Stability and natural frequencies of functionally graded stringer-reinforced panels

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Abstract

The paper presents an analysis of stability and free vibrations of rectangular functionally graded panels reinforced by a system of parallel stringers. The exact solution of the problem is illustrated for large aspect ratio panels with simply supported long edges and arbitrary boundary conditions along the short edges (hereafter the reference to an “exact solution” implies a closed-form solution in the content of the theory of plates). The spacing between the stringers and the cross sections of individual stringers can be arbitrary. In the particular case where identical stringers are equally spaced, the solution is simplified using the smeared stiffeners technique. The optimization problem concerned with the choice of stringers and their spacing in the situations where the buckling loads or fundamental frequencies are prescribed is also considered. The closed-form solution of the optimization problem is shown in the case of blade stringers.

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1. Introduction

Functionally graded materials (FGM) are composites formed of two or more phases that are varied throughout the domain occupied by the structure with the goal of optimizing its performance. Examples of such materials are particulate composites with a piece-wise or continuous variation of the volume fraction of particles and fiber-reinforced composites where the volume fraction or orientation of fibers depends on the location. Various aspects of mechanics, manufacturing, and properties of these materials have been considered in a number of reviews, including early studies [1,2] and more recent publications [3,4]. In addition, the latest review of the authors outlines recent work conducted on FGM materials [5].

Studies of FGM plates occupy a prominent position among papers concerned with functionally graded materials. While listing all these papers here is impossible (the reader is referred to review [5]), representative developments considering dynamics of FGM plates can be found in the papers [6,7], while stability of such plates was considered in [8,9]. However, to the best knowledge of the authors, the response of stringer-reinforced FGM plates has not been considered.

Through-the-thickness grading of material often leads to structures asymmetric about the middle surface resulting in a non-zero coupling stiffness. As a result, the advantages of FGM structures, such as their superior thermal characteristics and improved delamination resistance, have to be weighted against disadvantages including lower buckling loads and fundamental frequencies and higher stresses and deflections compared to equal-weight symmetric counterparts. One of the methods compensating for a reduced stiffness of FGM structures is based on strengthening them with stringers that can be bonded or riveted to the lower-stiffness surface (for example, metallic stringers on the metal-rich surface of a ceramic-metal panel). This dictates the necessity to analyze asymmetric stringer-reinforced structures.

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The first studies on the effect of stringers on the behavior of isotropic structures have been usually concerned with cylindrical shells. The earliest work in this area was published in the sixties and seventies of the last century by Block et al. [10,11], Singer et al. [12,13] and Fisher and Bert [14,15]. The most recent research on stiffened composite cylindrical shells can be found in the papers [16–18]. Studies of stiffened rectangular composite plates were published by Starnes et al. [19], Sheinman and Frostig [20], Birman [21,22] and Wang et al. [23,24]. These and other studies were conducted assuming that the plate is symmetrically laminated about its middle surface, i.e. they are not applicable to FGM panels considered here.

The present paper illustrates the solution of buckling and free vibration problems for a FGM panel reinforced by a system of stringers. The solution is applicable to the general case where both cross sections of individual stringers as well as their spacing are arbitrary. For the case of identical stringers with a relatively small spacing the smeared stiffeners technique can be applied resulting in uncoupled equations for each mode of buckling or vibration and a significantly simpler solution. Finally, optimization of the stringers is considered assuming the grading and thickness of the plate designed for required values of the buckling load or fundamental frequency are prescribed. In the case of blade stringers a closed-form solution yields the equation governing the optimum geometry and spacing.

2. Analysis

Consider a rectangular panel manufactured from a functionally graded material. The panel is reinforced by a system of stringers oriented parallel to the shorter edges \( y = 0, y = b \) as shown in Fig. 1. The problems considered in this paper include static stability under a system of biaxial in-plane loads and the natural frequencies of the panel.

![Fig. 1. A rectangular panel \((b/a > 1)\) reinforced by stringers and loaded by in-plane compressive forces. Although the solution does not superimpose a limitation on the spacing between stringers, they are shown equally spaced in the figure.](image)

The following assumptions are adopted in the subsequent solution:

- The panel is sufficiently thin to be analyzed by the classical plate theory;
- Geometrically and physically nonlinear effects are disregarded;
- Longer edges of the panel \( x = 0, x = a \) are simply supported by frames or beams that have negligible torsional stiffness and negligible stiffness in the direction perpendicular to the beam axis, while their axial stiffness is very high (such assumptions are almost universally acceptable for open-profile beams);
- The open-profile stringers have negligible torsional and out-of-plane stiffnesses;
- In-plane inertia terms are negligible in the free vibration problem.

The boundary conditions along the shorter edges can be arbitrary, though we specify them in the detailed analysis presented below as identical to the conditions along the long edges. The through-the-thickness variation of the constituent phases in the functionally graded panel can be arbitrary. There are no limitations on the grading of the stringers though they are often manufactured from a homogeneous isotropic material, as is the case in a ceramic-metal FGM panel reinforced by metallic stringers at the predominantly metallic surface.

2.1. Mathematical formulation

Various versions of governing equations for reinforced isotropic and symmetrically laminated or orthotropic plates and shells have been published in the references cited in Section 1. The present equations are different, accounting for the asymmetric distribution of constituent materials about the middle plane of a FGM panel.

