Onset of matrix cracking in angle-ply ceramic matrix composites

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Received 3 April 2001; received in revised form 8 April 2002; accepted 3 October 2002

Abstract

The problem of initial damage in angle-ply $[-\theta_m/0_n/\theta_m]$ and $[-\theta/0]$ ceramic matrix composites subjected to axial tension is considered in this paper. The damage is in the form of matrix cracks that may appear in either inclined ($-\theta$ and $\theta$ lamination angle) or longitudinal layers. As follows from the analysis, if the lamination angle of the inclined layers is small, the initial failure occurs in the 0-layers of $[-\theta_m/0_n/\theta_m]$ composites or in $[-\theta/0]$ composites in the form of bridging cracks. However, if the inclined layers form a larger angle with the load direction, they fail due to tunneling cracks. It is shown that the boundary between two different modes of failure in a representative SiC/CAS composite corresponds to a lamination angle equal to 35\textdegree in the case of $[-\theta_m/0_n/\theta_m]$ composites. In the case of $[-\theta/0]$ laminates, the boundary value of the lamination angle is equal to 45\textdegree, i.e. bridging cracks form if $\theta < 45\textdegree$ and tunneling cracks appear if $\theta > 45\textdegree$.

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0. Introduction

Ceramic matrix composites (CMC) have been employed in applications that require high toughness, environmental stability, fatigue resistance and damage endurance [1]. Typical applications include rocket motors, turbines, thermal protection blankets for re-entry vehicles, brake discs, etc.

A typical mode of failure in CMC is related to matrix cracking that results in a degradation of the strength and stiffness. Even more importantly, matrix cracks serve as pathways for oxygen...
Nomenclature

- \( A \) matrix of extensional stiffnesses
- \( a \) half-length of the crack
- \( a_{ij} \) compliance coefficients
- \( B \) matrix of coupling stiffnesses
- \( D \) matrix of bending stiffnesses
- \( E \) modulus of elasticity
- \( G \) energy release rate
- \( K \) stress intensity factor
- \( M^T \) residual thermal stress couple
- \( m \) number of layers with the lamination angle equal to \( +\theta \) or \(-\theta \)
- \( N^T \) residual thermal stress resultant
- \( n \) number of longitudinal layers (lamination angle equal to 0)
- \( p \) material constant in Eq. (19)
- \( [Q_k] \) matrix of transformed reduced stiffnesses of the \( k \)th layer
- \( q \) material constant in Eq. (19)
- \( r \) fiber radius
- \( T \) difference between the processing and operational temperatures
- \( V_i \) volume fraction of fibers \((i = f)\) and matrix \((i = m)\)

Greek letters

- \( \alpha_i \) coefficients of thermal expansion of the fibers \((i = f)\) and matrix \((i = m)\)
- \( \beta \) angle between the crack and the applied stress direction
- \( \gamma_F \) surface fracture energy density
- \( \gamma_m \) matrix fracture energy density
- \( \nu \) Poisson ratio
- \( \theta \) lamination angle
- \( \sigma_c \) applied composite stress
- \( \tau \) interfacial friction shear stress

to the fiber-matrix interfaces. The subsequent oxidation results in a dramatic embrittlement of the material. A comprehensive review of work performed on matrix cracking in CMC can be found in [1], mentioned here are experimental studies that illustrated the presence of matrix cracks in CMC [2,3].

The majority of investigations of matrix cracks have been confined to unidirectional and cross-ply composites. Long bridging cracks perpendicular to the fibers are a typical damage mode in the former case, while tunneling cracks propagating in transverse layers in the plane perpendicular to the load direction represent a typical initial failure mode in cross-ply laminates. Early studies on cracks in unidirectional and cross-ply composites are discussed in [1]. Examples of recent research can be found in [4,5].

Although extensive research has been conducted on matrix cracking in cross-ply laminates, their angle-ply counterparts have received less attention (some aspects of the phenomenon have been
considered in [6,7]). However, a widespread use of angle-ply composites justifies rigorous research of possible modes of damage in these materials. The present paper illustrates the solution for the onset of cracking in angle-ply CMC laminates with the layers arranged in \([- \theta_m/0_n/\theta_m]\) or \([- \theta/\theta]\) blocks. The cracks are assumed to be caused by the load acting in the axial \((x)\) direction. The solution includes the following steps:

1. Analysis of the mode of initial damage due to applied axial load: bridging or tunneling cracks.
2. Residual thermal stresses in the intact laminate (the knowledge of these stresses is necessary at the first step).
3. Analysis of the feasibility of initial matrix cracking in inclined layers with the cracks oriented at an angle to the fibers.

