglecting thermal expansion) are known.

2. GOVERNING EQUATIONS

Consider an orthotropic cylindrical shell formed from a large number of layers and...
reinforced by axial and ring stiffeners. Deformations of the shell being small, linear analysis is employed in the paper. The stiffeners in the mutually perpendicular directions are assumed to deform independently. Such an assumption is used, even if unspecified, in the majority of studies of reinforced shells, although some attempts to account for interaction of stiffeners via Poisson's effect are also known [14]. The equations of motion derived in this paper are based on a generalization of Donnell's theory of isotropic shells.

The shell is subjected to the action of axial pulsating forces uniformly distributed around the circumference and to a non-uniform thermal field. The strain-displacement relationships are given by

\[
\begin{align*}
\varepsilon_x^0 &= u_x, & \varepsilon_y^0 &= v_y - w/R, & \gamma_{xy}^0 &= u_y + v_x, \\
\kappa_x &= -w_{xx}, & \kappa_y &= -w_{yy}, & \kappa_{xy} &= -2w_{xy},
\end{align*}
\]  

(1)

where \(u, v, w\) are the axial, circumferential and radial displacements, respectively, \(x\) and \(y\) denote the axial and circumferential co-ordinates, \(\varepsilon_x^0, \varepsilon_y^0\) and \(\gamma_{xy}^0\) are strains in the middle surface, \(\kappa_x, \kappa_y\) and \(\kappa_{xy}\) are the changes of curvatures and twist, and \(R\) is the middle surface radius.

The in-surface strains at a distance \(z\) from the middle surface are

\[
\begin{align*}
\varepsilon_x &= \varepsilon_x^0 + z\kappa_x, & \varepsilon_y &= \varepsilon_y^0 + z\kappa_y, & \gamma_{xy} &= \gamma_{xy}^0 + z\kappa_{xy},
\end{align*}
\]  

(2)

In equations (1) and (2) positive directions of the radial displacement \(w\) as well as the co-ordinate \(z\) are inward. The stresses in the \(k\)th layer of the shell subjected to a temperature distribution \(T = T(x, y, z)\) measured from the strain-free temperature are the following functions of the strains:

\[
\begin{align*}
\sigma_x &= E_x(\varepsilon_x - \alpha_x T), & \sigma_y &= E_x(\varepsilon_y - \alpha_y T), \\
\tau_{xy} &= E_x(\gamma_{xy} - \alpha_{xy} T),
\end{align*}
\]  

(3)

Here \(Q_{ij}\) are transformed reduced stiffnesses for the layer and \(\alpha_{1k}, \alpha_{2k}\) and \(\alpha_{6k}\) are coefficients of thermal expansion. The stresses in the stiffeners in the axial and circumferential directions are

\[
\sigma_x = E_s(\varepsilon_x - \alpha_x T), \quad \sigma_y = E_s(\varepsilon_y - \alpha_y T),
\]  

(4)

correspondingly. In equations (4), \(E_s\) and \(E_x\) are the moduli of elasticity in the stiffener directions, and \(\alpha_s\) and \(\alpha_x\) are the respective coefficients of thermal expansion. Typically, \(\alpha_s = \alpha_x\) and \(E_s = E_x\).

Stress resultants and stress couples in the shell can be obtained by integration of equations (3) and (4) in the thickness direction (\(z\)):

\[
\begin{align*}
N_x &= A_{11} \varepsilon_x^0 + A_{12} \varepsilon_y^0 + \sum_s \delta(y - y_s)E_s A_s(\varepsilon_x^0 + z_s^0 \kappa_x) - N_x^T, \\
N_y &= A_{12} \varepsilon_x^0 + A_{22} \varepsilon_y^0 + \sum_r \delta(x - x_r)E_r A_r(\varepsilon_y^0 + z_r^0 \kappa_y) - N_y^T, \\
N_{xy} &= A_{66} \gamma_{xy}^0 - N_{xy}^T, & M_x &= D_{11} \kappa_x + D_{12} \kappa_y + \sum_s \delta(y - y_s)E_s A_s z_s^0 \varepsilon_x^0 + I_s \kappa_x) - M_x^T, \\
M_y &= D_{12} \kappa_x + D_{22} \kappa_y + \sum_r \delta(x - x_r)E_r A_r z_r^0 \varepsilon_y^0 + I_r \kappa_y) - M_y^T, \\
M_{xy} &= D_{66} \kappa_{xy} - \frac{1}{2} \sum_s \delta(y - y_s)G_s J_s w_{xy} - \frac{1}{2} \sum_r \delta(x - x_r)G_r J_r w_{xy} - M_{xy}^T,
\end{align*}
\]  

