BEHAVIOUR OF LAMINATED PLATES SUBJECTED TO CONVENTIONAL BLAST*

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Summary—Response of simply supported anti-symmetrically laminated angle-ply plates to explosive blast loading is considered. A closed-form solution is obtained for thick plates, it being assumed that the material remains in the elastic range. The effect of transverse shear deformations on the response of thick plates is taken into consideration. The behaviour of thin laminated plates subjected to blast is studied using geometrically non-linear theory. Initial imperfections, which can be important for thin plates, are included in the analysis. The solution for thin elastic laminated plates is obtained numerically using a Runge-Kutta method. The analysis yields the non-dimensional deflection vs time relationship which can be used to determine the stresses and strains in the layers of the plate.

I. INTRODUCTION

The damage occurring in structures subjected to a conventional air blast wave is due mainly to the overpressure in the shock wave. This overpressure, i.e. the excess over the atmospheric pressure, increases rapidly on the surface exposed to the blast wave and then decreases at a much slower rate. Studies concerned with the response of structures subjected to blast loading often tended to simplify the phenomenon by employing an idealized (usually rectangular) pressure impulse [1-6]. A realistic blast loading was considered by Houlston et al. [7]. They recorded the pressure–time history from a test using the pressure values at 70 time points to represent the loading function. Nagaya and Nagai [8] considered the dynamic response of an isotropic circular plate in contact with a fluid, the surface of which was excited by explosive pressure measured experimentally. They did not use an analytical expression for the pressure; instead values for a number of sample points were used in the analysis.

The response of simply supported orthotropic plates to blast-type loading was first discussed by Dobyns [9] who analyzed the response to pulses of different shapes (sine load, step load, triangular load, exponential load and stepped triangular load). Rajamani and Prabhakaran [10] considered the response of clamped composite plates to blast loading approximated by a rectangular overpressure pulse.

An analytical expression for the overpressure–time relationship was given recently by Gupta [11], who used the modified Friedlander exponential decay equation. An approximate solution was obtained for an elastic simply supported plate of homogeneous isotropic material. A similar problem was solved numerically using the ADINA finite-element code [12].

In the first part of this paper is studied the response of thick anti-symmetrically laminated angle-ply simply supported plates to conventional blast loading. The effect of shear deformation, which was shown to be significant in dynamic problems of thick composite plates [13,14], is taken into consideration. A closed-form solution is obtained using the

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convolution integral; the accuracy of this solution is limited only by the number of terms retained in the Fourier series representing displacements and bending slopes. Following the results of [13], the in-plane and rotatory inertias are neglected.

The response of geometrically non-linear imperfect anti-symmetrically laminated angle-ply simply supported plates is considered in the second part. The governing equations, formulated as a generalization of von Karman’s equations of the theory of isotropic plates, were first published by Stavsky and Hoff [15]. The non-dimensional version used in this paper was introduced by Hui [16,17]. The analysis results in the non-linear differential equation for the amplitude of out-of-plane displacement which is solved by the Runge–Kutta method.

2. ANALYSIS OF THICK LAMINATED PLATES

An anti-symmetrically laminated angle-ply plate of moderate thickness and rectangular planform is considered to be subjected to normal pressure \( q(x, y, t) \). The problem is taken to be physically and geometrically linear, and the plate is simply supported on all four sides. The equations of motion of such a plate in rectangular co-ordinates \((x, y)\) are:

\[
\begin{align*}
N_{1,x} + N_{6,y} &= 0; & N_{6,x} + N_{2,y} &= 0; & Q_{x,x} + Q_{y,y} &= \rho hw_{tt} - q(x, y, t) \\
M_{1,x} + M_{6,y} - Q_x &= 0; & M_{6,x} + M_{2,y} - Q_y &= 0,
\end{align*}
\]

(1)

where \( h \) is the plate thickness, \( M_i \) are the stress couples, \( N_i \) and \( Q_i \) are the respective in-plane and transverse shear stress resultants, \( t \) is time, \( w \) is normal deflection, \( \rho \) is material density and \( \partial/\partial x \) denotes \( \partial/(\partial x) \).