1. Mode of initial failure in \([- \theta_m/0_n/\theta_m]\) laminates

Consider a decomposed laminate subjected to stresses acting in the \(x\)-direction (Figs. 1 and 2). Fracture in longitudinal layers (0-layers) corresponds to Mode I cracking. On the other hand, \(\theta\)-layers experience a combination of stresses acting in the directions perpendicular and parallel to the fibers and a shear stress. Accordingly, their cracking is treated as a combination of Mode I and Mode II fracture problems. If the angle \(\theta\) is large, the laminate behaves in a manner similar to a cross-ply laminate. In this case, initial cracks appear in \(\theta\)-layers where they are parallel to the general fiber direction (tunneling cracks). However, if the angle \(\theta\) is small, initial cracks are likely to develop in longitudinal layers. These are so-called bridging cracks that are perpendicular to the fibers. Another possibility is related to bridging cracking in \(\theta\)-layers if the lamination angle is small. Finally, it is necessary to investigate whether cracks in \(\theta\)-layers may be oriented at an angle to the fibers (Fig. 3).

1.1. Bridging cracks in 0-layers

It is necessary for the subsequent analysis to relate the applied composite stress to the stresses acting in the layers of the intact laminate. The stress in 0-layers of a composite beam (negligible Poisson’s effect, all layers have an equal thickness) is found as

\[
\sigma^0_x = \sigma_c[(n + 2m)E^0_x/(nE^0_x + 2mE^0_x)]
\]

where \(\sigma_c\) is an applied composite stress, and \(E^0_x\) and \(E^0_\theta\) are the moduli of elasticity of 0 and \(\theta\)-layers, respectively. Note that if the laminate is wide, i.e. the Poisson effect cannot be neglected, the elastic moduli in Eq. (1) should be replaced with the transformed reduced stiffnesses of the corresponding layers.

The stresses in the principal directions of the \(\theta\)-layers of the intact laminate are found using the stress acting in the \(x\)-direction, i.e.

\[
\sigma^0_\theta = \sigma_c[(n + 2m)E^0_\theta/(nE^0_\theta + 2mE^0_\theta)].
\]

Subsequently, the stress transformation equations yield

\[
\{\sigma_1, \sigma_2, \tau_{12}\}^T = [P_0]\{\sigma^0_x, 0, 0\}^T,
\]
where \([P_0]\) is a transformation matrix. Now we have to compare the conditions for initial cracking in 0 and \(sDC2\)-layers (see Fig. 2). According to the Aveston–Cooper–Kelly theory [8], the applied matrix cracking initiation stress in a unidirectional CMC lamina is given by

\[
\sigma^0_{x,cr} = E_c\left((12\gamma_m\tau E_f V_f^2)/(E_c E_m^2 V_m r)\right)^{1/3} - \sigma^0_{x,r},
\]  

(4)

where \(E_f\) and \(E_m\) are the elasticity moduli of the fibers and matrix, respectively, \(V_f\) and \(V_m\) are the volume fractions of the fibers and the matrix, \(r\) is the fiber radius, \(\gamma_m\) is the matrix fracture energy density, and \(\tau\) is the interfacial friction shear stress that is assumed constant. The last term in (4) represents the contribution of the residual thermal stress that has to be determined based on the post-processing thermal stress analysis. The intact layer longitudinal modulus in Eq. (4) is obtained by the rule of mixtures

\[
E_c = V_f E_f + V_m E_m.
\]  

(5)

The contribution of the residual thermal stress in a unidirectional lamina can be determined as

\[
\sigma^0_{x,r} = E_f V_f (\alpha_f - \alpha_m) T,
\]  

(6)

where \(T\) is a difference between the processing and operational temperatures, and \(\alpha_f\) and \(\alpha_m\) are the coefficients of thermal expansion of the fibers and matrix, respectively. The stress calculated by
Eq. (6) reflects only thermal residual stresses in a unidirectional lamina and it does not reflect an additional contribution due to thermal stresses in the laminate. The latter contribution is found below and it is shown to be much smaller than the stress obtained by Eq. (6), so that it is possible to neglect it altogether.

Note that according to experiments of Wang et al. [9], Eq. (4) is a reliable lower-bound estimate of the matrix cracking initiation stress. The stress given by Eq. (4) is obtained for a single layer. In the case of a laminate, the applied composite stress \(\sigma_c\) that results in the onset of bridging cracks in 0-layers can be derived from the stress given by Eq. (4) using Eq. (1).

It is also necessary to note that Eq. (4) was derived neglecting the work of the interface debonding, in other words, it refers to the case of weakly bonded or debonded fibers. The cases where the fibers are fully bonded to the matrix or the interface debonds prior to matrix cracking were considered by Aveston and Kelly [10] and by Budiansky et al. [11]. As could be expected, these solutions yield higher critical stresses since they account for additional energy terms. For convenience, the corresponding equations are presented in Appendix A.

