Abstract

The present paper illustrates the effect of matrix cracks in longitudinal and transverse layers of cross-ply ceramic matrix composite (CMC) beams on their mechanical properties and vibration frequencies. Even in a geometrically linear problem considered in the paper, the physical non-linearity is introduced by matrix cracks and interfacial/overlay-matrix friction in longitudinal layers. A closed-form solution for mechanical properties of a cross-ply CMC beam with matrix cracks is developed in the paper. The frequency of free vibrations of a simply supported beam is derived as a function of the amplitude, accounting for the effect of matrix cracks. As shown in the paper, the prediction of the natural frequencies of cross-ply CMC beams with matrix cracks in both longitudinal and transverse layers is possible using simple, yet accurate, approximate equations. © 2002 Elsevier Science Ltd. All rights reserved.

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1. Introduction

As follows from numerous observations of matrix crack development in cross-ply beams, cracks appear in transverse layers at a relatively low load [1,2]. These cracks, called tunneling cracks, are perpendicular to the load direction and parallel to the fibers of the corresponding layer. Usually, tunneling cracks almost immediately propagate throughout the entire depth of the corresponding layer. As the load increases, the crack density increases as well, until the tunneling cracks reach saturation. After saturation, the density of tunneling cracks remains constant under increasing load.

As the load continues to increase, cracks begin to develop in longitudinal layers where they are perpendicular to the fibers. The cracks approaching the fiber-matrix interface may either break the fiber or pass it and continue their propagation in the matrix. The cracks propagating in the matrix of a longitudinal layer, without breaking the fibers, are called bridging cracks. Such cracks represent a typical mode of damage in CMC.

Note that the sequence of cracking explained here is not the only possible mode of damage. An alternative mode is associated with delaminations between the longitudinal and transverse layers that follows cracking in the latter layers. Examples of the analysis in this case are recent papers [3,4] that also contain a list of relevant references. However, it should be noted that such damage mode has not been observed in experiments conducted on typical CMC.
The density of cracks in transverse layers is often different than that in longitudinal layers. For example, Erdman and Weitsman [5] presented measurements of the matrix crack density in cross-ply SiC/CAS composites. Dependent on the lay-up, the saturation density in transverse layers varied between 20 and 40 cracks/in. However, the saturation density in longitudinal layers was between 122 and 238 cracks/in. This illustrates that the analysis should not be conducted by assumption that the densities of cracks in transverse and longitudinal layers are identical.

2. Analysis

2.1. Assumptions

The analysis utilizes the following assumptions.

1. The sequence of crack accumulation under a quasi-static loading includes cracking in the transverse layers, saturation of transverse cracks in these layers, and subsequently, cracking in the longitudinal layers that occurs without changes in the stiffness of the transverse layers.

2. Small-amplitude vibration cycling of an already damaged beam is not accompanied by a formation of additional cracks.

3. Cracks in longitudinal and transverse layers remain open during the tensile part of the cycle of motion. Under compression, the cracks in all layers are closed.

4. The mode shape of vibration of the beam is not affected by damage.

5. Energy dissipation due to damping, interfacial friction, thermoelastic effect, and closing and opening of the cracks is disregarded.

2.2. Mechanical properties of a cross-ply CMC beam with matrix cracks in longitudinal and transverse layers

2.2.1. Stiffness of transverse layers with open matrix cracks that reached saturation (intact longitudinal layers)

The expression for this stiffness is obtained for material where longitudinal layers are still intact, while tunneling cracks in transverse layers reached saturation. According to Han and Hahn [6], the modulus of the composite material with cracks in transverse layers is

\[ E' = E_i[1 + (h_l/s_T)(E_T/E_L)\tanh(\xi s_T/h_l)]^{-1}, \]

where \( s_T \) is a crack spacing (saturation spacing, in this case), \( h_l \) is a layer thickness, \( E_L \) and \( E_T \) are the longitudinal and transverse moduli of elasticity of a layer, and \( \xi \) is a shear lag parameter given by

\[ \xi = [6G_T E_i/E_L E_T]^{1/2}. \]

In Eqs. (1, 2), \( E_i = (E_L + E_T)/2 \) is the composite modulus of the intact material with a balanced cross-ply layup considered in this paper and \( G_T \) is a transverse shear modulus of the transverse layer. Note that the modulus given by Eq. (1) is used below for the analysis of vibration cycling, when the cracks already exist. Accordingly, as follows from the paper [6], residual thermal stresses do not affect this modulus.