The stress resultants and stress couples are related to the generalized displacements by the constitutive relations:

\[
\begin{bmatrix}
N_1 \\
N_2 \\
N_6 \\
M_1 \\
M_2 \\
M_6
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\
A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\
0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\
0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\
0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\
B_{16} & B_{26} & 0 & 0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_y \\
\psi_{x,x} \\
\psi_{y,y} \\
\psi_{y,x} + \psi_{x,y}
\end{bmatrix}
\]

(2)

\[
\begin{bmatrix}
Q_x \\
Q_y
\end{bmatrix} =
\begin{bmatrix}
k_2^4 A_{44} & 0 \\
0 & k_2^4 A_{55}
\end{bmatrix}
\begin{bmatrix}
w_x + \psi_x \\
w_y + \psi_y
\end{bmatrix},
\]

(3)

where \( k_2^4 \) and \( k_2^5 \) are the shear correction coefficients; \( A_{ij}, B_{ij}, \) and \( D_{ij} \) are the transformed reduced extensional, coupling and bending stiffnesses, respectively.

Substitution of the constitutive equations (2) and (3) into equations of motion (1) yields the set of equations:

\[
\begin{bmatrix}
u \\
v \\
w \\
\psi_y \\
\psi_x
\end{bmatrix} =
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
0 & \rho hw_{tt} - q(x, y, t) \\
0 & 0 \\
0 & 0
\end{bmatrix}
\]

(4)
where the elements of symmetric matrix $L_{kl}$ are

\[
\begin{align*}
L_{11} &= A_{11} \frac{d^2}{dx^2} + A_{66} \frac{d^2}{dy^2} & L_{13} &= 0 \\
L_{12} &= (A_{12} + A_{66}) \frac{d}{dx} \frac{d}{dy} & L_{14} &= (B_{16}/h) \frac{d^2}{dx^2} + (B_{26}/h) \frac{d^2}{dy^2} \\
L_{15} &= (2B_{16}/h) \frac{d}{dx} \frac{d}{dy} & L_{25} &= L_{14} \\
L_{22} &= A_{66} \frac{d^2}{dx^2} + A_{22} \frac{d^2}{dy^2} & L_{33} &= -k_2^2 A_{55} \frac{d^2}{dx^2} - k_2^2 A_{44} \frac{d^2}{dy^2} \\
L_{23} &= 0 & L_{34} &= -(k_2^2 A_{44}/h) \frac{d}{dy} \\
L_{24} &= (2B_{26}/h) \frac{d}{dx} \frac{d}{dy} & L_{35} &= -(k_2^2 A_{55}/h) \frac{d}{dy} \\
L_{33} &= -k_2^2 A_{55} \frac{d^2}{dx^2} - k_2^2 A_{44} \frac{d^2}{dy^2} & L_{44} &= (D_{66}/h^2) \frac{d^2}{dx^2} + (D_{22}/h^2) \frac{d^2}{dy^2} - (k_2^2 A_{44}/h^2) \\
L_{45} &= (D_{12} + D_{66}) \frac{d}{dx} \frac{d}{dy} & L_{55} &= (D_{11}/h^2) \frac{d^2}{dx^2} + (D_{22}/h^2) \frac{d^2}{dy^2} - (k_2^2 A_{55}/h^2) \\
\end{align*}
\]

The simple support boundary conditions considered here are generalizations of Almroth's S3 conditions [18] or Hoff's SS2 conditions [19]:

\[
\begin{align*}
&u(0, y) = u(a, y) = 0; \quad N_6(x, 0) = N_6(x, b) = 0 \\
&N_6(0, y) = N_6(a, y) = 0; \quad v(x, 0) = v(x, b) = 0 \\
&w(0, y) = w(a, y) = 0; \quad w(x, 0) = w(x, b) = 0 \\
&M_1(0, y) = M_1(a, y) = 0; \quad M_2(x, 0) = M_2(x, b) = 0 \\
&\psi_1(0, y) = \psi_1(a, y) = 0; \quad \psi_2(x, 0) = \psi_2(x, b) = 0.
\end{align*}
\]

These boundary conditions were used by Whitney and Leissa [20], Bert and Chen [13], and Bert and Birman [14]. Both the boundary conditions and the equations of motion (4) can be satisfied exactly by the following representation of displacements and bending slopes:

\[
\begin{align*}
u &= \sum_{m,n} U_{mn}(t) \sin \alpha_m x \cos \beta_n y & h\psi_y &= \sum_{m,n} V_{mn}(t) \sin \alpha_m x \cos \beta_n y \\
w &= \sum_{m,n} W_{mn}(t) \sin \alpha_m x \sin \beta_n y & h\psi_x &= \sum_{m,n} X_{mn}(t) \cos \alpha_m x \sin \beta_n y,
\end{align*}
\]

where $\alpha_m = m\pi/a$, $\beta_n = n\pi/b$, $m$ and $n$ being the number of half-waves along the $x$ and $y$ axes, respectively.