1.2. Tunneling cracks in \(\theta\)-layers

The condition for tunneling cracking in \(\theta\)-layers is based on the introduction of an equivalent matrix cracking stress \(\sigma_2\) acting in the direction perpendicular to the general fiber direction. It is assumed that the initial tunneling crack appears in the \(\theta\)-layer, if it is subject to a critical value of this stress. The value of \(\sigma_{2,cr}\) can be determined from one of numerous theories for transverse matrix cracking. For example, the theory of Han et al. [12] is used below.

The stress \(\sigma_2\) is related to the applied stress through the requirement that the critical energy release rate corresponding to the former stress and the energy release rate for the mixed mode of fracture associated with the actual applied stress \(\sigma_\theta\) should be equal. The stress intensity factors for Modes I and II fracture in the case where a crack is oriented at an angle \(\beta\) to the applied stress direction are [13]:

\[
K_I = \sigma_\theta^0 (\pi a)^{1/2} \sin^2 \beta, \\
K_{II} = \sigma_\theta^0 (\pi a)^{1/2} \sin \beta \cos \beta, \tag{7}
\]

where \(2a\) is the length of the crack. In the case of tunneling cracks considered here \(\beta = \theta\).

The expression for the energy release rate for a mixed fracture mode in the case where the crack propagates along the line of material property symmetry, as is the case with tunneling cracks, was suggested by Sih and Liebowitz [14]:

\[
G = f(a_{ij})[(a_{11}a_{22}/2)^{1/2}K_I^2 + (a_{11}/2)^{1/2}K_{II}^2], \tag{8}
\]

where \(f(a_{ij})\) is a function of the compliance coefficients in the Hookean relationships for a specially orthotropic material. These relationships are:

\[
\varepsilon_1 = a_{11}\sigma_1 + a_{12}\sigma_2, \\
\varepsilon_2 = a_{12}\sigma_1 + a_{22}\sigma_2, \\
\gamma_{12} = a_{66}\tau_{12}. \tag{9}
\]
Note that Eqs. (8) and (9) refer to the case where Mode I fracture is associated with $\sigma_2$. In the case of bridging cracks in $\theta$-layers considered below this fracture mode is due to $\sigma_1$ and notation in the corresponding equations will be modified accordingly.

It is required now that $G = G_c$ where $G_c$ is the fracture toughness of a transverse layer with tunneling cracks due to Mode I fracture, i.e. cracks that appear as a result of the stress $\sigma_2$. The expression for $G_c$ follows from Eq. (8) where $K_1 = \sigma_2 \pi a^{1/2}$ and $K_{\text{II}} = 0$. Equating $G_c$ to $G$ given by Eq. (8) yields the matrix cracking stress equivalent to the actually applied off-axis stresses:

$$\sigma_2 = \sigma^0_2 \sin \theta [\sin^2 \theta + (a_{11}/a_{22})^{1/2} \cos^2 \theta]^{1/2}. \quad (10)$$

This stress should be taken equal to the first-ply cracking stress in a transverse layer. For example, using the theory for transverse cracking in cross-ply laminates developed by Han et al. [12], the first-ply cracking stress in a transverse layer can be written as

$$\sigma_{2,\text{cr}} = (2\gamma_F/h_0)^{1/2}[12E_L E_T G_T/(E_L + E_T)]^{1/4} - \sigma^0_{2,r}, \quad (11)$$

where $\gamma_F$ is the surface fracture energy density (for cracks limited to the matrix, it may be taken equal to $\gamma_m$), $h_0$ is the total thickness of $m \theta$-layers, $G_T$ is a transverse shear modulus in the plane perpendicular to the fibers, and $\sigma^0_{2,r}$ is a residual thermal stress in the layer obtained from the thermal analysis of the intact laminate. It is emphasized here that while the residual thermal stress in Eq. (4) can be quite high due to a mismatch of the coefficients of thermal expansion of the fibers and matrix, the residual stress in Eq. (11) is much smaller being a ply stress that does not reflect the fiber-matrix mismatch.

Equating the right sides of (10) and (11), one obtains the applied stress resulting in cracking of $\theta$-layers (longitudinal 0-layers are intact):

$$\sigma^0_{x,\text{cr}} = \{(2\gamma_F/h_0)^{1/2}[12E_L E_T G_T/(E_L + E_T)]^{1/4} - \sigma^0_{2,r}\}/\{\sin \theta [\sin^2 \theta + (a_{11}/a_{22})^{1/2} \cos^2 \theta]^{1/2}\}. \quad (12)$$

Note that the stress in the left side of Eq. (12) is applied to the $\theta$-layers. This stress can be related to the applied composite stress acting in the $x$-direction by Eq. (2).