(5)

where

\[
(A_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z^2) \, dz
\]  

(6)
are extensional and bending stiffnesses, $\delta(x - x_c)$ and $\delta(y - y_c)$ are Dirac delta functions, and $x_c$ and $y_c$ are co-ordinates of the cross-sectional centroids of the ring and axial stiffeners, respectively. The cross-sectional areas of stiffeners are denoted by $A_r$ and $A_s$, and their moments of inertia about the middle surface of the shell are $I_r$ and $I_s$. The distances from the centroids of the stiffeners to the middle surface are $z_{r0}$ and $z_{s0}$. The terms with Dirac functions in $M_{xy}$ account for torsional stiffness of the reinforcements; $G_r$ and $G_s$ are their rigidity moduli, and $J_r$ and $J_s$ are torsional constants. Note that the terms containing $A_1, A_2, D_1, D_2, A_{1s}, D_{1s}$ are neglected in the expressions for stress resultants and stress couples. This simplification should be justified by comparison of extensional and bending stiffness except for the cases of antisymmetric laminates, cross-ply symmetric laminates, and composites formed from isotropic layers, where $A_1 = A_2 = D_1 = D_2 = 0$.

Thermal terms in equations (5) are given by

$$\begin{align*}
N_x^T &= \int_{-h/2}^{h/2} \left( Q_{11} \alpha_{1k} + Q_{12} \alpha_{2k} + Q_{16} \alpha_{6k} \right) T \, dz + \sum_{x} \delta(y - y_i) \int_{-h/2}^{h/2} E_r \alpha_r \beta_r(z) \, T \, dz, \\
N_y^T &= \int_{-h/2}^{h/2} \left( Q_{12} \alpha_{1k} + Q_{22} \alpha_{2k} + Q_{26} \alpha_{6k} \right) T \, dz + \sum_{x} \delta(x - x_i) \int_{-h/2}^{h/2} E_r \alpha_r \beta_r(z) \, T \, dz, \\
N_{xy}^T &= \int_{-h/2}^{h/2} \left( Q_{16} \alpha_{1k} + Q_{26} \alpha_{2k} + Q_{66} \alpha_{6k} \right) T \, dz, \\
M_x^T &= \int_{-h/2}^{h/2} \left( Q_{11} \alpha_{1k} + Q_{12} \alpha_{2k} + Q_{16} \alpha_{6k} \right) Tz \, dz + \sum_{x} \delta(y - y_i) \int_{-h/2}^{h/2} E_r \alpha_r \beta_r(z) \, Tz \, dz, \\
M_y^T &= \int_{-h/2}^{h/2} \left( Q_{12} \alpha_{1k} + Q_{22} \alpha_{2k} + Q_{26} \alpha_{6k} \right) Tz \, dz - \sum_{x} \delta(x - x_i) \int_{-h/2}^{h/2} E_r \alpha_r \beta_r(z) \, Tz \, dz, \\
M_{xy}^T &= \int_{-h/2}^{h/2} \left( Q_{16} \alpha_{1k} + Q_{26} \alpha_{2k} + Q_{66} \alpha_{6k} \right) Tz \, dz.
\end{align*}$$

In equations (7) the integrals $\int_{z_r}$ and $\int_{z_s}$ are taken through the depth of the corresponding stiffener, and $\beta_r(z)$ and $\beta_s(z)$ denote the widths of the stiffeners. Equations of motion of the shell subjected to pulsating axial loads of intensity $N_{xt} = N_{yt}(t)$, where $t$ is time, and to a static load $N^0_x$ are

$$\begin{align*}
N_{xx} + N_{xy,y} &= 0, \quad N_{xy,x} + N_{yy} = 0, \\
M_{xx} + 2M_{xy,y} + M_{yy} + (N_x/R) + (N_x'w_x + N_yw_y)_x \\
+ (N_{xx}w_{xx} + N_{yy}w_{yy})_y - (N_x + N^0_x)w_{xx} &= \rho_e h w_{xx}, \\
M_{xy} + \rho_e h w_{xy} &= \rho_e h w_{xy},
\end{align*}$$

where

$$\rho_e h = \rho h + \sum_{s} \delta(y - y_i) \rho_s A_s + \sum_{r} \delta(x - x_i) \rho_r A_r.$$  \hspace{1cm} (9)