Substitution of the series (7) into equations (4) results in the following independent sets of five equations for each set of modal parameters $m$ and $n$:

\[
\begin{align*}
C_{11} U_{mn} + C_{12} V_{mn} + C_{14} Y_{mn} + C_{15} X_{mn} &= 0 \\
C_{12} U_{mn} + C_{22} V_{mn} + C_{24} Y_{mn} + C_{14} X_{mn} &= 0 \\
(p/h^2/E_T) \tilde{W}_{mn} + C_{33} W_{mn} - C_{34} Y_{mn} - C_{35} X_{mn} &= (h/E_T)q_{mn}(t) \\
C_{14} U_{mn} + C_{24} V_{mn} + C_{34} W_{mn} + C_{44} Y_{mn} + C_{45} X_{mn} &= 0 \\
C_{15} U_{mn} + C_{14} V_{mn} + C_{35} W_{mn} + C_{45} Y_{mn} + C_{55} X_{mn} &= 0,
\end{align*}
\]

where $\tilde{\cdot} = d^2(\cdot)/dt^2$. 


The non-dimensional coefficients $C_{ij}$ coincide with those in [14] except $C_{33}$ (see the Appendix),

$$q_{mn}(t) = \frac{4}{ab} \int_0^a \int_0^b q(x, y, t) \sin \alpha_m x \sin \beta_n y \, dx \, dy. \quad (9)$$

Transformations similar to those described in [14] reduce the set (8) to a single differential equation:

$$\left(\frac{\rho h^2}{E T}\right) \ddot{W}_{mn} + \left(k_2^2 \bar{A}_{55} \alpha_m^2 + k_4^2 \bar{A}_{44} \beta_n^2 - F_1 - F_2\right) W_{mn} = \left(\frac{h}{E T}\right) q_{mn}(t), \quad (10)$$

where $\alpha_m = \frac{mnh}{a}$, $\beta_n = \frac{mnh}{b}$. The non-dimensional stiffness $\bar{A}_{44}$ and $\bar{A}_{55}$ and the non-dimensional functions $F_1$ and $F_2$ are given in the Appendix.

The squared natural frequency of vibration associated with $m$ and $n$ half-waves of deformation along the respective $x$ and $y$ axes is immediately obtained from equation (10) being

$$\omega_{mn}^2 = \left(\frac{E T}{\rho h^2}\right) \left(k_2^2 \bar{A}_{55} \alpha_m^2 + k_4^2 \bar{A}_{44} \beta_n^2 - F_1 - F_2\right). \quad (11)$$

The equation of motion (10) can be represented in the non-dimensional form:

$$\frac{d^2 \bar{W}_{mn}}{d\tau^2} + \bar{\omega}_{mn}^2 \bar{W}_{mn} = \bar{q}_{mn}(\tau), \quad (12)$$

where

$$\bar{W}_{mn} = W_{mn}/h \quad \tau = \omega_{11} t$$

$$\bar{\omega}_{mn} = \omega_{mn}/\omega_{11} \quad \bar{q}_{mn} = q_{mn}/\rho h^2 \omega_{11}^2. \quad (13)$$

The response $\bar{W}_{mn}(\tau)$ is obtained using the convolution integral:

$$\bar{W}_{mn}(\tau) = A_{mn} \sin \bar{\omega}_{mn} \tau + B_{mn} \cos \bar{\omega}_{mn} \tau + \frac{1}{\bar{\omega}_{mn}} \int_0^\tau \bar{q}_{mn}(v) \sin \bar{\omega}_{mn}(\tau - v) \, dv, \quad (14)$$

where $A_{mn}$ and $B_{mn}$ are constants of integration. The plate being at rest when loaded by the blast pressure, one can express the initial conditions as

$$\bar{W}_{mn}(0) = 0, \quad \frac{d\bar{W}_{mn}(0)}{d\tau} = 0. \quad (15)$$

The first condition (15) yields $B_{mn} = 0$.