### 1.3. Bridging cracks in $\theta$-layers

Consider now a possibility that bridging cracks perpendicular to the fibers may develop in $\theta$-layers if the angle $\theta$ is small (Fig. 2). The approach is similar to that employed above to determine an equivalent Mode I fracture stress in the case of tunneling cracks. Considering that in this case Mode I fracture is due to the stress acting along the 1-axis, it is easy to show that $\beta = \pi/2 - \theta$, and $a_{11}$ and $a_{22}$ should be rotated in Eq. (8). Then using the cracking stress given by Eq. (4), the applied crack initiation stress is obtained as

$$\sigma^0_{x,\text{cr}} = \{E_{c1}[(12\gamma_m \tau(E T^2)/(E_{c1} E_m^2 V_m r)]^{1/3} - \sigma^0_{1,r}\}/\{\cos \theta [\cos^2 \theta + (a_{22}/a_{11})^{1/2} \sin^2 \theta]^{1/2}\}, \quad (13)$$

where $\sigma^0_{1,r}$ is a residual thermal stress. The latter stress contains a contribution due to the mismatch of the coefficients of thermal expansion of the fibers and matrix and a contribution related to the thermal laminate analysis. The former contribution is often dominant as is shown in numerical examples.
below. This contribution is obtained by Eq. (6) where the subscript “x” should be replaced with the subscript “1”. The contribution obtained from the laminate analysis is obtained in the next section.

Note that it can immediately be predicted whether the initial damage will occur in the form of bridging cracks in 0-layers or in \( \theta \)-layers. The former situation occurs if

\[
\cos \theta [\cos^2 \theta + (a_{22}/a_{11})^{1/2} \sin^2 \theta] < \left\{ E_c [(12\gamma_1 r E_1 V_1^2)/(E_c E_m V_m r)]^{1/3} - \sigma_{1,r}^0 \right\} / \left\{ E_c [(12\gamma_1 r E_1 V_1^2)/(E_c E_m V_m r)]^{1/3} - \sigma_{x,r}^0 \right\}.
\]

This condition is often satisfied in CMC laminates. In particular, if \( \sigma_{x,r}^0 \) is approximately equal to \( \sigma_{1,r}^0 \), as is often the case, the right side of (14) is equal to unity. Then it is easy to find the ratio \( a_{22}/a_{11} = E_L/E_T \) as a function of the lamination angle. The boundary value of the ratio \( E_L/E_T \) necessary for bridging cracks to develop in 0-layers, rather than in \( \theta \)-layers is shown in Fig. 4 for a representative SiC/CAS material analyzed in numerical examples as a function of the lamination angle. Bridging cracks form in 0-layers if \( E_L/E_T \) is less than this ratio shown in Fig. 4. Note that this condition is usually satisfied in CMC where the stiffness of the fibers is comparable to that of the matrix.

The smallest value of the applied composite stresses obtained using Eqs. (1) and (2) and based on the results given by Eqs. (4), (12) and (13) defines the mode of initial cracking, i.e. bridging cracks in 0-layers or in \( \theta \)-layers or tunneling cracks in \( \theta \)-layers. Also, it may be necessary to compare the result given by Eq. (4) to those that follow from Eqs. (A1) and (A3).

2. Residual thermal stresses in the intact laminate

These stresses are needed for the analysis of initial matrix cracking. Note that the effect of temperature on material properties is neglected in the subsequent analysis.

The matrices of extensional, coupling, and bending stiffnesses of the intact laminate, i.e. A, B, and D, can be determined in terms of engineering constants using standard equations of the theory of composite materials. Subsequently, thermal residual strains in a composite laminate that has not been subject to mechanical loading are found as [15]:

\[
\varepsilon^0 = A'N^T + B'M^T,
\]

\[
\kappa = B'N^T + D'M^T,
\]

where \( \varepsilon^0 \) and \( \kappa \) are the vectors of middle plane strains and changes of curvature and twist, respectively. The vectors of thermal stress resultants and stress couples are given by

\[
N^T = \Sigma [Q_k]\{z_k\}(z_k - z_{k-1})T,
\]

\[
M^T = (1/2) \Sigma [Q_k]\{z_k\}(z_k^2 - z_{k-1}^2)T.
\]

In Eq. (16), \( [Q_k] \) is a matrix of transformed reduced stiffnesses of the \( k \)th layer, \( \{z_k\} \) is a vector of thermal expansion coefficients of this layer, and \( z_i \) are the coordinates of its interfaces. The summation is performed over all layers constituting the laminate. The superscript “T” in the left
side of Eq. (16) indicates that the corresponding vector is due to a temperature variation (not a transpose of a vector).