Once the value of \( E' \) has been determined, the modulus of the transverse layers can be obtained from

\[ E'_T = 2E' - E_L, \]

where \( E_L \) is the modulus of intact longitudinal layers. The modulus of transverse layers remains unaffected by the stress during cycling, as long as the cracks remain open, i.e. the material is subjected to tension. Under compression, the cracks are closed and \( E_T \) corresponds to the modulus of the intact material.

2.2.2. Stiffness of longitudinal layers with bridging matrix cracks and stiffness of the laminate (saturation cracking in transverse layers)

The stress–strain curve of a unidirectional CMC lamina with bridging matrix cracks exhibits noticeable hysteresis [7]. The average modulus corresponding to reverse fatigue loading with the stress varying from zero to a maximum value was given in [7], based on the model of Pryce and Smith [8]

\[ E'_L = \tau/[E_L + (r/4s_L)(\Delta \sigma_L/E_L)(V_m E_m/V_f E_L)^2]^{-1}. \]
In Eq. (4), \( \tau \) is an interfacial shear stress, \( r \) is the fiber radius, \( s_L \) is the matrix crack spacing in longitudinal layers, \( \Delta \sigma_L \) is the range of stresses applied to the longitudinal layer, \( E_L \) and \( E_m \) are the moduli of fibers and matrix, respectively, and \( V_f \) and \( V_m \) are the volume fractions of the fibers and matrix, respectively.

Residual thermal stresses do not affect the average modulus \( E'_L \), as can easily be shown using the solution [8]. The interfacial shear stress does not remain constant during cycling. Rather, it decreases with cycling due to wear, as was shown by Holmes and Cho [9]. Other factors, such as lubrication, strain rate, and temperature have also been shown to affect this stress. However, during small-amplitude steady state vibrations, the changes of the factors affecting the interfacial stress are slow. Accordingly, the value of the interfacial stress during one cycle may be assumed constant.

Eq. (4) was obtained by assumption of a partial slip along the fiber-matrix interface during cycling. The limits of applicability of this equation are

\[
E'_L > V_f E_L/(1 - V_m E_m/2 E_L). \tag{5}
\]

If inequality (5) is violated, a partial-full slip occurs along the interface and Eq. (4) should be replaced with a different relationship [7]. However, this situation is not considered here since a partial-full slip is unlikely in the case of small amplitude vibrations.

The range of stresses acting in the longitudinal layer can be determined keeping in mind that the stiffness of transverse layers remains constant during the tensile part of the vibration cycle, as is also reflected in Eqs. (1) and (3). Accordingly, the ratio between the stress ranges in adjacent longitudinal and transverse layers is

\[
\frac{\Delta \sigma_L}{\Delta \sigma_T} = \frac{E'_L (\Delta \sigma_L)}{E'_T}, \tag{6}
\]

where \( E'_T \) is constant. At the same time, the range of the applied composite stress is

\[
\Delta \sigma = \frac{\Delta \sigma_L + \Delta \sigma_T}{2}. \tag{7}
\]

Combining (6) and (7) one can eliminate \( \Delta \sigma_T \). Subsequently, solving (6) together with (4), it is possible to obtain the stress range \( \Delta \sigma_L \) and the average modulus for the longitudinal layers \( E'_L \) as functions of the range of the applied composite stress \( \Delta \sigma \).