In equation (9) $\rho$, $\rho_s$ and $\rho_r$ denote densities of the materials of the shell and of the axial and ring stiffeners, respectively and the in-surface, coupling and rotatory inertias have been neglected. Substitution of equations (5) into equations (8) yields the following set of equations of motion where only linear terms are retained:

$$\begin{align*}
&\left[ A_{11} + \sum_{x} \delta(y - y_i) A_x \right] u_{xx} + A_{66} u_{yy} + (A_{12} + A_{66}) v_{xy} - (A_{12}/R) w_x - \sum_{x} \delta(y - y_i) A_x z_{x0}^0 w_{xxx} = N_{xx}^T + N_{xy,y}^T, \\
&- \sum_{x} \delta(y - y_i) E_r \alpha_r x_{s0}^0 w_{xxx} = N_{xy,x}^T + N_{yy,x}^T.
\end{align*}$$
(A_{12} + A_{66}) u_{xy} + \left[ A_{22} + \sum_r \delta(x - x_r) E_r A_r \right] v_{ry} + A_{66} v_{xx} + \delta(x - x_r) E_r A_r z_0^0 w_{yy} = N_{x,x}^T + N_{x,y}^T,

\rho_{eq} \left[ \frac{(N_x + N_x^0) w_{xx}}{R} + D_{11} w_{xxxx} + 2(D_{12} + 2D_{66}) w_{xyxy} + D_{22} w_{yyyy} \right] \left[ -\frac{(A_{12}/R) u_x}{R} - \frac{(A_{22}/R) v_y}{R} + \frac{(A_{22}/R^3) w}{R} \right] + \sum_r \delta(x - x_r) \left[ G_r I_r w_{xyy} + E_r I_r w_{yyy} \right] + E_r A_r \left[ -\left( v_{ry} / R \right) + \left( w / R \right) - \left( 2z_0^0 w_{yyy} / R \right) - z_0^0 w_{xyy} \right] + \sum_y \delta(y - y_r) \left[ G_j I_j w_{xxy} + E_j I_j w_{xxx} - E_j A_j z_0^0 u_{xxx} \right] + M_{xxx}^T + 2M_{xxy,y}^T + M_{yyy}^T + (N_x^T / R) + (N_y^T w_x + N_y^T w_y)_x + (N_x^T w_x + N_y^T w_y)_y = 0. \quad (10)

A non-linear version of equations (10) was obtained by Birman by assuming that torsional stiffnesses of reinforcements are negligible [15]. In the present paper non-linear effects are neglected, since it is concerned with the distribution of instability regions rather than determining the motion within these regions. However, torsional stiffness of reinforcements is taken into consideration, since it may be significant for certain geometries.

3. DYNAMIC STABILITY ANALYSIS

If the motion of the shell and the thermal field are represented by

\begin{align*}
  u &= U(t) f_u(x) \bar{f}_u(y), \quad v = V(t) f_v(x) \bar{f}_v(y), \\
  w &= W(t) f_w(x) \bar{f}_w(y), \quad T = T_0 f_T(x) \bar{f}_T(y) T(z),
\end{align*}

\quad (11)

where \( f_i(x) \) and \( \bar{f}_i(y) \) are modal functions which satisfy all of the appropriate boundary conditions, the substitution of expressions (11) into the first two of equations (10) and use of the Galerkin procedure yield

\begin{align*}
  U &= U_1(W) + U_2(T_0), \quad V = V_1(W) + V_2(T_0).
\end{align*}

\quad (12)

The functions \( U_i \) and \( V_i \) can be evaluated for particular boundary conditions. Note that \( U_2 \) and \( V_2 \) represent static functions which do not affect dynamic stability of the shell in a linear problem. Therefore, for the purpose of the present analysis only \( U_1 \) and \( V_1 \) have to be retained. Similarly, the contribution of the terms \( M_{x,xx}^T + 2M_{x,xy,xy}^T + M_{y,yy,y}^T + N_y^T / R \) in the third of equations (10) can be disregarded.