Equations of free vibrations of a thin panel including in-plane compressive static loads and transverse inertia are

\[
N_{x} + N_{y} = 0
\]

\[
M_{xx} + M_{yy} + N_{x}w_{xx} + N_{y}w_{yy} = -m\omega^{2}w
\]

where \( N_{x}, N_{y} \) are applied in-plane stress resultants, \( m \) is the mass per unit area, \( \omega \) is the frequency of free vibrations, and \( w \) is the transverse deflection.

The vectors of stress resultants and stress couples are given by well known constitutive relationships:

\[
\begin{pmatrix}
N \\
M
\end{pmatrix} =
\begin{bmatrix}
A & B \\
B & D
\end{bmatrix}
\begin{pmatrix}
\mathbf{\varepsilon} \\
\mathbf{\kappa}
\end{pmatrix}
\]  

(2)

where \( A, B, D \) are extensional, coupling and bending stiffness matrices,
\[N = \{ N_x \ N_y \ N_{xy} \}^T\]
\[M = \{ M_x \ M_y \ M_{xy} \}^T\]
\[e^0 = \{ u_{xz} \ v_y \ u_{yz} + v_z \}^T\]
\[\kappa = \{-w_{xxx} \ -w_{yy} \ -2w_{xyz}\}^T\]

In (3), \(u\) and \(v\) are in-plane displacements in the \(x\) and \(y\) directions, respectively, and the superscript \(T\) refers to a transpose of the vector. Obviously, according to the theory of thin plates adopted in the analysis, in-plane displacements are linear functions of the thickness \((z)\) coordinate and the total strain vector is given by \(\{ \varepsilon \} = \{ \varepsilon^0 \} + z\{ \kappa \}\), where \(z\) is counted from the middle plane of the panel.

The expressions for the stiffness matrices in (2) should account for grading of the panel material in the thickness direction and the presence of stringers. In general, grading can be an arbitrary function of the thickness coordinate. The local material constants of a two-phase particulate FGM (such as a ceramic-metal material), i.e. the moduli of elasticity and shear and the Poisson ratio, can be determined by one of micromechanical theories, such as the self-consistent model, the Mori–Tanaka theory, etc. Moreover, some of the recent solutions enable us to account for the effect of a continuous variation of the volume fraction through the thickness at the micromechanical level. A comprehensive review of this subject is outside the scope of the present paper and the reader is referred to the review of the authors [5] that includes a section on homogenization methods in FGM.

The material of the panel that includes spherical or multiple randomly oriented particles embedded within an isotropic matrix is quasi-isotropic with continuously varying through the thickness properties. The non-zero elements of the stiffness matrices contributed by the panel without the stringers are given by

\[A = \begin{bmatrix}
  A_{11} & A_{12} & 0 \\
  A_{12} & A_{22} & 0 \\
  0 & 0 & A_{66}
\end{bmatrix}, \quad B = \begin{bmatrix}
  B_{11} & B_{12} & 0 \\
  B_{12} & B_{22} & 0 \\
  0 & 0 & B_{66}
\end{bmatrix}, \quad D = \begin{bmatrix}
  D_{11} & D_{12} & 0 \\
  D_{12} & D_{22} & 0 \\
  0 & 0 & D_{66}
\end{bmatrix}
\]

where \(A'_{11} = A_{22}, \ B'_{11} = B_{22}, \ D'_{11} = D_{22}\) and

\[\{ A'_{ij}, B'_{ij}, D'_{ij} \} = \int_{-h/2}^{h/2} Q_{ij}\{ 1, \ z, \ z^2 \} dz\]  

In (5), \(h\) is the thickness of the panel, and \(Q_{ij}\) are reduced stiffnesses calculated in terms of the material constants by standard equations.

According to the assumptions introduced above, the contribution of the stringers affects only the stiffness in the direction of their axes as is customary in the analyses of reinforced structures. Therefore, the corresponding stiffness terms affected by the presence of stringers manufactured from an isotropic material are

\[\{ A''_{11}, B''_{11}, D''_{11} \} = \sum_i \delta(y - y_i)E_i\{ A_{s}, \ F_s, \ I_s \}\]

where \(\delta(y - y_i)\) is the Dirac delta function, \(y_i\) is the coordinate of the \(i\)th stringer, \(E_i\) is the modulus of elasticity of the stringer material, and \(A_s, \ F_s, \ I_s\) are the area, and first and second moments of the \(i\)th stringer about the middle plane of the panel, respectively. In case where the modulus of elasticity of the stringer varies in the \(z\)-direction (composite or functionally graded stringer), the previous equation is replaced with

\[\{ A'_{i1}, B'_{i1}, D'_{i1} \} = \sum_j \delta(y - y_j) \int b(z)E(z)\{ 1, \ z, \ z^2 \} dz\]

where \(b(z)\) is the width of the stringer and the coordinate is counted from the middle plane of the panel. There are no restrictions on either variation of the modulus \(E(z)\) or the shape of the cross section of the stringer if their contribution is introduced by (7). The contributions of the stringers to other elements of the stiffness matrices are negligible, i.e. \(A''_{ij} = B''_{ij} = D''_{ij} = 0 \ (ij \neq 11)\).