The average composite modulus of the material with matrix cracks in both longitudinal and transverse layers, \( E = E(\Delta \sigma) \), is available as

\[
E = (E'_L + E'_T)/2. \tag{8}
\]

This modulus remains constant during the tensile part of the cycle when the stress varies from zero to a maximum value. Note that during the compressive part of the cycle the modulus \( E'_L = E_L \), according to Eq. (4). This is easy to comprehend if one sets the stress range in (4) equal to zero which results in \( E'_L = E_L \). Accordingly, the stress range in (4) cannot be negative implying that the cracks are closed under compression, in agreement with assumption 3 above. Physically this is justified since the crack closing strain could be estimated as a ratio of the crack opening displacement to the crack spacing. This ratio is very small and it may be assumed equal to zero. An exact analysis of the shear lag phenomenon may yield a condition for the crack closing in transverse layers, in cases where the cracks in longitudinal layers have already been closed under compression. However, this analysis is not necessary since a correction to the modulus of elasticity of the cross-ply laminate will be small. Therefore, it is assumed here that under compression the composite modulus is equal to that of the intact material, i.e. \( E_c \), as follows from assumption 3.

Note that if the cracks are due to static bending, the modulus calculated by Eq. (8) varies through the depth of the beam. In this case, a density of cracks in both longitudinal and transverse layers depends on the thickness coordinate and Eq. (8) is sufficiently accurate if the number of layers is large (say, larger than 20). Of course, if matrix cracks are due to uniform tension, the modulus given by Eq. (8) is constant throughout the cross section.

2.2.3. Stiffness as a function of local strain

Consider a non-linear relationship between the range of the applied stresses and the range of strains for the section of a cross-ply CMC beam with matrix cracks that is subject to tension during reversed bending. The permanent offset strain that remains in the material after removal of the load that caused cracking during initial loading does not explicitly affect the average modulus of elasticity during the motion. However, the effects of preloading are
incorporated in the solution via the matrix cracks spacing. As shown above, we can determine the modulus $E$ as a non-linear function of the applied stress range $\Delta \sigma$. Subsequently, it is possible to find the corresponding strain range from

$$\Delta \varepsilon = \Delta \sigma / E(\Delta \sigma).$$

(9)

It is necessary to characterize the curve $E = E(\Delta \varepsilon)$ by an analytical expression that can be used in the frequency analysis. One possibility is to use a curve fitting procedure. A different approach may be based on minimization of the squared error in the energy densities evaluated using the curve $E = E(\Delta \varepsilon)$ and an analytical function for the modulus of elasticity of the cross-ply laminate. For example, if one selects the function

$$E_a = E_c + a \Delta \varepsilon + b \Delta \varepsilon^n,$$

(10)

where $E_c$ is the modulus of the material subjected to a negligible tensile strain that does not affect the modulus of longitudinal layers, $a$ and $b$ are constants that have to be determined and $n$ is an arbitrary integer (as follows from numerical examples, it can be taken equal to 2), the square error in the energy density is

$$R = (M - N)^2.$$

(11)

In (11), $M$ is the energy density obtained as

$$M = \int E_a \Delta \varepsilon \, d(\Delta \varepsilon).$$

(12)

The value of $N$ can be found from the counterpart of Eq. (12) where $E_a$ is replaced with $E(\Delta \varepsilon)$ obtained using Eq. (9) and a relationship between the average composite modulus and the applied stress range that follows from (8).

The minimization of the squared error implies that

$$\partial R / \partial a = \partial R / \partial b = 0.$$

(13)

The values of $a$ and $b$ are available from Eq. (13).

Note that $E_c$ in (10) is independent of the crack density in longitudinal layers. This is because the modulus of elasticity of longitudinal layers with bridging cracks based on the model of Pryce and Smith [8] is affected by the crack density only in the presence of the applied stress. Physically, such a model is justified since the opening of a crack in a longitudinal layer requires the application of tension due to the necessity to overcome friction along the damaged section of the fiber-matrix interface.

A solution is also available using collocations. The requirement $E_a(\Delta \varepsilon_i) = E(\Delta \varepsilon_i)$ applied at two representative values of dynamic strain range ($i = 1, 2$) yields the values of $a$ and $b$.

2.2.4. Non-linear small-amplitude free vibrations of a simply supported cross-ply CMC beam with matrix cracks in longitudinal and transverse layers

The solution is obtained by the energy method, based on the assumption that the energy dissipation is negligible. Accordingly, in the case of harmonic vibrations, the squared frequency is obtained from

$$\omega^2 = U_{\text{max}} / T_{\text{max}},$$

(14)

where $U$ and $T$ are the strain and kinetic energies, respectively.

