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NON-LINEAR BEAM-TYPE VIBRATIONS OF LONG CYLINDRICAL SHELLS

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Abstract—This paper presents a closed-form solution of the problem of free beam-type vibration of a long cylindrical shell subject to uniform axial tension, uniform internal pressure, and elastic axial restraint. The shell is assumed to be clamped at the ends. The bending moment-curvature relationship employed in this paper is non-linear due to the effect of ovalization (flattening) of initially circular cross sections. Another non-linear effect taken into account is the stretching of the shell axis which is caused by the axial restraint.

The analysis results in a cubic differential equation; the frequency of the solution of this equation is found exactly using elliptic integrals.

INTRODUCTION

Large-amplitude flexural vibration of thin-walled circular cylindrical shells has been considered by numerous investigators [1–10]. Vibrations of perfect shells were studied by Reissner [1], Nowinski [2], Evensen [4] and Kildibekov [5], as well as many others. The effect of initial imperfections on vibrations was analyzed by Lipovskii and Tokarenko [3], Rosen and Singer [6, 7], and Watawala and Nash [8]. Experimental studies of non-linear vibrations were reported by Evensen [9] and Olson [10].

It is noted that vibrations of cylindrical shells are usually associated with a multi-half-wave mode shape of the elastic surface. On the contrary, the shape of a long vibrating shell or tube can be associated with the global deformation, i.e. the single half-wave along the shell axis and uniform cross sections which are sometimes assumed to remain circular. In this case the non-linear analysis of the shell can be performed using beam theory [11].

However, the assumption that the cross sections of a deformed shell remain circular is not accurate. Brazier found that the cross sections of a shell subjected to pure bending are ovalized [12]. The "flattening" of the cross sections contributes to non-linearity of the curvature-bending moment relationship. This fact was confirmed and generalized by Wood [13] and Reissner [14] for pressurized shells. The analysis of non-linear vibration of long shells including the Brazier effect was reported by Hu and Kirmser [15] who considered an unpressurized shell free at both ends.

The influence of elastic axial restraint on non-linear free vibrations of structures is usually studied for the limiting case, i.e. complete axial restraint. However, Ray and Bert [16] and Wrenn and Mayers [17] investigated the effect of elastic axial restraint on non-linear vibrations of beams.

In this paper are considered the free vibrations of long circular cylindrical shells which are both pressurized and subjected to tensile forces. The non-linearities due to cross sections flattening and the elastic axial restraint are included in the analysis. The mode shape of vibrating shell is assumed to be represented by the single-mode displacement function. Then the resulting non-linear differential equation is solved exactly using elliptic integrals. The effects of flattening, stretching, pressurizing, and tension on the frequency of the fundamental mode of free vibrations are considered in numerical examples.

GOVERNING EQUATIONS

Consider free large-amplitude vibrations of a long cylindrical shell with a mean radius a, length L, wall thickness h, and the cross sectional area F. The direct force T acts along the shell axis; the shell is pressurized, and the gauge pressure, i.e. the difference between the internal and the external pressure, is P.

The equation of global transverse vibrations of the shell is

$$\frac{\partial^2 M}{\partial z^2}+\left[T-\frac{kFE}{2L}\int_0^L\left(\frac{\partial U}{\partial z}\right)^2 \mathrm{d}z\right]\frac{\partial^2 U}{\partial z^2}+m\frac{\partial^2 U}{\partial t^2}=0 \tag{1}$$

where m is the mass of the shell per unit length, E is the modulus of elasticity, M is the bending moment, $U = U(z,t)$ is transverse displacement, z is axial position, and t is time. The non-linear term in equation (1) represents the stretching of the shell axis caused by the axial restraint expressed by the coefficient k. If the axial deformations of the shell are not restricted $k = 0$. The value $k = 1$ corresponds to the shell having ends that are immovable in the axial direction.

It is usually assumed that the circular cross sections remain unchanged during vibrations. Then bending moment is proportional to curvature and equation (1) coincides with the non-linear equation of motion of a solid rod whose solution was obtained by Woinowsky-Krieger [18] and Burgreen [19] for $k = 1$. However, as it was shown by Brazier [12], Wood [13], and Reissner [14] the relationship between the bending moment and curvature of long circular shells is non-linear. In the first approximation this relationship is:

$$M=\frac{EI}{\rho}\left(1-\frac{C}{\rho^2}\right). \tag{2}$$

Here I is the moment of inertia of the undeformed circular cross section, ρ is the axial radius of curvature, and C is the coefficient given by

$$C=\frac{1}{8}\left[\frac{\dfrac{a^4Eh}{D}}{1+\dfrac{1}{3}\dfrac{Pa^3}{D}}\right] \tag{3}$$

where D is the flexural rigidity given by

$$D = Eh^3/12(1-\nu^2) \tag{4}$$

and ν is Poisson's ratio.