Therefore, the elements of the overall stiffness matrices of the panel in (2) are

\[\{ A_{ij}, B_{ij}, D_{ij} \} = \{ A'_{ij}, B'_{ij}, D'_{ij} \} + \{ A''_{ij}, B''_{ij}, D''_{ij} \}\]

The substitution of (2) and (3) into (1) yields the system of equations of motion in terms of displacements accounting both for the grading of the panel material in the thickness direction and for the presence of stringers:

\[A_{11}u_{xx} + A_{66}u_{yy} + (A_{12} + A_{66})v_{xy} - B_{11}w_{xxx} - (B_{12} + 2B_{66})w_{yy} = 0\]
\[(A_{12} + A_{66})u_{xy} + A_{66}v_{xx} + A_{22}v_{yy} - (B_{12} + 2B_{66})w_{xxy} - B_{22}w_{yyy} = 0\]
\[D_{11}w_{xxx} + 2(D_{12} + 2D_{66})w_{xxy} + D_{22}w_{yyy} - B_{11}u_{xxx} - (B_{12} + 2B_{66})u_{xyy} + v_{xxy} = 0\]
\[B_{22}v_{yy} - \nabla_x w_{xx} - \nabla_y w_{yy} = \rho_0 w\]

Note that while this system has been presented in numerous references considering laminated plates [25], the elements of the matrix of stiffness coefficients in (9) are different due to the presence of stringers.

The mass per unit surface area of the plate is given by

\[m = m_p + \delta(y - y_p)\rho_s A_s\]

where \(m_p\) is the mass of the panel without the stringers per unit surface area and \(\rho_s\) is the mass density of the stringer material. In the case where the stringers are manufactured from graded materials with the density varying in the thickness direction,

\[m = m_p + \delta(y - y_p) \int \rho_s(z)b(z)dz\]

where \(b(z)\) is the width of the stringer and the coordinate is counted from the middle plane of the panel. There are no restrictions on either variation of the density \(\rho(z)\) or the shape of the cross section of the stringer if their contribution is introduced by (7). The contributions of the stringers to other elements of the stiffness matrices are negligible, i.e. \(A''_{ij} = B''_{ij} = D''_{ij} = 0 \ (ij \neq 11)\).

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\[D_{11}w_{xxx} + 2(D_{12} + 2D_{66})w_{xxy} + D_{22}w_{yyy} - B_{11}u_{xxx} - (B_{12} + 2B_{66})u_{xyy} + v_{xxy} = 0\]
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\[m = m_p + \delta(y - y_p) \int \rho_s(z)b(z)dz\]
The boundary conditions along the edges $x = 0$, $x = a$ corresponding to the assumptions introduced above imply that
\[
w = M_x = N_x = v = 0 \quad (12)
\]

Boundary conditions along the edges $y = 0$, $y = b$ can be arbitrary but in the closed-form solution for the case of closely spaced stringers obtained by the smeared stiffeners technique that is shown in Section 2.3 they will be assumed similar to (12), so that
\[
w = M_y = N_y = u = 0 \quad (13)
\]

2.2. Solution of the eigenvalue problem for the case of discrete stiffeners and arbitrary boundary conditions along the edges $y = 0$, $y = b$

The solution can be obtained both for the combination of the buckling loads as well as for the fundamental frequency of free vibrations assuming that the panel deforms in the form
\[
W = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{nm} \sin \frac{n \pi x}{a} \sin \frac{m \pi y}{b}
\]

where $w_{nm}$ are the corresponding functions yielding
\[
U_{yy} = C_1 U - C_2 V_{yy} - C_4 W - C_3 W_{yy}
\]
\[
V_{yy} = C_5 U_{yy} - C_6 V - C_7 W_{yy} + C_8 W_{xyy}
\]
\[
W_{xyy} = C_3 U - C_{40} U_{yy} - C_{11} V_{yy} + C_{12} V_{xyy} - C_{13} W + C_{14} W_{yy}
\]

where
\[
C_1 = \frac{x^2 A_{11}}{A_{66}}, \quad C_2 = \frac{2}{A_{66}}(A_{11} + A_{66}), \quad C_3 = \frac{x^2 B_{11}}{A_{66}},
\]
\[
C_4 = \frac{x(B_{12} + 2B_{66})}{A_{66}}, \quad C_5 = \frac{x(A_{12} + A_{66})}{A_{22}}, \quad C_6 = \frac{x^2 A_{66}}{A_{22}},
\]
\[
C_7 = \frac{x^4 B_{12}}{D_{22}}, \quad C_8 = \frac{B_{22}}{A_{22}}, \quad C_9 = \frac{x^4 B_{11}}{D_{22}},
\]
\[
C_{10} = \frac{x^2 C_4 D_{12} + 2B_{66}}{D_{22}}, \quad C_{11} = \frac{C_4}{D_{22}}, \quad C_{12} = \frac{C_2}{D_{22}}, \quad C_{13} = \frac{x^4 A_{11} + x^2 C_4 - C_4}{D_{22}},
\]
\[
C_{14} = \frac{C_4}{D_{22}}
\]

Of course, in the static problem the dynamic term is omitted.