The denominator in the right side of the above equation is

$$T_{\text{max}} = (m/2) \int_0^L w^2 \, dx,$$

(15)

where $m$ is the mass of the beam per unit length, $w$ is a deflection corresponding to the maximum deviation from equilibrium, and $x$ is the axial coordinate ($0 < x < L$), $L$ being the length of the beam.

The evaluation of the maximum value of the strain energy is complicated due to the fact that the modulus of elasticity varies throughout the depth and the length of the beam. As explained above, if the dynamic strains are compressive, the modulus of the material remains constant and equal to $E_l$. Therefore, the cross-ply beam material behaves in a manner that resembles a non-linear material of “bimodular” type, though in the present problem the material behavior is more complicated. The material with cracks exhibits physical non-linearity during the tensile part of the cycle but retains constant stiffness under compression. The previous research has shown that a deviation of the neutral axis from the centroidal axis of the beam due to different responses in tension and compression is very
small if the cracks are limited to transverse layers
[10]. However, this deviation should be larger if the
Cracks appear in the longitudinal layers since this
case corresponds to a larger stiffness degradation.
The location of the neutral curve of the beam
can be determined from the condition that the net
axial force in the beam should be equal to zero and
from the requirement of zero strain along the neutral
curve. These conditions yield
\[ z_n = B/A, \]
(16)
where \( A \) and \( B \) are extensional and coupling stiffnesses, respectively. For example, if the lower part of the cross section of a unit-width beam that has a thickness \( h \), i.e. \( z_n < z < h/2 \), is subject to tension and the crack densities in both longitudinal and transverse layers are independent of the thickness \( z \)-coordinate, these stiffnesses become
\[
\{A, B\} = \int_{z_n}^{h/2} \left[ E_c + a \Delta \varepsilon(z) + b \Delta \varepsilon(z)^n \right] \{1, z\} \, dz + \int_{-h/2}^{z_n} E_i \{1, z\} \, dz
\]
(17)
If the crack density depends on the \( z \)-coordinate, Eq. (17) is still valid, but in this case \( E_c = E_c(z) \), \( a = a(z) \) and \( b = b(z) \). If the crack distribution is independent of the \( z \)-coordinate but varies along the beam axis, \( A = A(x) \), \( B = B(x) \), and accordingly, \( z_n = z_n(x) \).

Note that \( \Delta \varepsilon \) at each location is equal to the maximum dynamic strain achieved during the cycle. Given the value of \( \Delta \varepsilon_{\text{max}} = \Delta \varepsilon(h/2) \), it is possible to obtain the dynamic strain range as
\[
\Delta \varepsilon(z) = [(z - z_n)/(h/2 - z_n)] \Delta \varepsilon_{\text{max}}.
\]
(18)
Therefore, Eq. (16) in conjunction with (17,18) can be used to determine \( z_n \) as long as \( \Delta \varepsilon_{\text{max}} \) is prescribed.