The radius of curvature is

$$\frac{1}{\rho}=\frac{\partial^2 U/\partial z^2}{[1+(\partial U/\partial z)^2]^{3/2}}. \tag{5}$$

This expression can be simplified since it is assumed that $(\partial U/\partial z)^2 \ll 1$. Therefore

$$\frac{1}{\rho}=\frac{\partial^2 U}{\partial z^2}. \tag{6}$$

The substitution of equations (2), (3), (4), and (6) into equation (1) yields the following equation

$$m\frac{\partial^2 U}{\partial t^2}+\left[T-\frac{kFE}{2L}\int_0^L\left(\frac{\partial U}{\partial z}\right)^2 \mathrm{d}z\right]\frac{\partial^2 U}{\partial z^2} + EI\frac{\partial^4 U}{\partial z^4}-3CEI\left[2\frac{\partial^2 U}{\partial z^2}\left(\frac{\partial^3 U}{\partial z^3}\right)^2+\left(\frac{\partial^2 U}{\partial z^2}\right)^2\frac{\partial^4 U}{\partial z^4}\right]=0. \tag{7}$$

This equation can be conveniently represented in the non-dimensional form:

$$\frac{mL^4}{EI}\frac{\partial^2 u}{\partial t^2}+\frac{TL^2}{EI}\frac{\partial^2 u}{\partial s^2}-\frac{1}{2}k\bar{F}\left[\int_0^1\left(\frac{\partial u}{\partial s}\right)^2 \mathrm{d}s\right]\frac{\partial^2 u}{\partial s^2}+\frac{\partial^4 u}{\partial s^4}-\gamma\left[2\frac{\partial^2 u}{\partial s^2}\left(\frac{\partial^3 u}{\partial s^3}\right)^2+\left(\frac{\partial^2 u}{\partial s^2}\right)^2\frac{\partial^4 u}{\partial s^4}\right]=0 \tag{8}$$

where

$$s = z/L$$

$$u = U/a$$

$$\gamma = \frac{4.5(1-\nu^2)\left(\frac{a}{h}\right)^2\left(\frac{a}{L}\right)^4}{1+\bar{P}} \tag{9}$$

$$\bar{F} = \frac{Fa^2}{I} = \frac{4a^2}{2a^2+h^2/2}.$$

Note that if $h \ll a$, $\bar{F} \simeq 2$.

Non-dimensional pressure in equation (9) is

$$\bar{P} = P/|P_{cr}| \tag{10}$$

where P_{cr} is a critical external pressure given by

$$P_{cr} = -\frac{E}{4(1-\nu^2)}\left(\frac{h}{a}\right)^3. \tag{11}$$

FREE VIBRATIONS OF A SHELL CLAMPED AT THE ENDS

If the shell is clamped at both ends, the transverse displacement can be approximated by

$$u = A(t)\sin^2 \pi s. \tag{12}$$

The substitution of equation (12) in equation (8) and use of the Galerkin procedure yield

$$\frac{3}{16}\frac{mL^4}{EI}\frac{\mathrm{d}^2 A}{\mathrm{d}t^2}+\left(\pi^4-\frac{\pi^2 TL^2}{4EI}\right)A-\pi^4\left(\pi^4\gamma-\frac{1}{16}k\bar{F}\right)A^3=0. \tag{13}$$

The natural frequency of linear transverse vibration of the shell can be immediately found from equation (13) ($T=0$)

$$\lambda_o = 4\frac{\pi^2}{L^2}\sqrt{\frac{EI}{3m}}. \tag{14}$$

The axial buckling load of the shell is

$$T_{cr} = 4\left(\frac{\pi}{L}\right)^2 EI. \tag{15}$$

Introducing the non-dimensional time parameter

$$\tau = \lambda_0 t \tag{16}$$

and the non-dimensional axial load

$$\bar{T} = T/T_{cr} \tag{17}$$

one may rewrite equation (13) as

$$\frac{d^2A}{d\tau^2} + (1 - \bar{T})A - \left(\pi^4\gamma - \frac{1}{16}k\bar{F}\right)A^3 = 0. \tag{18}$$