The inverted matrices $A'$, $B'$ and $D'$ are given as [15]:

$$A' = A^{-1} + A^{-1}B(D - BA^{-1}B)^{-1}BA^{-1},$$

$$B' = -A^{-1}B(D - BA^{-1}B)^{-1},$$

$$D' = (D - BA^{-1}B)^{-1}. \tag{17}$$

It is emphasized that Eqs. (15), (16) and (17) are written in the $x$–$y$–$z$ coordinates system. For convenience, the transformation equations for reduced stiffnesses and for the vectors of the coefficients of thermal expansion are provided in Appendix B.

Now thermal residual stresses in the $k$th layer can be determined from

$$\{\sigma_k\} = [Q_k](\{\epsilon^{0}\} + z\{\kappa\} - \{z_k\}T). \tag{18}$$

The thermal residual stresses associated with the laminate response to post-processing cooling obtained by Eqs. (18) in the $x$–$y$ coordinate system can be transformed into the layer principal stresses in the 1–2 coordinate system using the stress transformation equations presented in Appendix B.

3. On the possibility of initial inclined cracks in $\theta$-layers

Consider now whether initial cracks in $s_{DC2}$-layers can be oriented at an angle $\beta$ relative to the fiber direction that is different from either zero or 90° (Fig. 3). One possibility is to use the criterion introduced by Wu [16] and generalized by Spencer and Barnby [17]. According to this criterion, initial cracks are formed in a lamina if

$$(K_I/K_{Ic})^p + (K_{II}/K_{IIc})^q = 1, \tag{19}$$

where $p$ and $q$ are material constants and the critical values of the stress intensity factors for Modes I and II fracture, i.e. $K_{Ic}$ and $K_{IIc}$, are assumed known for the material of the lamina.

Substituting stress intensity factors given by Eqs. (7) we can evaluate the smallest value of the applied layer stress corresponding to failure as a function of angle $\beta$. Note however, that the crack initiation stress available from Eq. (19) is affected by the crack half-length $a$. Therefore, we will consider using a different criterion.

![Fig. 3. Initial cracks (broken lines ------) formed in a $\theta$-layer at an arbitrary angle $\beta$ different from $\theta$.](image-url)
In the case of a combination of Modes I and II, the energy release rate is given by [13]

\[ G = E'(K_I^2 + K_{II}^2), \] (20)

where \( E' \) is a plane-strain modulus of elasticity of the lamina. The angle of initial cracks relative to the \( x \)-axis is obtained from the requirement that they are oriented so that the energy release rate is minimum. Then substituting Eq. (7) into Eq. (20) and requiring

\[ \frac{\partial G}{\partial \beta} = 0 \] (21)

one obtains \( \beta = 0 \) or \( \pi/2 \). Therefore, initial cracking occurs either in the form of bridging cracks or in the form of tunneling cracks.

Note that the solution outlined above refers to the case of a “self-similar” crack propagation. This is expected in case of an initial mixed mode cracking in a lamina. The solution suggested by Sih and Chen [18,19] could be used to analyze the direction of propagation of the already formed initial tunneling crack, as a result of a gradually increasing load. However, this analysis is beyond the scope of the present paper. It should also be noted that the solution developed in [18,19] is based on the model assuming the crack propagation in a layer of matrix sandwiched between a smeared fiber-matrix medium. The validity of this model may be questioned on the basis of the fact that once the direction of the crack changes from that aligned with the fiber direction, the crack penetrates into the orthotropic medium consisting of the fibers and matrix.

4. Mode of initial failure in \([ -\theta/\theta] \) laminates

The mixed-mode energy release rate corresponding to formation of tunneling cracks \((G = G_t)\) is given by Eq. (8). In the case of bridging cracks, the strain energy release rate given by (8) should be modified, as is clear from the comparison of Figs. 2b and c, i.e. the subscripts 1 and 2 should be rotated in Eqs. (8) and (9). Accordingly, for bridging cracks,

\[ G_b = f(a_{ij})[(a_{11}a_{22}/2)^{1/2}K_1^2 + (a_{22}/2^{1/2})K_{II}^2]. \] (22)