The maximum strain energy can be evaluated as
\[
U_{\text{max}} = U_1 + U_u,
\]
(19)
where the two components in the right side correspond to the contribution of the lower (tension) and upper (compression) parts of the beam cross sections, respectively. In the following analysis, the cracks are assumed uniformly distributed along the beam. Then the neutral curve becomes the neutral axis, i.e. \( z_n \) is independent of the axial coordinate. Accordingly,
\[
U_1 = (1/2) \int_{z_n}^{h/2} \int_0^L \left[ E_c + a \Delta \varepsilon(z) + b \Delta \varepsilon(z)^n \right] \Delta \varepsilon^2 \, dx \, dz.
\]
(20)
The energy of the upper part of the beam is
\[
U_u = (1/2) \int_{-h/2}^{z_n} \int_0^L E_i \Delta \varepsilon(z)^2 \, dx \, dz.
\]
(21)
It remains to prescribe the cyclic strain range. Assuming that the mode shape of vibrations is unaffected by damage (assumption 4), and solving for the fundamental frequency, the deflection can be written as
\[
w = W \sin(\pi x/L) f(t),
\]
(22)
where \( W \) is the amplitude of motion and \( f(t) \) is a periodic harmonic function normalized so that \( f(t)_{\text{max}} = 1 \). Accordingly, one can write the dynamic strain range (coinciding with the maximum strain) as
\[
\Delta \varepsilon = (z - z_n)(\pi/L)^2 W \sin(\pi x/L).
\]
(23)
This yields the solution for the fundamental frequency. It is usually more convenient to operate with a non-dimensional value of the frequency denoted here as \( F \) and defined as a ratio of the frequency in the presence of damage to the frequency of the intact beam obtained for the same amplitude of motion. The square of this ratio is immediately available in the form
\[
F^2 = (U_1 + U_u)/2U_c
\]
(24)
where \( U_c \) is the half-energy of the intact beam that is obtained from (21) with \( z_n = 0 \).
As follows from Eq. (24), the squared non-dimensional frequency is the following non-linear function of the amplitude of free vibrations:
\[
F^2 = f_0 + f_1 W + f_n W^n,
\]
(25)
where \( f_0, f_1 \) and \( f_n \) are constants dependent on geometry, material, and damage.
If the cracks are symmetric about the middle axis of a symmetrically laminated beam, the
non-dimensional frequency $F$ represents the solution. However, if the crack distribution is asymmetric, it is necessary to determine the half-periods of motion corresponding to tension in the upper and lower parts of the beam cross section. Subsequently, the period of vibration of the beam is found as a sum of these half-periods and a non-dimensional frequency can be specified.

In a particular case with $n = 2$ Eq. (25) obtained by substituting the energy terms given by Eqs. (20, 21) into Eq. (24) becomes

$$F^2 = [(1 + 2Z_n)^3 + E_n(1 - 2Z_n)^3 + (\pi/2L_n)^4a_nW_n + (9/160)(\pi/L_n)^4(1 - 2Z_n)^5b_nW_n^2)/2, \quad (26)$$

where $E_n = E_c/E_i, L_n = L/h, W_n = W/h, a_n = a/E_i, b_n = b/E_i$.

2.2.5. Position of the neutral axis as a function of the applied strain

This position can be found from Eq. (16). The evaluation of the extensional and coupling stiffnesses in terms of $z_n$ and transformations yield a quadratic equation for $Z_n = z_n/h$ (the case of $n = 2$):

$$n_2Z_n^2 + n_1Z_n + n_0 = 0, \quad (27)$$

where

$$n_2 = 2(E_c - E_i) + (8/3)a\Delta e_{max} + b\Delta e_{max}^2$$
$$n_1 = -2(E_c + E_i) - (2/3)a\Delta e_{max} - b\Delta e_{max}^2$$
$$n_0 = (E_c - E_i)/2 - a\Delta e_{max}/3 + (1/4)b\Delta e_{max}^2 \quad (28)$$

In these equations, the strain range $\varepsilon_{max}$ corresponds to the maximum tensile strain at $z = h/2$.

3. Numerical examples and discussion

The material considered in the following examples is SiC/CAS with the following properties [11]: $E_i = 200$ GPa, $E_m = 97$ GPa, $r = 8 \mu$m, $\tau = 5$ MPa, $V_f = 0.35$. The saturation matrix crack spacing in the longitudinal layers of such material is close to 125 $\mu$m. The layer thickness adopted in these examples is 125 $\mu$m. The longitudinal and transverse moduli of this material are $E_L = 133$ GPa and $E_T = 118$ GPa, respectively. The transverse shear modulus $G_T$ was taken equal to 45 GPa. It should be noted that a relatively low value of the interfacial shear stress is adopted here to reflect the experimentally observed fact that these stresses decrease during vibrations due to a gradual smoothening of the fiber-matrix interface [9].

In the following examples, the saturation matrix crack spacing in transverse layers was assumed equal to either twice or four times the layer thickness (250 and 500 $\mu$m, respectively) corresponding to typical values for CMCs. The crack spacing in longitudinal layers was equal to 125, 250 or 500 $\mu$m. It is emphasized that the results presented below refer to laminates with preexisting matrix cracks caused by uniform tension.

