Physics 413: Statistical Mechanics - Final Exam

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16 May 2002

Problem 1: Two-level system (100 points)

A quantum mechanical system has two single-particle states $|-1\rangle$ and $|1\rangle$ with energies $\epsilon_1 = -\epsilon_{-1} = \epsilon$.

a) The system is occupied by just one particle. Using the canonical ensemble calculate the Helmholtz free energy, the entropy, the internal energy and the specific heat as functions of temperature.

b) The system is occupied by two identical particles. Write down all possible states, the corresponding energies and the canonical probabilities for these states for bosons (S=0) and for fermions (S=1/2, but both particles being in the $\uparrow$ state).

c) Consider an additional interaction between the particles of the form $U\ n_{-1}\ n_1$. where U is the interaction energy and n_{-1} and n_1 are the particle numbers of the two single-particle states. How do the canonical probabilities for the two-boson states from b) change as a result of U? Discuss the limits $U \to \infty$ and $U \to -\infty$.

Problem 2: Blackbody radiation in one dimension (100 points)

Consider photons in a one-dimensional cavity of length L. The Hamiltonian is $H = \sum_i c|p_i|$.

a) Calculate the density of states $a(\epsilon)$.

b) Calculate the internal energy U and the specific heat C_V as functions of L and the temperature T. (Hint: $\int_0^\infty dx x/(e^x - 1) = \pi^2/6$)

c) Calculate the entropy S, Helmholtz free energy A and the pressure p.

d) Calculate and discuss the isothermal compressibility $\kappa = (\partial V/\partial p)_T/V$.

Problem 3: Classical hard sphere gas (100 points)

Consider a three-dimensional classical gas of hard spheres, i.e., with an interaction potential

$$V(r) = \begin{cases} \infty & r < R \\ 0 & r > R \end{cases}$$

a) Calculate the Helmholtz free energy A, internal energy U, and the entropy for the ideal gas case, i.e., $R = 0$.

b) Use the virial expansion to second order,

$$\frac{pV}{Nk_BT} = 1 + a_2(T)\left(\frac{N}{V}\lambda^3\right) ,$$

where

$$a_2(T) = -\frac{1}{2\lambda^3} \int d^3r (e^{-\beta V(r)} - 1)$$

to calculate the equation of state of the interacting gas ($R \neq 0$).

c) Rewrite the equation of state in a van-der-Waals like form (for $NR^3 \ll V$). Compare with the van-der-Waals equation and discuss the difference.