It is obvious that bridging cracks can be anticipated in the case of a small lamination angle, while tunneling cracks will appear if the lamination angle is large. The boundary between two modes of cracking is defined from the condition that the energy release rates corresponding to these modes are equal to each other, i.e. \( G_t = G_b \). This condition yields the equation for the critical value of the lamination angle \( \theta_{ct} \):

\[ (a_{11}a_{22})^{1/2} \{ [f(a_{ij}) - f(a_{ji})] - [f(a_{ij}) + f(a_{ji})] \cos 2\theta_{ct} \} = [a_{22}f(a_{ji}) - a_{11}f(a_{ij})] \sin 2\theta_{ct}. \] (23)

Numerical solution of this equation can easily be obtained. Note that in the present case, i.e. if the laminate consists of the layers oriented at either \( \theta \) or \(-\theta\) angle to the direction of the applied load, the magnitude of the applied stress does not affect the critical value of the lamination angle.
5. Numerical analysis

The properties of the SiC/CAS material considered in the following examples are chosen based on the data reported in Refs. [20–23]:

\( E_f = 200 \) GPa, \( E_m = 97 \) GPa, \( V_f = 0.35 \), \( r = 8 \times 10^{-6} \) m, \( \tau = 17 \) MPa, \( \alpha_f = 4 \times 10^{-6} \) C\(^{-1}\), \( \alpha_m = 5 \times 10^{-6} \) C\(^{-1}\), \( T = -1000 \)°C, layer thickness = \( 180 \times 10^{-6} \) m, \( \gamma_m = 25 \) J/m\(^2\), \( \gamma_d = 4 \) J/m\(^2\) (Mode II interface debonding critical energy release rate). The fiber and matrix Poisson ratios were chosen equal to 0.2 and 0.35, respectively. Based on this data, \( E_L = 133 \) GPa, \( E_T = 118 \) GPa, \( G_T = 46 \) GPa, \( \nu_{LT} = 0.2975 \), \( \alpha_L = 4.65 \times 10^{-6} \) C\(^{-1}\), \( \alpha_T = 4.60 \times 10^{-6} \) C\(^{-1}\).

The \([-\theta/0/\theta]\) laminate was considered in the examples, i.e. \( m = n = 1 \). The lamination angles were equal to 15\(^{\circ}\), 30\(^{\circ}\), 45\(^{\circ}\) and 60\(^{\circ}\). The analysis is conducted by assumption that the laminate is constrained against twist under a uniaxial tension. The other possibility is that the laminate consists of \([-\theta/0/\theta]\) blocks that are symmetrically arranged about the middle plane, in which case, the stretching–twisting coupling is eliminated.

First of all, the laminate thermal residual stresses were determined according to Eqs. (15)–(18). These stresses in the \( x–y \) and 1–2 coordinate systems are listed in Table 1.

Note that the stresses shown in Table 1 are related to the laminate post-processing residual effects and they do not include additional stresses accumulated at the micromechanical level due to a thermal mismatch between the fibers and matrix. The magnitude of the stresses in Table 1 is very small, it is negligible compared to a typical matrix cracking stress for all lamination angles. The reason is related to a small difference both between the coefficients of thermal expansion and between the moduli of elasticity of a lamina in the longitudinal and transverse directions. Under these conditions, the laminate tends to behave similar to a unidirectional lamina, even if the lamination angle \( \theta \) varies from 0 to 90\(^{\circ}\). Accordingly, the laminate residual stresses are very small (they are equal to zero in a unidirectional lamina), although the internal lamina stresses due to a mismatch between the coefficients of thermal expansion of the fibers and matrix can be significant. Indeed, the stresses calculated by Eq. (6) were equal to 70 MPa, so that they had to be accounted for in the subsequent analysis.

The mode of initial failure due to bridging matrix cracks in \( \theta \)-layers was discounted since the ratio \( E_L/E_T = 1.13 \) is smaller than the boundary values given in Fig. 4. Therefore, the modes of failure that have to be compared are bridging cracks in 0-layers and tunneling cracks in \( \theta \)-layers. Furthermore, it was found that the model based on weakly bonded fibers provides a more conservative and

![Table 1](image-url)

**Table 1**

Laminate thermal residual stresses in \([-\theta/0/\theta]\) laminates

<table>
<thead>
<tr>
<th>Angle ( \theta ) (degrees)</th>
<th>15</th>
<th>30</th>
<th>45</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_{11,0} ) (MPa)</td>
<td>Negligible</td>
<td>0.743</td>
<td>1.765</td>
<td>2.826</td>
</tr>
<tr>
<td>( \sigma_{02,0} ) (MPa)</td>
<td>Negligible</td>
<td>−0.629</td>
<td>−1.765</td>
<td>−2.553</td>
</tr>
<tr>
<td>( \tau_{12,0} ) (MPa)</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( \sigma_{11,1} ) (MPa)</td>
<td>Negligible</td>
<td>0.400</td>
<td>0</td>
<td>−1.208</td>
</tr>
<tr>
<td>( \sigma_{02,1} ) (MPa)</td>
<td>Negligible</td>
<td>−0.286</td>
<td>0</td>
<td>1.481</td>
</tr>
<tr>
<td>( \tau_{12,1} ) (MPa)</td>
<td>Negligible</td>
<td>0.594</td>
<td>1.765</td>
<td>2.329</td>
</tr>
</tbody>
</table>

(absolute value)
Fig. 4. The boundary ratio $E_L/E_T$ for the mode of failure corresponding to bridging cracks. The cracks develop in 0-layers if $E_L/E_T$ is smaller than the boundary value. Otherwise, the cracks develop in $\theta$-layers.