It is obvious that boundary conditions (12) are identically satisfied. The substitution of (14) into (9) yields the system of ordinary differential equations with respect to unknown functions $U(y)$, $V(y)$, $W(y)$:

\[
F_1 = -x^2 A_{11} U + A_{66} U_{yy} + x(A_{12} + A_{66}) V_{yy}
\]
\[
+ x^2 B_{11} W - x(B_{12} + 2B_{66}) W_{yy} = 0
\]
\[
F_2 = -x(A_{12} + A_{66}) U_{yy} - x^2 A_{66} V + x^2 C_4 V_{yy}
\]
\[
+ x^2 C_4 U_{yy} - x^2 W_{yy} = 0
\]
\[
F_3 = x^2 D_{12} W - x^2 (B_{12} + 2B_{66}) W_{xyy} + D_{22} W_{xyyy} - x^2 B_{11} U
\]
\[
+ (B_{12} + 2B_{66})(x U_{yy} + x^2 V_{yy})
\]
\[
- B_{22} V_{xyy} + x^2 \mathcal{N}_x W - \mathcal{N}_y W_{yy} - \cos^2 W = 0
\]

where $x = \pi/a$.

2.2.1. Exact solution of the eigenvalue problem

Following the approach of Reddy to the analysis of laminated un-reinforced plates [25], the system (15) can be solved with respect to the highest derivatives of the corresponding functions yielding

\[
U_{yy} = C_1 U - C_2 V_{yy} - C_4 W - C_3 W_{yy}
\]
\[
V_{yy} = C_5 U_{yy} - C_6 V - C_7 W_{yy} + C_8 W_{xyy}
\]
\[
W_{xyy} = C_3 U - C_{40} U_{yy} - C_{11} V_{yy} + C_{12} V_{xyy} - C_{13} W + C_{14} W_{yy}
\]

where

\[
C_{15} = (C_0 - C_1 C_{10} + C_1 C_5 C_{12}) A
\]
\[
C_{16} = (C_2 C_{10} - C_{11} - C_1 C_5 C_{12} + C_6 C_{12}) A
\]
\[
C_{17} = (C_1 C_{10} - C_7 C_5 C_{12} - C_{13}) A
\]
\[
C_{18} = (C_{14} - C_4 C_{10} - C_7 C_{12} + C_4 C_5 C_{12}) A
\]
\[
A = \frac{1}{1 - C_5 C_{12}}
\]
The system of Eq. (19) can be solved using the approach illustrated by Dorf [26] and other investigators working in control theory. The solution is sought in the form

$$\{L\} = e^{\mathbf{T}Y}\{K\}$$

(22)

where \(\{K\}\) is a vector of unknown coefficients and

$$e^{\mathbf{T}Y} = \sum_{r=0}^{\infty} \frac{[\mathbf{T}Y]^r}{r!} = [I] + [\mathbf{T}]Y + \frac{[\mathbf{T}]Y^2}{2!} + \frac{[\mathbf{T}]Y^3}{3!} + \cdots$$

(23)

\([I]\) being a unit matrix. The series (23) is convergent for all square matrices [26], i.e. the accuracy of the solution is limited only by the number of terms retained in these series.

The substitution of (22) into (19) yields an identity. Therefore, equations of motion are satisfied and eight constants of integration in the vector \(\{K\}\) can be considered accounting for the boundary conditions along the short edges of the panel.

The elements of the matrix determined from (23) are functions of the \(y\)-coordinate so that the exponential term can be represented by an 8 * 8 matrix

$$e^{\mathbf{T}Y} = [P(y)]$$

(24)

For example, if series (23) are truncated retaining only terms dependent on \(y\) to the third and lower powers, the elements of \([P(y)]\) are

$$P_{ij}(y) = I_{ij} + T_{ij}y + \frac{1}{2} \sum_{k=1}^{8} T_{ik} T_{kj} y^2 + \frac{1}{6} \sum_{i=1}^{8} \sum_{k=1}^{8} T_{ii} T_{ik} T_{kj} y^3$$

(25)

Using (24) the solution of the equations of motion can be written as

$$\{L\} = [P(y)]\{K\}$$

(26)

Subsequently, the eigenvalues (buckling loads or natural frequencies) can be determined applying the boundary conditions along \(y = 0\) and \(y = b\). This results in a system of eight homogeneous algebraic equations and the characteristic equation immediately follows from the requirement that \(\{K\} \neq 0\).

Consider for example, the case of simple supported short edges given by (13). It is convenient to represent the stress resultant \(N_y\) and stress couple \(M_y\) in the form

$$N_y = \sum_{n=1}^{8} R_n(y) K_n \sin \frac{\pi n y}{a} e^{i\omega t}$$

$$M_y = \sum_{n=1}^{8} H_n(y) K_n \sin \frac{\pi n y}{a} e^{i\omega t}$$

(27)

where \(K_n\) are the elements of the normalized eigenvector \(\{K\}\) and

$$R_n(y) = -\frac{\pi}{a} A_{12} P_{1a}(y) + A_{22} P_{4a}(y) + \left(\frac{\pi}{a}\right)^2 B_{12} P_{2a}(y) - B_{22} P_{7a}(y)$$

$$H_n(y) = -\frac{\pi}{a} B_{12} P_{1a}(y) + B_{22} P_{4a}(y) + \left(\frac{\pi}{a}\right)^2 D_{12} P_{5a}(y) - D_{22} P_{8a}(y)$$