Exact solution of this equation is available in terms of elliptic functions [20, 21]. Multiplication of equation (18) by dA and integration yields

$$\left(\frac{dA}{d\tau}\right)^2 = -(1 - \bar{T})A^2 + \frac{1}{2}\left(\pi^4\gamma - \frac{1}{16}k\bar{F}\right)A^4 + \bar{C} \tag{19}$$

where $\bar{C}$ is a constant of integration. When the shell displacement is maximum ($\bar{A}$), the velocity is equal to zero:

$$A(0) = \bar{A}$$
$$\frac{dA(0)}{d\tau} = 0. \tag{20}$$

The substitution of equations (20) into equation (19) yields

$$\bar{C} = (1 - \bar{T})\bar{A}^2 - \frac{1}{2}\left(\pi^4\gamma - \frac{1}{16}k\bar{F}\right)\bar{A}^4.$$

Now equation (19) can be written as

$$\frac{d\xi}{d\tau} = \sqrt{(1 - \bar{T})(1 - \xi^2) - \frac{1}{2}\left(\pi^4\gamma - \frac{1}{16}k\bar{F}\right)\bar{A}^2(1 - \xi^4)} \tag{21}$$

where $\xi = A/\bar{A}$. (22)

The solution of equation (21); obtained for the initial condition that $\xi = 0$ when $\tau = 0$, is

$$\tau = \frac{1}{\pi^2\bar{A}}\sqrt{\frac{2}{\gamma}}\int_0^{\xi}\frac{d\xi}{\sqrt{(1 - \xi^2)(b^2 - \xi^2)}} \tag{23}$$

where

$$b^2 = \frac{1 - \bar{T} - \frac{1}{2}\left(\pi^4\gamma - \frac{1}{16}k\bar{F}\right)\bar{A}^2}{\frac{1}{2}\left(\pi^4\gamma - \frac{1}{16}k\bar{F}\right)\bar{A}^2}. \tag{24}$$

The restoring force in equation (18) is an odd function of A. Therefore, the non-dimensional period of motion can be calculated as

$$\bar{\tau} = 4\bar{\tau}' \tag{25}$$

where $\bar{\tau}'$ is time between $\xi = 0$ and $\xi = 1$, i.e.

$$\bar{\tau}' = \frac{1}{\pi^2 \bar{A}} \sqrt{\frac{2}{\gamma b^2}} \int_0^1 \frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-n\xi^2)}} \tag{26}$$

$$n = 1/b^2. \tag{27}$$

It is noted that the integral in equation (26) is a complete elliptic integral of the first kind:

$$K(n) = \int_0^1 \frac{\mathrm{d}\xi}{\sqrt{(1-\xi^2)(1-n\xi^2)}}. \tag{28}$$

Therefore, the non-dimensional frequency of vibration is

$$\omega = \frac{\pi}{2} \sqrt{\frac{1 - \bar{T} - \frac{1}{2}\pi^4\gamma\left(1 - \frac{k\bar{F}}{16\pi^4\gamma}\right)\bar{A}^2}{1 - \frac{k\bar{F}}{16\pi^4\gamma}}} \frac{1}{K(n)}. \tag{29}$$

If the effects of flattening of the cross sections and non-linear stretching on the frequencies are neglected the non-dimensional frequency given by equation (29) is reduced to the classical result available also from equation (18) when $\gamma = 0$, $k = 0$:

$$\omega_0 = \sqrt{1 - \bar{T}}. \tag{30}$$

Finally it is noted that the non-dimensional frequency given by equation (29) depends on the non-dimensional axial load $\bar{T}$, the parameter γ (which reflects the geometry of the tube and the gauge pressure), the coefficient of the axial restraint k, and the non-dimensional amplitude of vibration $\bar{A}$.

NUMERICAL RESULTS AND DISCUSSION

The non-dimensional frequency–amplitude relationships for long cylindrical shells without axial restraint are plotted in Figs 1 and 2 for different values of the parameter

$$q = \frac{1}{2}\pi^4\gamma. \tag{31}$$

The relationships given in Fig. 1 correspond to the shell without external tension ($\bar{T} = 0$). The curves in Fig. 2 represent the response of the shell subjected to the tensile load $\bar{T} = -1$.

It can be noted that the effect of flattening of the cross sections is negligible for very small values of q. Indeed, the frequency–amplitude relationship for $q = 0.01$ almost coincides with the classical linear results which are given by $\omega_0 = 1.0$ (Fig. 1) and $\omega_0 = 1.41$ (Fig. 2).

The effect of flattening of the cross sections increases with q. This means that the increase of internal pressure reduces the effect of flattening, as would be expected.