The average modulus of longitudinal layers experiences a noticeable decrease as a result of a larger range of tensile stresses acting in these layers, as shown in Fig. 1. This reduction is more pronounced if the matrix crack spacing in longitudinal layers is smaller. The reader is reminded that this and subsequent results are shown for the stress ranges from zero to a maximum tensile value. As explained above, the cracks are closed under compression and the modulus corresponds to that for the intact material.
The ranges of stresses in longitudinal and transverse layers differ significantly, as the range of the applied composite stress increases, as follows from Figs. 2 and 3. A difference between these ranges increases if the matrix crack spacing in longitudinal layers becomes larger. This is predictable since if damage in the longitudinal layers is relatively small, they absorb a larger fraction of the applied load.

Relationships between the composite modulus and the modulus of longitudinal layers on one hand and the range of applied stresses on the other hand are presented in Figs. 4 and 6. The same moduli are shown versus the range of the composite strain in Figs. 5 and 7. It is emphasized that the longitudinal modulus can be calculated by Eq. (4) only if inequality (5) is satisfied. In the present examples this means that $E_L' > 91.7 \text{ GPa}$. Accordingly, the results obtained when this condition is violated are questionable. This explains a strange situation shown in Figs. 6 and 7 for the case where $s_L = 125 \text{ mm}$ and $s_T = 500 \text{ mm}$ where the modulus of longitudinal layers and the composite modulus converge as the range of applied stress approaches 100 MPa.

As follows from the results shown in Figs. 5 and 7, a very accurate approximation of the relationship $E = E(\Delta \varepsilon)$ can be obtained from Eq. (10) using $n = 2$. The coefficients $a$ and $b$ in Eq. (10) are listed in Table 1 for various matrix crack spacings in the longitudinal layers. The coefficients $a$ and $b$ appeared insensitive to the matrix crack spacing $s_T$. However, the modulus $E_c$ varied dependent on this spacing. In particular, it was equal to 95.85 GPa for $s_T = 250 \text{ mm}$ and 105.25 GPa for $s_T = 500 \text{ mm}$. The non-linear quadratic representation of the composite modulus-applied strain relationship is very accurate. Maximum deviations of Eq. (10) with $n = 2$ and the coefficients listed in Table 1 from the actual curves were under 1%. It is also evident from Table 1 that the quadratic term in Eq. (10) is negligible in the case of a small strain range. This is reflected in Figs. 5 and 7 where the composite modulus is almost a linear function of the range of the applied strain.

Relationships between the composite stress and strain ranges are presented in Fig. 8. As shown in these figures, these relationships remain practically linear, in spite of extensive cracking in both longitudinal and transverse layers. However, the material becomes more compliant if the crack density in longitudinal and/or transverse layers is higher.
Fig. 3. (a) Relationships between the range of the applied composite stress and the range of stress in longitudinal and transverse layers ($s_L = 0.125$ mm, $s_T = 0.500$ mm). The stress is measured in MPa; (b) Relationships between the range of the applied composite stress and the range of stress in longitudinal and transverse layers ($s_L = 0.250$ mm, $s_T = 0.500$ mm). The stress is measured in MPa; (c) Relationships between the range of the applied composite stress and the range of stress in longitudinal and transverse layers ($s_L = 0.500$ mm, $s_T = 0.500$ mm). The stress is measured in MPa.

Fig. 4. Relationships between the elasticity modulus of longitudinal layers and the composite modulus and the range of applied composite stress ($s_T = 0.250$ mm).

The positions of the neutral axis are shown for representative cases where $s_T = 250$ and 500 $\mu$m in Figs. 9 and 10, respectively. These results were generated for the case where the lower part of the beam was subject to tension. Accordingly, the stiffness of this part is degraded due to open cracks, while the stiffness of the upper part of the beam corresponds to that for the intact material. This results in the neutral axis located closer to the compressed surface of the beam, i.e. $Z_n < 0$ in all cases.

As follows from Figs. 9 and 10, the effect of the strain on the position of the neutral axis is
While in the case of larger density of matrix cracks in longitudinal layers the neutral axis moves closer to the centroidal axis as a result of a large composite strain ($s_L = 125 \mu m$), if this density is smaller, the tendency is reversed. However, in all cases shown in Figs. 9 and 10 the changes in the position of the neutral axis due to a larger strain remain very small, not exceeding 5%, even as the tensile strain on the surface $z = h/2$ reaches 1%.