(28)

The boundary conditions (13) yield a system of homogeneous algebraic equations:

$$[C] \{K\} = 0$$

(29)

The elements of the square matrix in the left side of (29) are

$$C_{ij} = P_{ij}(y = 0), \quad C_{ij} = P_{ij}(y = b)$$

$$C_{5j} = P_{5j}(y = 0), \quad C_{5j} = P_{5j}(y = b)$$

$$C_{6j} = R_{ij}(y = 0), \quad C_{6j} = R_{ij}(y = b)$$

$$C_{7j} = H_{ij}(y = 0), \quad C_{7j} = H_{ij}(y = b)$$

(30)

The requirement of a non-trivial solution of (29) results in either the buckling condition (for static buckling problems) or the frequency equation (for the free vibration problem):

$$\det[C] = 0$$

(31)

Obviously, it is impossible to analytically evaluate the buckling loads or natural frequencies from (31). However, the numerical solution is straightforward; although this is not a closed-form solution, it is exact in the sense that it does not rely on any assumptions, except for those specified in the beginning of the paper.

2.2.2. Approximate solution of the eigenvalues problem

In numerous situations a reliable estimate of the eigenvalues can be obtained by approximate methods, such as the Galerkin procedure illustrated below. Even if the static boundary conditions in terms of stress couples and stress resultants are violated, the solution can be sought by the Generalized Galerkin procedure [27,28], although this case is not considered here. Let

$$U(y) = \sum_k U_k u_k(y), \quad V(y) = \sum_k V_k v_k(y)$$

$$W(y) = \sum_k W_k w_k(y)$$

(32)

where \(U_k, V_k, W_k\) are unknown constant coefficients and the terms in the series satisfy all boundary conditions.

The Galerkin procedure yields the system of 3\(k\) homogeneous algebraic equations:

$$\sum_n (f_{1nk} U_n + g_{1nk} V_n + s_{1nk} W_n) = 0$$

$$\sum_n (f_{2nk} U_n + g_{2nk} V_n + s_{2nk} W_n) = 0$$

$$\sum_n (f_{3nk} U_n + g_{3nk} V_n + s_{3nk} W_n) = 0$$

(33)

where in the case of isotropic stringers the coefficients are given by
\[ f_{1nk} = -x^2 A_{11} \int_0^b u_n(y)u_n(y)dy - x^2 \sum s E_s A_{ss} u_n(y)u_n(y) + A_{66} \int_0^b u_n(y)_{yy}u_n(y)dy \]

\[ g_{1nk} = x(A_{12} + A_{66}) \int_0^b v_n(y)_{yy}u_n(y)dy \]

\[ s_{1nk} = x^3 B_1' \int_0^b w_n(y)u_n(y)dy + x^3 \sum s E_s F_s w_n(y)u_n(y) \]

\[ - x(B_{12} + 2B_{66}) \int_0^b w_n(y)_{yy}u_n(y)dy \]

\[ f_{2nk} = -x(A_{12} + A_{66}) \int_0^b u_n(y)_{yy}v_n(y)dy \]

\[ g_{2nk} = -x^2 A_{66} \int_0^b v_n(y)_{yy}v_n(y)dy + A_{22} \int_0^b v_n(y)_{yy}v_n(y)dy \]

\[ s_{2nk} = x^2 (B_{12} + 2B_{66}) \int_0^b w_n(y)_{yy}v_n(y)dy - B_{22} \int_0^b w_n(y)_{yy}v_n(y)dy \]

\[ f_{3nk} = -x^3 B_1' \int_0^b u_n(y)w_n(y)dy - x^3 \sum s E_s F_s u_n(y)w_n(y) \]

\[ + x(B_{12} + 2B_{66}) \int_0^b u_n(y)_{yy}w_n(y)dy \]

\[ g_{3nk} = x^2 (B_{12} + 2B_{66}) \int_0^b v_n(y)_{yy}w_n(y)dy \]

\[ s_{3nk} = x^4 D_{11} \int_0^b w_n(y)w_n(y)dy + x^4 \sum s E_s J_s w_n(y)w_n(y) \]

\[ - 2x^2 (D_{12} + 2D_{66}) \int_0^b w_n(y)_{yy}w_n(y)dy \]

\[ + D_{22} \int_0^b w_n(y)_{yy}w_n(y)dy \]

\[ + x^2 N_s \int_0^b w_n(y)w_n(y)dy \]

\[ - N_s \int_0^b w_n(y)_{yy}w_n(y)dy \]

\[ - m_p \omega^2 \int_0^b w_n(y)w_n(y)dy \]

\[ - \sum s \rho_s A_s \omega^2 w_n(y)w_n(y) \]

\[ (34) \]

Both the static critical loads as well as the fundamental frequency can be determined from the requirement that the determinant of the system (33) must be equal to zero.