The frequency–amplitude relationship of a shell having cross sections which flatten during vibration is softening. The frequency of the shell with $q = 0.5$ approaches zero when the amplitude increases. This means that the shell is unstable when vibrating with such amplitudes. It will snap-through to a curved equilibrium position; vibration takes place about this position.

The influence of tension on frequencies of vibration is shown in Fig. 3 for different values of $q\bar{A}^2$ ($k = 0$). The curves ω correspond to the solution obtained in this paper. The curve ω_0 calculated from equation (30) represents the linear result. In all cases an increase in tension increases the frequencies. It is noted that the curves ω and ω_0 lie very close if

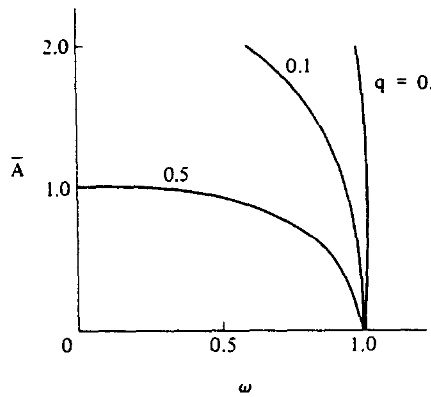


Fig. 1. Frequency–amplitude relationship for long cylindrical shells; no external tension ($\bar{T} = 0$).

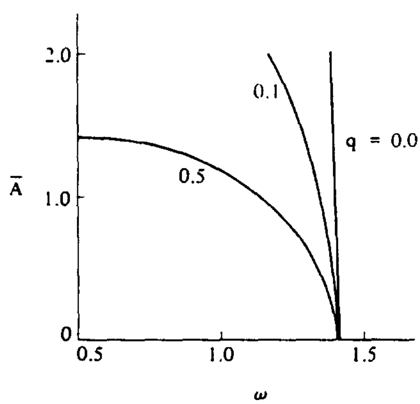


Fig. 2. Frequency–amplitude relationship for long cylindrical shells; $\bar{T} = -1$.

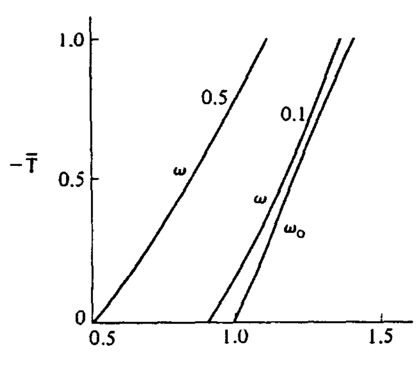


Fig. 3. Influence of tension on frequency–amplitude relationship of the shell; $q\bar{A}^2 = 0.1$, $q\bar{A}^2 = 0.5$.

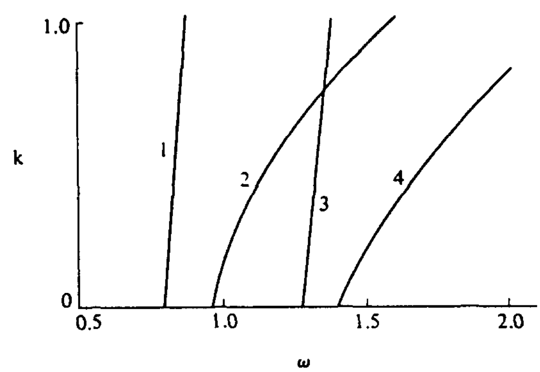
$q\bar{A}^2 \leqslant 0.1$. In the case of $q\bar{A}^2 = 0.5$ the frequency is far from that obtained from the linear solution; however, the tendency of increase of frequency with tension is preserved.

The effect of elastic axial restraint on the frequencies is illustrated in Fig. 4. Both shells free of external axial forces (curves 1, 2) and shells subjected to tension (curves 3, 4) are considered. In all cases axial restraint is shown to increase the frequencies of vibration. This tendency was also found for beams in [16, 17]. If the shell is subject to large internal pressure (curves 2, 4) the effect of axial restraint is considerably stronger.

The frequency–amplitude relationships of a shell with $q = 0.3$, $\bar{T} = 0$, for two limiting cases of axial restraint, are shown in Fig. 5. Although the frequencies of a completely restrained shell ($k = 1$) are larger, the curves representing the frequency–amplitude relationships are quite similar.