This results in an important conclusion that it is possible to assume that the position of the neutral axis is unaffected by the strain. Accordingly, this position can be calculated from a simplified version of Eq. (27). The latter equation can be simplified even more due to the observation that the quadratic term has little effect on the value $Z_n$. Therefore, given the values of $E_c$ and $E_i$, we can determine the position of the neutral axis from a simple formula

$$Z_n = - (1/4)(1 - E_n)/(1 + E_n).$$

(29)
Fig. 7. Relationships between the elasticity modulus of longitudinal layers and the composite modulus and the range of composite strain ($s_T = 0$):

(a) $s_L = 0.125\text{mm}$

(b) $s_L = 0.250\text{mm}$

(c) $s_L = 0.500\text{mm}$

The analysis of Eq. (26) illustrates that the effect of the amplitude of motion on the frequency is negligible, as could be expected in a geometrically linear program. For example, if $s_L = 125\ \mu\text{m}$ and $s_T = 250\ \mu\text{m}$ (the case where the effect of the amplitude is most pronounced, as compared to other cases considered in the examples) and $L_n = 25$, $F = 0.933$ for $W_n = 0$ and $F = 0.931$ for $W_n = 1.0$. A larger amplitude of motion is meaningless in this analysis since it corresponds to a geometrically non-linear problem that is not considered here. On the other hand, a shorter beam that exhibits a higher non-linearity, as follows from (26), should be analyzed including transverse shear effects. In addition, a short beam would typically fail due to high bending stresses at a much smaller amplitude of motion than $W_n = 1.0$. 

Table 1

Coefficients ($a, b$) in the non-linear composite modulus-applied strain relationships (10) obtained using $n = 2$

<table>
<thead>
<tr>
<th>$s_L (\mu\text{m})$</th>
<th>125</th>
<th>250</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a, b$ (GPa)</td>
<td>-268.83</td>
<td>-153.0</td>
<td>-81.46, 940.07</td>
</tr>
</tbody>
</table>

Fig. 8. (a) Relationships between the ranges of composite stress and strain ($s_T = 0.250\ \text{mm}$); (b) Relationships between the ranges of composite stress and strain ($s_T = 0.500\ \text{mm}$).
The above discussion enables us to develop “master curves” valid for cross-ply CMC beams. These curves can be obtained if the ratio $E_n$ has been determined. Then the position of the neutral axis can be obtained from (29), while the non-dimensional fundamental frequency is available from the simplified version of (26), i.e.

$$F^2 = \frac{[1 + 2Z_n]^3 + E_n(1 - 2Z_n)^3]}{2}.$$  \hspace{1cm} (30)

The master curves are shown in Figs. 11 and 12. These curves can be convenient for the analysis of free vibrations of beams with matrix cracks in longitudinal and transverse layers experiencing small-amplitude vibrations.

4. Conclusions

The paper presents a closed-form solution for mechanical properties and natural frequencies of cross-ply CMC beams with preexisting matrix cracks in longitudinal and transverse layers. The problem is physically non-linear due to friction
along the damaged sections of the fiber-matrix interface in longitudinal layers. The analysis is complicated due to a difference in the stiffness of the cross-ply material under tensile and compressive stresses. In the former case, the cracks are open and the modulus of elasticity is a non-linear function of strain, while in the latter case, the cracks are closed and the modulus corresponds to that for the intact material. Accordingly, one of the problems that has to be addressed is a position of the neutral curve (neutral axis, if the cracks are uniform along the beam axis). This position has been derived as a non-linear function of the applied strain. However, as shown in the paper based on the numerical analysis, the effect of strain is negligible and the position of the neutral axis or neutral curve can be determined from a simple formula (29). The natural frequency of small-amplitude vibrations is a non-linear function of the amplitude due to a physically non-linear material response when the cracks are open (Eq. 26). However, numerical analysis illustrates that the non-linear terms have a negligible effect on the frequency, as long as the amplitude of motion remains small. Based on this observation, a simple, yet accurate, formula is proposed for the natural frequency (Eq. 30).

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