For example, in case where the boundary conditions along the short edges are represented by (13), these conditions are satisfied by

\[ u_k = \sin \frac{k\pi y}{b}, \quad v_k = \cos \frac{k\pi y}{b}, \quad w_k = \sin \frac{k\pi y}{b} \]

Then the coefficients in (34) are can be explicitly presented in the form

\[ f_{1nk} = (x^2 A_{11} + \beta_n^2 A_{66}) \delta_{nk} + \frac{2}{b} x^2 \sum s E_s A_{ss} P_{nk}(y) \]

\[ g_{1nk} = f_{2nk} = x\beta_n(A_{12} + A_{66}) \delta_{nk} \]

\[ s_{1nk} = f_{3nk} = -x[\beta_n^2 B_{11} + \beta_n^2 (B_{12} + 2B_{66})] \delta_{nk} \]

\[ - \frac{2}{b} x^2 \sum s E_s F_s P_{nk}(y) \]

\[ (36) \]

\[ g_{2nk} = x^2 A_{66} + \beta_n^2 A_{22} \delta_{nk} \]

\[ s_{2nk} = g_{3nk} = \beta_n(x^2 B_{12} + 2B_{66}) + \beta_n^2 B_{22} \delta_{nk} \]

\[ s_{3nk} = [x^4 D_{11} + 2x^2 \beta_n^2 (D_{12} + 2D_{66}) + \beta_n^2 D_{22} + x^2 N_s + \beta_n^2 N_s - m_p \omega^2] \delta_{nk} \]

\[ + \frac{2}{b} \sum s (x^2 E_s J_s - \rho_s A_s \omega^2) P_{nk}(y) \]

\[ (37) \]

where

\[ \delta_{nk} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases} \]

\[ P_{nk}(y) = \frac{\sin \frac{n\pi y}{b}}{\sin \frac{k\pi y}{b}} \]

2.3. Closed-form solution for the buckling loads and natural frequencies by the smeared stiffeners technique in case of a large number of closely-spaced stringers

The previous solution enables us to analyze stability and vibration problems without assuming the shape of the deformed surface in the cross sections \( x = \text{const} \). Moreover, this solution is applicable to panels with arbitrary boundary conditions along the short edges. In the case of simply supported short edges the solution can be significantly simplified if the number of stringers is sufficiently large so that the spacing is small. Then the contribution of stringers to the stiffness can be smeared over the surface of the panel. As was shown by Birman [21], even plates with only three equally-spaced stringers can be adequately analyzed by the smeared stiffeners technique.

Using the smeared stiffeners approach in the situation where the stringers are equally spaced and homogeneous, Eq. (6) becomes

\[ \{ A'_{11}, B'_{11}, D'_{11} \} = \frac{E_s}{I} \{ A_s, F_s, I_s \} \]

(38)

where \( I \) is the spacing. The mass per unit surface area of the panel given by (10) is also modified so that \( m = m_p + \rho_s A_s / I \).

By applying the smeared stiffeners approximation, the terms in series (14) where the \( y \)-dependent functions are represented by (32) and (35) are uncoupled. For the static case where the panel is compressed in the \( x \)-direction as
well as for the dynamic case concerned with the fundamental
frequency, \( k = 1 \), the system of Eqs. (15) yields
\[
[S\{U\} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} - \mu_0^2 \end{bmatrix} \begin{bmatrix} U_1 \\ V_1 \\ W_1 \end{bmatrix} = 0
\tag{39}
\]

The elements of the matrix \([S]\) are known from the problems of unreinforced laminated plates, but the stiffness terms in the present problem are different, accounting for the presence of stringers. These elements are reproduced here for convenience:
\[
S_{11} = \pi^2 A_{11} + \beta^2 A_{66}, \quad S_{12} = \pi \beta (A_{12} + A_{66}) \\
S_{13} = -[\pi^2 B_{11} + \pi \beta^2 (B_{12} + 2B_{66})] \\
S_{22} = \pi^2 A_{66} + \beta^2 A_{22} \\
S_{23} = -[\pi^2 (B_{12} + 2B_{66}) + \beta^3 B_{32}] \\
S_{33} = \pi^2 D_{11} + 2\pi^2 \beta^2 (D_{12} + 2D_{66}) + \beta^4 D_{22} + \pi^2 \beta \pi_0
\tag{40}
\]
where \( \beta = \pi / b \).

The solution of the static buckling and free vibration problems is easily available from (39) using the requirement of non-trivial solutions, i.e. \( \det[S] = 0 \).

2.4. Example on optimization of the stringers for a FGM panel with prescribed buckling load or fundamental frequency (smeared stiffeners solution)

In the general case, independent variables in the problem under consideration include

(a) Grading of the FGM panel;
(b) Thickness of the FGM panel;
(c) Spacing of the stringers;
(d) Geometry of the stringers;
(e) Material of the stringers (including possible grading).

In a number of applications the material and geometry of the panel are prescribed and it is necessary to design stringers providing the required stability or fundamental frequency with a minimum additional weight.

The following optimization problem is solved assuming the stringers are homogeneous and their material cannot be altered. The only variables considered in the optimization process are the cross section and spacing of the stringers. This situation is found in ceramic-metal panels with metallic stringers manufactured from the same metal as the panel. In the case where the smeared stiffeners technique is applicable, the problem of minimizing the weight of the stringers implies the requirement for a minimum ratio \( F_1 = A_{1d} / l \) (this refers to the situation where the cross section does not vary in the axial direction).