The blast wave reaches the peak value in such a short time that the structure is usually assumed to be loaded instantly. Furthermore, experiments indicate that the pressure can be considered uniformly distributed over the plate [7]. The overpressure vs time relationship can be represented by the modified Friedlander exponential decay equation [11,12]:

$$q(x, y, t) = q_p (1 - t/t_p) e^{-a' t/t_p}, \quad (16)$$

where $q_p$ is the peak reflected pressure in excess of the ambient pressure, $t_p$ is the positive phase duration of the impulse and $a'$ is a coefficient. Therefore, the non-dimensional pressure appearing in equation (14) can be expressed as

$$\bar{q}_{mn}(v) = \frac{16}{mn\pi^2} \frac{q_p}{t_p^2} \left(1 - \frac{v}{t_p}\right) e^{-a' v/t_p}, \quad (17)$$
where \( \check{q} = \frac{q_p}{\rho h^2 \omega_{11}^2} \), \( \tau_p \) being the non-dimensional duration of the positive phase, i.e. \( \tau_p = \omega_{11} t_p \). Substitution of equation (17) into equation (14) and the satisfaction of the second condition (15) yield

\[
\dot{\omega}_{mn}(\tau) = \frac{16\check{q}}{\pi^2 \omega_{mn}} \sum_{1}^{n} \left\{ e^{-\frac{\tau}{\tau_p}} \left( 1 - \frac{\tau}{\tau_p} \right) \tilde{\omega}_{mn} \right. \\
+ \left( \frac{\tau}{\tau_p} \right) \sin \tilde{\omega}_{mn} - \cos \tilde{\omega}_{mn} \\
- \frac{1}{[\left( \frac{\tau}{\tau_p} \right)^2 + \tilde{\omega}_{mn}^2]^{\frac{3}{2}}} \left[ 2\left( \frac{\tau}{\tau_p} \right) \tilde{\omega}_{mn} e^{-\frac{\tau}{\tau_p}} \\
+ \left( \frac{\tau}{\tau_p} \right)^2 \sin \tilde{\omega}_{mn} - 2\left( \frac{\tau}{\tau_p} \right) \tilde{\omega}_{mn} \cos \tilde{\omega}_{mn} \right] \}.
\]

(18)

3. PLATE STRENGTH: THICK PLATE

The present solution is valid as long as the plate remains in the elastic range. Therefore, it is important to detect the instant of time when the stresses in a part of the plate reach the combination prescribed by yield criteria. In the case of simply supported plates, plastic effects first appear in the centre: \( x = a/2, y = b/2 \). It can be seen that the non-zero stress couples in the centre are \( M_1 \) and \( M_2 \); the only non-zero stress resultant is \( N_6 \). Therefore, transverse shear deformability does not influence directly the yield condition. However, this influence exists through the deflection \( w(t) \) which depends on shear correction.

Maximum stresses at the faces of the plate are:

\[
\sigma_x = \pm 6M_1/h^2; \quad \sigma_y = \pm 6M_2/h^2; \quad \tau_{xy} = N_6/h.
\]

(19)

These stresses are related to the stresses in the principal material directions by the transformation relation:

\[
\begin{pmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{pmatrix} =
\begin{bmatrix}
c^2 & s^2 & 2sc \\
s^2 & c^2 & -2sc \\
-sc & sc & c^2 - s^2
\end{bmatrix}
\begin{pmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{pmatrix},
\]

(20)

where \( c = \cos \theta, s = \sin \theta \). Once the stresses \( \sigma_1, \sigma_2, \tau_{12} \) are calculated, the Tsai–Hill yield criterion can be used to detect the instant of plasticity initiation [21]:

\[
(\sigma_1^2/X^2) - (\sigma_1 \sigma_2/X^2) + (\sigma_2^2/Y^2) + (\tau_{12}^2/S^2) = 1,
\]

(21)

where \( X \) and \( Y \) are failure strengths for a layer in the fibre and transverse directions, respectively; \( S \) is the shear failure strength.