Fig. 5. Applied composite stress ($\sigma_0$) that results in the onset of bridging cracks in 0-layers in $[-\theta_n/\theta_n/\theta_m]$ laminates.

accurate estimate for the stress corresponding to the onset of bridging cracking in 0-layers than the counterparts based on the assumptions of fully bonded fibers or the initial fiber debond. Indeed, a comparison of the stresses given by Eqs. (4), (A1) and (A3) yields $(\sigma_{0,cr}^0 + \sigma_{cr}^0) = 356, 1052, \text{and } 440 \text{ MPa}$, respectively. Note that the stress acting in the longitudinal layers at the onset of cracking $\sigma_{0,cr}^0 = 286 \text{ MPa}$ calculated based on Eq. (4) is equal to the value of 286 MPa that was experimentally found by Domergue et al. [20] for a similar unidirectional material.

The values of the applied composite stress that results in the onset of bridging cracks in 0-layers were calculated using $\sigma_{0,cr}^0 = 286 \text{ MPa}$ and Eq. (1) where the moduli of elasticity are replaced with the reduced stiffnesses. For example, $E_{11}^0$ should be replaced with $Q_{11} = E_L/(1 - v_{LT}v_{TL})$, etc. The results are shown in Fig. 5 as functions of the lamination angle. The applied composite stress that causes tunneling cracks in the inclined $\theta$-layers is compared with the applied stress producing bridging cracks in 0-layers in Fig. 6. The former stress in this figure is calculated by Eqs. (12) and (2), the moduli of elasticity being replaced with the reduced stiffnesses in the latter equation.

As follows from Fig. 6, the mode of initial failure is affected by the lamination angle of the inclined layers. If the layers are predominantly oriented in the longitudinal direction, the initial
failure occurs in the 0-layers in the form of bridging cracks. However, if the inclined layers form a larger angle with the load direction, they fail first due to tunneling cracks. In the case of material and geometry considered here, the boundary between two different modes of failure corresponds to a lamination angle equal to $35^\circ$. It is interesting that this conclusion is in a qualitative agreement with FEA results [6] where the initial failure in the form of bridging cracks perpendicular to the load direction was found to be more likely to occur in angle-ply CMC laminates ($\theta = 30^\circ$) than the interfacial failure. In the case of symmetrically laminated $[-\theta/\theta]$ composites, the corresponding boundary value of the lamination angle was found equal to $45^\circ$, i.e. bridging cracks formed in the layers at smaller values of the lamination angle, while tunneling cracks represented the initial mode of failure at larger values of $\theta$.

6. Conclusions

The paper presents the solution for the initial matrix cracking in angle-ply $[-\theta_m/0_n/\theta_m]$ and $[-\theta/\theta]$ ceramic matrix composites. The modes of failure considered in the paper include bridging matrix cracks in the longitudinal and inclined ($-\theta$ and $\theta$) layers, tunneling cracks in the inclined layers and cracks oriented at an arbitrary angle to the load direction in the inclined layers. If the angle of lamination of the inclined layers is small, initial failure occurs in the form of bridging cracks in the longitudinal layers of $[-\theta_m/0_n/\theta_m]$ composites or in $[-\theta/\theta]$ composites. However, as the lamination angle increases, initial cracks appear in the inclined layers of $[-\theta_m/0_n/\theta_m]$ composites or in $[-\theta/\theta]$ composites where they propagate in the planes parallel to the fiber orientation (tunneling cracks). The critical lamination angle corresponding to the alternation in the mode of the initial failure for a typical SiC/CAS material considered in the examples was equal to $35^\circ$ in $[-\theta/0/\theta]$ composites and to $45^\circ$ for in $[-\theta/\theta]$ composites. An additional conclusion from the numerical analysis was that the model based on weakly bonded fibers provides a more accurate estimate for the stress corresponding to the onset of bridging cracking in 0-layers than the counterparts based on the assumptions of fully bonded fibers or on the initial fiber debond. The results obtained by the numerical analysis were found in good agreement with available experimental data.
Acknowledgements

This research has been supported by the Air Force Office of Scientific Research through the contract F49620-93-C-0063. The program manager is Dr. H. Thomas Hahn.