For stringers of an arbitrary cross section, the variables that affect the buckling or frequency equation (39) are the cross sectional area, first and second moments of the stringers about the middle plane of the panel and the spacing of the stringers. Using the smeared stiffeners technique the number of these variables can be reduced to three, i.e. \( F_1, F_2 = F_2/l, F_3 = I_1/l \). The mass of the plate per unit area is \( m = m_p + \rho_s F_1 \).

The eigenvalue problem becomes
\[
\begin{vmatrix} a_{11} + b_{11} F_1 & a_{12} & a_{13} - b_{13} F_2 \\ a_{12} & a_{22} & a_{23} \\ a_{13} - b_{13} F_2 & a_{23} & a_{33} + b_3 F_3 - c_3 F_1 \end{vmatrix} = 0
\tag{41}
\]
where the terms \( a_{ij} \) (\( i \neq j \)) are obtained from (40) using \( a_{ij} = S_{ij}, A_{11} = A_{11}, B_{11} = B_{11} \). Furthermore, if \( \pi_0 = 0 \),
\[
a_{13} = \pi^2 D_{11} + 2\pi^2 \beta^2 (D_{12} + 2D_{66}) + \beta^4 D_{22} + \pi^2 \beta \pi_0 - m_p \\
b_{11} = \pi^2 E_{11}, \quad b_{13} = \pi^2 E_{13}, \quad b_{33} = \pi^2 E_{33}, \quad c_3 = \rho_s \pi^2
\tag{42}
\]

The optimization procedure can now be conducted for prescribed eigenvalues \( \pi_0, \alpha \) varying the profile of the stringer (P). For every profile the corresponding geometric properties \( A_{1d}(P), F_1(P), I_1(P) \) and their ratios \( F_1(P)/A_{1d}(P) \) and \( I_1(P)/A_{1d}(P) \) can be calculated. Therefore, \( F_2 \) and \( F_3 \) can be expressed in terms of \( F_1 \) that is subsequently found from (41). The optimum stringer system corresponding to prescribed eigenvalues is specified using \( F_1(P) = \text{min}[F_1(P)] \) subject to constraints on the profile \( P \) and spacing \( l \).

If the optimization is performed for stringers with a rectangular cross section (blade stringers), the solution for both static and dynamic optimization problems is even simpler. If the stringers are sufficiently high, it is possible to neglect a square of half-thickness of the panel compared to a square of the distance between the centroid of the stringer to the middle plane of the plate \( d \). Then the variables available for optimization are reduced to only two, i.e. \( F_1 \) and \( d \). Accordingly, Eq. (41) becomes
\[
\begin{vmatrix} a_{11} + b_{11} F_1 & a_{12} & a_{13} - b_{13} F_1 d \\ a_{12} & a_{22} & a_{23} \\ a_{13} - b_{13} F_1 d & a_{23} & a_{33} + (4/3) b_3 F_1 d^2 - c_3 F_1 \end{vmatrix} = 0
\tag{43}
\]

The quadratic relationship between \( F_1 \) and \( d \) is available from (43). The minimization with respect to \( F_1 \) can be conducted either analytically or numerically. Based on the calculated values of \( F_1 \) and \( d \) it is easy to select optimum combinations of the height, width and spacing between the stringers.

The weight of the stringers can be more efficiently reduced if their cross sectional area varies along the axis and their spacing is variable (indeed, there is no need in stringers in the vicinity to the short edges of the panel \( y = 0, y = h \). The optimization problem is formulated for this case by assumption that all stringers are identical and the smeared stiffeners technique is applicable. The solution obtained by the Galerkin procedure yields a modified version of Eq. (41):
inner cycle for a specified spacing where the variables are the profile of the stringer and its variation along the axis, i.e. \( P(x) \). The outcome from the inner cycle is the optimum profile \( P_\text{opt}(y) \) for a prescribed \( l(y) \) subject to constraints on geometry. The comparison of these outcomes yields the optimum profile and spacing, i.e. \( P_\text{opt}, l(y)_\text{opt} \).

3. Numerical examples

The panels considered in examples were manufactured from titanium boride/titanium (TiB/Ti). The panels had seven 0.254 mm thick layers with varying volume fraction of the constituent materials; the properties of the layers are listed in Table 1 where the content of TiB increases from layer 1 to layer 7. The thickness of titanium blade stringers located on the titanium-rich surface of the panel was equal to the thickness of the panel (1.778 mm). The maximum allowable height of the stringers was limited to 20 times the thickness. In all examples concerned with buckling the load was applied in the \( x \)-direction. The aspect ratio of panels considered in examples varied but the mode shape of buckling in all cases was dominated by a single half-wave in both \( x \) and \( y \) directions (of course, this is also the mode corresponding to the fundamental frequency of the panel).

The non-dimensional buckling load is shown as a function of the height of the stringers in Fig. 3 where the ordinate is a ratio of the buckling load for a panel with ribs to that of the unstiffened counterpart. Fig. 3 also shows the buckling mode shape found using the finite element program ABAQUS [29] for the case \( \lambda / h = 10 \). Two-node, linear beam elements were used to model the ribs while composite shell elements modeled the FGM skin in the ABAQUS solution. The numerical solution was in an excellent agreement with results obtained from the smeared stiffeners model with the difference in the buckling loads being less than 0.5%.