Substitution of equations (12) and the expressions for \( w \) and \( T \) given in equations (11) into the third of equations (10) and use of the Galerkin procedure result in a single differential equation for \( W(t) \):

\begin{align*}
  W_{,tt} + \Omega^2 (1 - \bar{N}_x - \bar{N}_{xt} - \bar{N}_T) W = 0.
\end{align*}

\quad (13)

In equation (13) \( \Omega \) is a natural frequency of the unloaded shell vibrating with the mode shape given by equations (11). The terms \( \bar{N}_x \) and \( \bar{N}_{xt} \) are non-dimensional static and pulsating loads, and \( \bar{N}_T \) is a dimensionless thermal term given by

\begin{align*}
  (\bar{N}_x, \bar{N}_{xt}, \bar{N}_T) = (N_x^0, N_{xt}, N^T) / N_{xcr},
\end{align*}

\quad (14)
where $N_{cr}$ is the static critical load corresponding to the chosen modal functions. The thermal term $N^T$ is given by

\[ N^T = \int_0^L \int_0^{2\pi R} \left\{ N_x^T \frac{\partial f_w(x)}{\partial x} \cdot \tilde{f}_w(y) + N_y^T \frac{\partial \tilde{f}_w(y)}{\partial y} \right\} f_w(x) \tilde{f}_w(y) \, dx \, dy \times \rho_{eq} h \int_0^L \int_0^{2\pi R} f_w^2(x) \tilde{f}_w^2(y) \, dx \, dy \]  

(15)

If a dynamic load is represented by

\[ \tilde{N}_{st} = \tilde{N} \cos \omega t \]  

(16)

equation (13) is a Mathieu equation. The behavior of solutions of Mathieu equations depends on relationships between the coefficients [16, 17]. If the equation is represented in the form

\[ W_{st} + \Omega_1^2(1 - 2\mu \cos \theta t) W = 0, \]  

(17)
the boundaries of the principal, second and third dynamic instability regions can be approximately defined in the $\Omega_1 - \mu$ plane by the following relationships [17]:

\[ \theta = 2\Omega_1 \sqrt{1 \pm \mu} \quad \text{(principal region)}, \]  

(18)

\[ \theta = \Omega_1 \sqrt{1 \pm \frac{\mu^2}{2}}, \quad \theta = \Omega_1 \sqrt{1 - 2\mu^2} \quad \text{(second region)}, \]  

(19)

\[ \theta = \frac{3}{2} \Omega_1 \sqrt{1 - 9\mu^2/(8 + 9\mu)} \quad \text{(third region)}. \]  

(20)

Note here that the applicability of equations (18)-(20) is limited to small values of the excitation parameter $\mu$. In the problem considered here

\[ \mu = \tilde{N} / (1 - \tilde{N}_r - \tilde{N}^T). \]  

(21)

The relationships between the static load, temperature and the amplitude of the pulsating force are usually such that $\mu < 1$, in which case equations (18)-(20) can be used. The comparison of equations (13), (16) with equations (17)-(20) yields the following boundaries of the instability regions:

\[ \tilde{\omega} = \sqrt{1 - \tilde{N}_x - \tilde{N}_r} \pm \tilde{N}/2, \]  

(22)

\[ \tilde{\omega} = \frac{1}{2} \sqrt{1 - \tilde{N}_x - \tilde{N}_r^T + \frac{3}{2} \tilde{N}^2/(1 - \tilde{N}_r - \tilde{N}^T)}, \]  

(23)

\[ \tilde{\omega} = \frac{1}{2} \sqrt{1 - \tilde{N}_x - \tilde{N}_r^T - \frac{3}{2} \tilde{N}^2/(1 - \tilde{N}_r - \tilde{N}^T)}, \]  

(24)

where

\[ \tilde{\omega} = \omega / \Omega. \]  

(25)
The boundaries of the first three instability regions are shown in the $\vec{N} - \vec{\omega}$ plane in Figure 1. The first (principal) region calculated by using equations (23) is the widest. The widths of the higher regions decrease in agreement with the theory of Mathieu equations. As follows from Figure 1, larger pulsating loads are more likely to cause dynamic instability of shells subjected to a combination of a static axial load and a thermal field which does not cause buckling.