4. ANALYSIS OF THIN LAMINATED PLATES

Thin anti-symmetrically laminated plates are studied here using geometrically non-linear theory. The effect of transverse shear deformation which is important for thick plates can be neglected for thin geometrically non-linear plates. However, initial imperfections which are usually negligible in thick plates can affect the behaviour of thin plates. The non-dimensional dynamic version of the governing equations for imperfect plates is [16,17]:

\[
L_q(\ddot{w}) + L_p(f) + \pi^* \ddot{w}_{x\xi} = f_{,j\xi}(w + \ddot{w}_0)_{,x\xi} + f_{,x\xi}(w + \ddot{w}_0)_{,j\xi} - 2f_{,j\xi}(w + \ddot{w}_0)_{,j\xi} + \pi^* \ddot{q}
\]

\[
L_q(f) = L_p(\ddot{w}) + (\ddot{w} + 2\ddot{w}_0)_{,j\xi} \ddot{w}_{,x\xi} - (w + \ddot{w}_0)_{,x\xi} \ddot{w}_{,j\xi} - \ddot{w}_{,j\xi} \ddot{w}_{,x\xi},
\]

(22)
where
\[ w = \frac{w}{h}; \quad \bar{w}_0 = \frac{w_0}{h}; \quad f = \frac{F}{E_f h} \]

are the non-dimensional out-of-plane deflection from the imperfect position, the non-dimensional initial imperfection and the non-dimensional stress function \((F = \text{Airy-type stress function})\), respectively. The non-dimensional time and the position co-ordinates are
\[ \bar{t} = \omega t \quad \omega = \pi^2 \sqrt{E_f h^3/\rho b^4} \]
\[ \bar{x} = \frac{x}{b} \quad \bar{y} = \frac{y}{b}. \]
The non-dimensional transverse load is
\[ q = \frac{q(x, y, t)}{E_f} \left( \frac{b}{h} \right)^4. \]
The non-dimensional lengths of the plate sides are \( \lambda = a/b \) in the \( \bar{x} \) direction and unity in the \( \bar{y} \) direction.

The linear operators in equations (22) are given by
\[ L_a(\cdot) = a_{22}(\cdot,xxxx) + (2a_{12} + a_{66})(\cdot,xxxx) + a_{11}(\cdot,yyyy) \]
\[ L_b(\cdot) = (2b_{26} - b_{61})(\cdot,xxxx) + (2b_{16} - b_{62})(\cdot,yyyy) \]
\[ L_d(\cdot) = d_{11}(\cdot,xxxx) + 2(d_{12} + 2d_{66})(\cdot,yyyy) + d_{22}(\cdot,yyyy), \]
where \( a_{ij}, b_{ij} \) and \( d_{ij} \) are the elements of the non-dimensional matrices \([\bar{A}_{ij}],[\bar{B}_{ij}]\) and \([\bar{D}_{ij}]\) defined by
\[ [\bar{A}_{ij}] = E_f h [A_{ij}]^{-1} \]
\[ [\bar{B}_{ij}] = -[A_{ij}]^{-1} [B_{ij}] / h \]
\[ E_f h^3 [\bar{D}_{ij}] = [D_{ij}] - [B_{ij}][A_{ij}]^{-1} [B_{ij}], \]
\([A_{ij}],[B_{ij}]\) and \([D_{ij}]\) being the matrices of extensional, coupling and bending stiffnesses. Substitution of the non-dimensional initial imperfection and deflection
\[ \bar{w}_0 = W_0 \sin \frac{m\pi \bar{x}}{\lambda} \sin n\pi \bar{y} \]
\[ \bar{w} = W(\bar{t}) \sin \frac{m\pi \bar{x}}{\lambda} \sin n\pi \bar{y} \]
into equations (22) and application of the Galerkin procedure yield the following equation of motion for odd numbers \( m, n \):
\[ W(\bar{t})_{\bar{t}\bar{t}} + k_1 W(\bar{t}) + k_2 W^2(\bar{t}) + k_3 W^3(\bar{t}) = q^*(\bar{t}), \]

*Note that \( \omega \) in equation (24) is different from the corresponding values in the papers of Hui [16,17] because Hui defined \( \rho \) as mass per unit area.*
where the coefficients \( k_i \) are defined as in [22]:

\[
\begin{align*}
  k_1 &= \frac{1}{\pi^4} \left[ C_a \left( \frac{m}{\lambda}, n \right) + \frac{C_b \left( \frac{m}{\lambda}, n \right)^2}{C_a \left( \frac{m}{\lambda}, n \right)} \right] + 4 \left( \frac{m}{\lambda} \right)^2 n^2 (c_1 + c_2)W_0^2 \\
  k_2 &= 6 \left( \frac{m}{\lambda} \right)^2 n^2 (c_1 + c_2)W_0 \\
  k_3 &= 2 \left( \frac{m}{\lambda} \right)^2 n^2 (c_1 + c_2),
\end{align*}
\]

\[\text{(30)}\]