The results of the optimization in both buckling and dynamic problems are shown in Figs. 4 and 5. It was found that the lightest design can be achieved using a small number of high stringers. It should be noted that the present solution is limited to the analysis of global buckling, while neither local buckling of the plate between widely spaced stringers nor buckling of stringers were considered. These alternative modes of failure may become important in certain configurations [30]. The necessity to avoid local buckling between the stringers was reflected in the upper limit

\[
\begin{bmatrix}
a_{11} + \Phi_1(P, l) & a_{12} & a_{13} - \Phi_2(P, l) \\
a_{12} & a_{22} & a_{23} \\
a_{13} - \Phi_3(P, l) & a_{23} & a_{33} + \Phi_4(P, l) - \Phi_5(P, l) a^2
\end{bmatrix} = 0
\]

(44)

where

\[
\begin{align*}
\Phi_1(P, l) &= \frac{4\pi^3 E_s ab}{ab} \int_0^b \int_0^{l(y)} \frac{A_s(x)}{l(y)} \cos^2 \frac{\pi x}{l} \sin^2 \frac{\pi y}{l} \, dx \, dy \\
\Phi_2(P, l) &= \frac{4\pi^3 E_s ab}{ab} \int_0^b \int_0^{l(y)} \frac{F_s(x)}{l(y)} \cos^2 \frac{\pi x}{l} \sin^2 \frac{\pi y}{l} \, dx \, dy \\
\Phi_3(P, l) &= \frac{4\pi^3 E_s ab}{ab} \int_0^b \int_0^{l(y)} \frac{I_s(x)}{l(y)} \sin^2 \frac{\pi x}{l} \sin^2 \frac{\pi y}{l} \, dx \, dy \\
\Phi_4(P, l) &= \frac{4\pi^3 E_s ab}{ab} \int_0^b \int_0^{l(y)} \frac{I_s(x)}{l(y)} \sin^2 \frac{\pi x}{l} \sin^2 \frac{\pi y}{l} \, dx \, dy
\end{align*}
\]

(45)

A possible optimization procedure for specified eigenvalues is shown in Fig. 2. The procedure involves two cycles, the outer cycle varying the spacing of the stringers and the

![Fig. 2. Optimization of the variable stringer profile and variable spacing.](image-url)

Table 1

<table>
<thead>
<tr>
<th>Layer</th>
<th>( \rho ) (kg/m³)</th>
<th>( E ) (GPa)</th>
<th>( v )</th>
</tr>
</thead>
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<tr>
<td>1</td>
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<td>106.9</td>
<td>0.34</td>
</tr>
<tr>
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<td>117</td>
<td>0.31</td>
</tr>
<tr>
<td>3</td>
<td>4567</td>
<td>153</td>
<td>0.28</td>
</tr>
<tr>
<td>4</td>
<td>4565.5</td>
<td>159</td>
<td>0.25</td>
</tr>
<tr>
<td>5</td>
<td>4564</td>
<td>193</td>
<td>0.22</td>
</tr>
<tr>
<td>6</td>
<td>4562.5</td>
<td>237</td>
<td>0.19</td>
</tr>
<tr>
<td>7</td>
<td>4561</td>
<td>274</td>
<td>0.17</td>
</tr>
</tbody>
</table>
on the spacing superimposed on the buckling optimization procedure and chosen equal to 0.125 m. This spacing corresponds to optimum stringer solutions in Figs. 4 and 5. In general, the consideration of Figs. 3–5 confirms that reinforcing the less stiff surface of a FGM panel with stringers is an exceptionally effective tool increasing stability and raising the fundamental frequency.

Note that optimum stringers in Figs. 4 and 5 are limited to a particular (blade) configuration. The stringers could be further optimized using either one or a combination of the following improvements:

(a) Stringers with a flange (T-shapes);
(b) Graded stringers with the flange manufactured from a stiffer material (such as TiB in the examples considered in this paper);
(c) Variable cross section of stringers providing a higher overall stiffness of the panel in the central region;
(d) Variable spacing of stringers, concentrating them in the central part of the panel;
(e) Non-straight stringers characterized by the length coordinate \( s = s(x, y) \) and providing enhanced resistance against the dominant mode of buckling or vibration.

4. Conclusions

The paper presents a formulation and solution of the eigenvalue problem (buckling load or fundamental frequency) for FGM panels reinforced by a system of stringers of an arbitrary spacing and geometry. The exact solution can be obtained for rectangular panels with stringers oriented along arbitrary supported short edges that deform forming one half-wave in the direction parallel to these edges. In the case where the stringer spacing is relatively small, i.e. the smeared stiffener technique is applicable, the solution for the buckling load and fundamental frequency is also exact.

Optimization problems are formulated both for panels with smeared equally-spaced identical stringers and for the case where the stringer cross section varies along its axis and the spacing is non-uniform. Examples are presented for the case of titanium blade stringers bonded to a metal-rich surface of a ceramic-metal (TiB/Ti) panel. As follows from these examples, both the buckling load as well
as the fundamental frequency can be significantly increased using a relatively light system of stringers. Further improvements are possible if cross sections and spacing of stringers are appropriately varied.

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[29] ABAQUS ver. 6.5, ABAQUS Inc. Rising Sun Mills, 116 Valley St, Providence, RI.