REFERENCES

1. E. Reissner, Non-linear effects in vibrations of cylindrical shells, Ramo-Wooldridge Corp., Guided Missile Res. Div., Rept. AM 5-6, Aug. (1955).
2. J. L. Nowinski, Non-linear transverse vibrations of orthotropic cylindrical shells. *AIAA J.* 1, 617–620 (1963).
3. D. E. Lipovskii and V. M. Tokarenko, Effect of initial imperfection on free oscillation frequencies of cylindrical shells. In *VI All-Union Conference on the Theory of Shells and Plates*, pp. 542–547. Nauka, Moscow, in Russian (1966).
4. D. A. Evensen, Nonlinear flexural vibrations of thin-walled circular cylinders. *NASA TN* D-4090 (1967).
5. I. G. Kildibekov, Free non-linear vibrations of a circular cylindrical shell. In *VIII All-Union Conference on the Theory of Shells and Plates*, pp. 488–491. Nauka, Moscow, in Russian (1973).
6. A. Rosen and J. Singer, The effect of axisymmetric initial imperfections on the vibrations of cylindrical shells. *AIAA J.* **12**, 995–997 (1974).
7. A. Rosen and J. Singer, Influence of asymmetric imperfections on the vibrations of axially compressed cylindrical shells. *Isr. J. Tech.* **14**, 23–36 (1976).



1 $\bar{T} = 0$, $q = 0.5$
2 $\bar{T} = 0$, $q = 0.1$
3 $\bar{T} = -1.0$, $q = 0.5$
4 $\bar{T} = -1.0$, $q = 0.1$

Fig. 4. Effect of elastic axial restraint on frequencies of free vibration of long cylindrical shells ($\bar{A} = 0.7$).

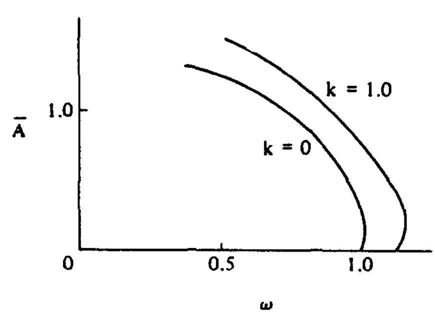


Fig. 5. Frequency–amplitude relationship for long cylindrical shells unrestrained ($k = 0$) and completely restrained ($k = 1$) in axial direction; $q = 0.3$, $\bar{T} = 0$.

8. L. Watawala and W. A. Nash, Influence of initial geometric imperfections on vibrations of thin circular cylindrical shells. *Comput. Struct.* **16**, 125–130 (1983).
9. D. A. Evensen, Some observations on the non-linear vibrations of thin cylindrical shells. *AIAA J.* **1**, 2857–2858 (1963).
10. M. D. Olson, Some experimental observations on the non-linear vibrations of thin cylindrical shells. *AIAA J.* **3**, 1775–1777 (1965).
11. B. E. Bennett and M. F. Metcalf, Non-linear dynamic analysis of coupled axial and lateral motions of marine risers. In *Proceedings of the 9th Offshore Technology Conference*, Paper OTC 2776, Houston, Texas (1977).
12. L. G. Brazier, On the flexure of thin cylindrical shells and other thin sections. *Proc. R. Soc., Lond. A* **116**, 104–114 (1927).
13. J. D. Wood, The flexure of a uniformly pressurized circular cylindrical shell. *J. appl. Mech.* **25**, 453–458 (1958).
14. E. Reissner, On finite bending of pressurized tubes. *J. appl. Mech.* **26**, 386–392 (1959).
15. K. K. Hu and P. G. Kirmser, On the non-linear vibrations of free-free beams. *J. appl. Mech.* **38**, 461–466 (1971).
16. J. D. Ray and C. W. Bert, Non-linear vibrations of a beam with pinned ends. *J. Engrg Indust.* **91B**, 997–1006 (1969).
17. B. G. Wrenn and J. Mayers, Non-linear beam vibration with variable axial boundary restraint. *AIAA J.* **8**, 1718–1720 (1970).
18. S. Woinowsky-Krieger, The effect of an axial force on the vibration of hinged bars. *J. appl. Mech.* **17**, 35–36 (1950).
19. D. Burgreen, Free vibrations of a pin-ended column with constant distance between the pin ends. *J. appl. Mech.* **18**, 135–139 (1951).
20. H. Jeffreys and B. Swirles, *Methods of Mathematical Physics*, 3rd edn, Chap. 25. Cambridge University Press, Cambridge (1980).
21. A. P. Filippov, *Vibrations of Deformable Systems*, pp. 462–463. Mashinostroenie, Moscow, in Russian (1970).