where

\[
\begin{align*}
  C_a \left( \frac{m}{\lambda}, n \right) &= \left[ a_{22} \left( \frac{m}{\lambda} \right)^4 + (2a_{12} + a_{66}) \left( \frac{m}{\lambda} \right)^2 n^2 + a_{11}n^4 \right] \pi^4 \\
  C_b \left( \frac{m}{\lambda}, n \right) &= \left[ (2b_{26} - b_{61}) \left( \frac{m}{\lambda} \right)^3 n + (2b_{16} - b_{62}) \frac{m}{\lambda} n^3 \right] \pi^4 \\
  C_d \left( \frac{m}{\lambda}, n \right) &= \left[ d_{11} \left( \frac{m}{\lambda} \right)^4 + 2(d_{12} + 2d_{66}) \left( \frac{m}{\lambda} \right)^2 n^2 + d_{22}n^4 \right] \pi^4 \\
  c_1 &= n^2 \left[ 32 \left( \frac{m}{\lambda} \right)^2 d_{22} \right], \quad c_2 = \left( \frac{m}{\lambda} \right)^2 / (32n^2a_{11}).
\end{align*}
\]

\[\text{(31)}\]

The dimensionless load parameter \( \tilde{q}^* (\tau) \) obtained after substitution of equation (16) into equations (22) is

\[
\tilde{q}^* (\tau) = \frac{16 q_0}{mn\pi^2 E_Y h} \left( \frac{b}{h} \right)^4 (1 - \bar{\tau}/\bar{\tau}_p) e^{-\sigma \tilde{q}^* \bar{\tau}/\bar{\tau}_p},
\]

\[\text{(32)}\]

where

\[
\bar{\tau}_p = \omega t_p.
\]

\[\text{(32)}\]

The solution of equation (29) can be obtained numerically.

5. PLATE STRENGTH: THIN PLATE

The Tsai–Hill criterion can be used for thin plates as well as for thick ones. Therefore, equations (19)–(21) are applicable in this case as well. However, the constitutive relations used to obtain the stress resultants and stress couples are different from equations (2) and (3).

\[
\begin{bmatrix}
  A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\
  A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\
  0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\
  0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\
  0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\
  B_{16} & B_{26} & 0 & 0 & 0 & D_{66}
\end{bmatrix}
\begin{bmatrix}
  u_x + \frac{1}{2}w_{xy}^2 + w_{0,x}w_{x} \\
  v_y + \frac{1}{2}w_{xy}^2 + w_{0,y}w_{y} \\
  u_y + v_x + w_{x}w_{y} + w_{0,x}w_{y} + w_{x}w_{0,y} \\
  -w_{,xx} \\
  -w_{,yy} \\
  -2w_{,xy}
\end{bmatrix}
\]

\[\text{(34)}\]
Note that due to neglect of transverse shear deformations, the transverse shear stress resultants \( (Q_x, Q_y) \) must be obtained from the last two equilibrium equations (1), instead of constitutive equations (3).

6. NUMERICAL EXAMPLES AND DISCUSSION

The response of thick laminated plates with an aspect ratio equal to 2 and lamination angle \( \pm 30^\circ \) is shown in Figs 1 and 2. The material of the plates is high-modulus graphite-epoxy with the following non-dimensional characteristics: \( E_L/E_R = 40 \), \( G_{LT}/E_R = 0.5 \), \( G_{LT}/E_T = G_{TZ}/E_T = 0.6 \), \( \nu_{LT} = 0.25 \). The shear correction coefficients are \( k_1^2 = k_2^2 = 5/6 \). The blast parameters appearing in equation (16) are taken as \( a' = 1.98 \), \( t_P = 0.004 \text{ s} \). The response of two plates consisting of four super layers* with a total laminate thickness of 1 in. is shown in Fig. 1. The response of 1-in.-thick multi-layer plates which do not exhibit bending-stretching coupling (i.e. many alternating angle-ply layers) is shown in Fig. 2. In all cases, the non-dimensional amplitude of the load is \( \bar{q} = 0.1 \). Note that the curves in Figs 1 and 2 are not comparable since the values of \( \bar{q} \) and \( \tau \) depend on the plate natural frequencies \( \omega_{11} \) which are different. However, the general tendency is the same for all plates. The deflection reaches a maximum value, after which it decreases. If one continues the analysis for larger values of the non-dimensional time \( \tau \), the response appears to be oscillatory and the effect of damping should be taken into account. These oscillations are not shown in Figs