Appendix A

A.1. Applied stresses that cause matrix cracking in a unidirectional CMC lamina: the cases of fully bonded fibers and interface debond

Case 1: Fully bonded fibers

\[
\sigma_{x,cr}^0 = BE\varepsilon\{(6E_fV_f^2)/[E_cV_m^2(1 + \nu_m)]\}^{1/4}[2\gamma_m/(rE_m)]^{1/2} - \sigma_{x,r}^0,
\]  

where \( \nu_m \) is a matrix Poisson’s ratio and

\[
B = \{(2V_m^3)/[-6\ln(V_f) - 3V_m(3 - V_f)]\}^{1/4}.
\]

Case 2: Debond of fibers from the interface precedes matrix cracking

The applied matrix cracking stress is obtained from the following equations (Course “Ceramic Matrix Composites” developed with the support of the National Science Foundation. Dr. N.A. Yu, University of Tennessee, Knoxville, Tennessee: web site: www.engr.utk.edu/~cmc):

\[
\sigma_{x,cr}^0 = \lambda \sigma_{m,cr}' - \sigma_{x,r}^0,
\]  

where

\[
\sigma_{m,cr}' = E_c[(6\gamma_m\tau E_f V_f^2)/(E_c E_m^2 V_m r)]^{1/3}
\]

and \( \lambda \) is found from

\[
\lambda^3 + (3\beta^2)\lambda - \beta^3 = 1 \quad \text{if } \alpha < \beta < (1/3)^{1/3},
\]

\[
\lambda^3 - [3\alpha(\alpha - 2\beta)]\lambda + \alpha^2(2\alpha - 3\beta) = 1 \quad \text{if } \beta < \alpha < (3\beta)^{-1/2},
\]

\[
\lambda = (3\beta)^{-1/2} \quad \text{if } \beta > (1/3)^{1/3} \text{ or } \alpha > (3\beta)^{-1/2}.
\]

In Eqs. (A4),

\[
\alpha = \sigma_D/\sigma_{m,cr}',
\]

\[
\beta = \sigma_s/\sigma_{m,cr}',
\]

where

\[
\sigma_D = 2V_f[(E_f E_c \gamma_d)/(V_m E_m r)]^{1/2}
\]

is the fiber–matrix interface debonding stress and \( \gamma_d \) is the Mode II interface debonding critical energy release rate.
The stress $\sigma_s$ is determined as
\[ \sigma_s = \frac{2VfE_c}{(V_mE_m\rho)} \]
\[ \rho = \left( \frac{B^2}{V_m} \right) \left\{ \frac{6E_c}{[E_f(1 + \nu_m)]} \right\}^{1/2}. \] (A7)

Appendix B

B.1. Transformation equations

The transformation of the reduced stiffnesses in the principal lamina axes (1–2) into the coordinate system $x$–$y$ that forms an angle $\theta$ with the system 1–2 is given by the following equations:

\[ Q'_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}s^4, \]
\[ Q'_{12} = (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(s^4 + c^4), \]
\[ Q'_{22} = Q_{11}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 + Q_{22}c^4, \]
\[ Q'_{16} = (Q_{11} - Q_{12} - 2Q_{66})sc^3 + (Q_{12} - Q_{22} + 2Q_{66})s^3c, \]
\[ Q'_{26} = (Q_{11} - Q_{12} - 2Q_{66})s^3c + (Q_{12} - Q_{22} + 2Q_{66})sc^3, \]
\[ Q'_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4), \] (A8)

where the prime denotes the transformed stiffnesses in the $x$–$y$ coordinate system, $s = \sin \theta$ and $c = \cos \theta$.

The transformation of the coefficients of thermal expansion is given by ($\alpha_{12} = 0$)
\[ \alpha_x = \alpha_1c^2 + \alpha_2s^2, \]
\[ \alpha_y = \alpha_1s^2 + \alpha_2c^2, \]
\[ \alpha_{xy} = 2(\alpha_1 - \alpha_2)cs. \] (A9)

The stress transformation equations from the $x$–$y$ coordinate system to the 1–2 coordinate system are:
\[ \sigma_1 = \sigma_xc^2 + \sigma_ys^2 + 2\tau_{xy}sc, \]
\[ \sigma_2 = \sigma_xs^2 + \sigma_yc^2 - 2\tau_{xy}sc, \]
\[ \tau_{12} = (\sigma_y - \sigma_x)sc + (c^2 - s^2)\tau_{xy}. \] (A10)
References