The principal instability region is usually most important in studies of structures, both because of its width as well as due to structural damping which often neutralizes higher regions. The effect of temperature on the principal instability region is shown in Figure 2 for a fixed amplitude of pulsating loading. It is obvious that higher temperatures result in an increase of the width of the excitation-frequency bands corresponding to the principal region. Higher temperatures also shift the principal instability region to smaller excitation frequencies. These conclusions also can be obtained from Figure 3, where the principal instability region is shown in $\vec{N} - \vec{\omega}$ space for different temperatures.

Figure 1. Boundaries of the first three regions of instability; $\vec{N}_i = 0$, $\vec{N}^T = 0.2$, $i =$ number of a region.

Figure 2. Boundaries of the principal region of instability; $\vec{N} = 0.2$. Cases: 1, $\vec{N}_i = 0$ (R); 2, $\vec{N}_i = 0$ (L) and $\vec{N}_i = 0.2$ (R); 3, $\vec{N}_i = 0.2$ (L) and $\vec{N}_i = 0.4$ (R); 4, $\vec{N}_i = 0.4$ (L). The left and right boundaries of the instability region are denoted by (L) and (R), respectively.

Figure 3. Boundaries of the principal instability region ($\vec{N}_i = 0$). The region from right to left correspond to $\vec{N}^T = 0$, 0.2 and 0.4, respectively.
The influence of structural damping on the regions of dynamic instability can be significant. Bolotin showed that viscous damping usually neutralizes instability corresponding to small excitation parameters $\mu$ [17]. The critical values of the excitation parameters, i.e., the smallest values $\mu$ for which instability is still possible, were obtained by Bolotin as

$$\mu_i = (\Delta / \pi)^{1/4},$$  \hspace{1cm} (26)

where $i$ is the number of the instability region and $\Delta$ is the decrement of damping of the free vibrations of a structure loaded by a constant component of the longitudinal load. In this problem

$$\Delta = 2\pi \varepsilon / \Omega \sqrt{1 - \bar{N}_x - \bar{N}^T},$$ \hspace{1cm} (27)

where $\varepsilon$ is a coefficient of viscous damping for radial vibrations of the shell. Using expression (21) for $\mu$, one obtains the following expression for the critical (minimum) amplitude of the pulsating load necessary for realization of the $i$th region of instability:

$$\bar{N} = 2(2\varepsilon / \Omega)^{1/4}(1 - \bar{N}_x - \bar{N}^T)^{(1/2i)}.$$ \hspace{1cm} (28)

Equation (28) is plotted in Figure 4 for the first three regions of dynamic instability. As follows from Figure 4, the second and higher regions of instability can be realized only for very small ratios $\varepsilon / \Omega$. The effect of temperature on the critical amplitude of the pulsating load for the principal region is illustrated in Figure 5. As could be expected,
higher temperatures decrease the amplitudes of pulsating loads which can cause an unstable motion.

### 4. CONCLUSIONS

The problem of dynamic stability of discretely reinforced composite cylindrical shells subjected to the action of a thermal field has been considered. Linear governing equations of motion of a shell with a finite torsional stiffness of reinforcements have been derived by using a Donnell type theory. These equations have been written for shells reinforced by discrete stiffeners in both the axial and circumferential directions.

Dynamic stability has been analyzed in terms of the static critical load and the natural frequency corresponding to the evaluated mode shape of motion. A shell subjected to high temperatures is shown to become dynamically unstable at smaller amplitudes and frequencies of the driving force than the same shell at room temperature. Structural damping has a significant effect on the regions of dynamic instability. In particular, higher regions can be realized only for shells with small damping or with large frequencies of free vibrations (Ω). High temperatures reduce the amplitude of pulsating loads necessary to overcome damping and to cause unstable motion of the shell.

### REFERENCES