*The term 'super layer' is used here to mean a set of many parallel unidirectional layers. For example, if each super layer consists of 10 layers, then the total laminate thickness is equal to 40 layers.
The influence of initial imperfections on the non-linear response of the plates with the aspect ratio $\lambda = 2$ is shown in Fig. 3. The difference between the response of the perfect plate and that of the plate with a large initial imperfection is important only at the peak displacement and beyond. The difference in the maximum deflection is only about 4% of the plate thickness although the initial imperfection at the plate centre was equal to the thickness of the laminate. Therefore, the effect of initial imperfections is not very significant in this problem.

The responses of plates with different relative thickness are compared in Fig. 4. The decrease of the relative thickness $h/a$ by the factor 2 results in the sharp increase of the maximum deflection of geometrically non-linear plates (compare curves for $h/a = 0.01$ and 0.005).

The effect of the plate aspect ratio is shown in Fig. 5. It appears that the displacements of plates with larger aspect ratios are much larger. This can be explained in part by the fact that
FIG. 5. Effect of aspect ratio on non-linear response of thin laminated plates. \( w_0 = 0, \ h/b = 0.02, \ m = n = 1, \ h = 1 \text{ in.} \)

### Table 1. Comparison of different modes in response of laminated plates to blast loading

\[
\begin{array}{cccccccccccccccc}
\lambda & \xi & 0.113 & 0.226 & 0.339 & 0.451 & 0.564 & 0.675 & 0.788 & 0.900 & 1.012 & 1.125 & 1.238 & 1.351 & 1.464 \\
2 & \bar{W} & 0.063 & 0.126 & 0.186 & 0.242 & 0.295 & 0.341 & 0.383 & 0.417 & 0.445 & 0.464 & 0.475 & 0.478 & 0.473 \\
& (m = n = 1) & & & & & & & & & & & & & \\
& \bar{W} & 0.020 & 0.036 & 0.043 & 0.040 & 0.028 & 0.020 & 0.010 & & & & & & & \\
& (m = 3, n = 1) & & & & & & & & & & & & & \\
3 & \bar{W} & 0.063 & 0.126 & 0.188 & 0.248 & 0.305 & 0.359 & 0.410 & 0.457 & 0.500 & 0.537 & 0.570 & 0.597 & 0.619 \\
& (m = n = 1) & & & & & & & & & & & & & \\
& \bar{W} & 0.021 & 0.040 & 0.056 & 0.068 & 0.075 & 0.076 & 0.071 & 0.061 & 0.046 & 0.028 & 0.007 & & & & \\
& (m = 3, n = 1) & & & & & & & & & & & & & \\
\end{array}
\]

The ratio \( h/b \) was kept constant and, therefore, an increase of the aspect ratio corresponds to an increase of the plate planform area.

All results presented in Figs 3–5 were obtained for the mode shape of deformation with a single half-wave along both axes \( (m = n = 1) \). Therefore, the contribution of other modes has to be estimated. This is done in Table 1 where the amplitudes of the mode \( m = n = 1 \) are compared with those of the mode \( m = 3, n = 1 \) for two different aspect ratios. Note that the modal interaction was neglected, i.e. each mode was considered independently. Thus, the results in Table 1 represent an estimation rather than exact values of deflections. However, even these results illustrate that the mode \( m = 3, n = 1 \) becomes essential only if the aspect ratio \( \lambda > 3 \). The contribution of other modes is even smaller. Note that this conclusion is not universal; it applies to the particular plate arrangement considered in these numerical examples.

### References


**APPENDIX**

**SOME COEFFICIENTS**

\[ C_{33} = k_1^2 \alpha_{15}^2 \alpha_{16} + k_2^2 \alpha_{44} \beta_1^2, \]  
\[ (A1) \]

where

\[ \bar{A}_{ij} = \sum_{k=1}^{n} \bar{A}_{ij}(k)/n \]  
\[ (A2) \]

\[ F_1 = C_{35}(C_{35}S_6 - C_{34}S_5) / \bar{S} \]  
\[ F_2 = C_{34}(C_{34}S_7 - C_{35}S_5) / \bar{S}, \]  
\[ (A3) \]

where \( S_5, \ldots, S_8 \) are as given in Ref. [14] and

\[ \bar{S} = S_5 S_8 - S_6 S_7. \]  
\[ (A4) \]