Physics 481: Condensed Matter Physics - Homework Solutions 4

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(1)

Problem 4.1

a) random directions

$$S = \frac{1}{4\pi}\int d\varphi \, \sin\theta \, d\theta \, \left(\cos^2\theta - 1/3\right)$$

$$= \frac{1}{2}\int_0^{\pi} d\theta \, \sin\theta \, \left(\cos^2\theta - \frac{1}{3}\right)$$

$$= \frac{1}{2}\int_{-1}^{1} d\eta \, \left(\eta^2 - \frac{1}{3}\right) = \frac{1}{2}\left(\frac{1}{3}\eta^3 - \frac{1}{3}\eta\right)\Big|_{-1}^{1} = 0$$

perfect nematic order

$$\theta = \text{const}$$

$$S = 1 - \frac{1}{3} = \frac{2}{3}$$

b) $\langle \cos\theta \rangle$ is not an order parameter for the nematic because the nematic order is unchanged by $\theta \to \theta + \pi$ (symmetric rods) while $\langle \cos\theta \rangle$ changes sign.

c) order parameter must be invariant under 90° rotations

4.1c) continued

$$S' = \langle \cos(4\theta) + c \rangle$$

to find c, consider random directions

$$S' = \frac{1}{4\pi} \int d\varphi \int \sin\theta \, d\theta \left(\cos 4\theta + c \right)$$

$$= \frac{1}{2} \int_0^{\pi} d\theta \, \sin\theta \left(\cos 4\theta + c \right)$$

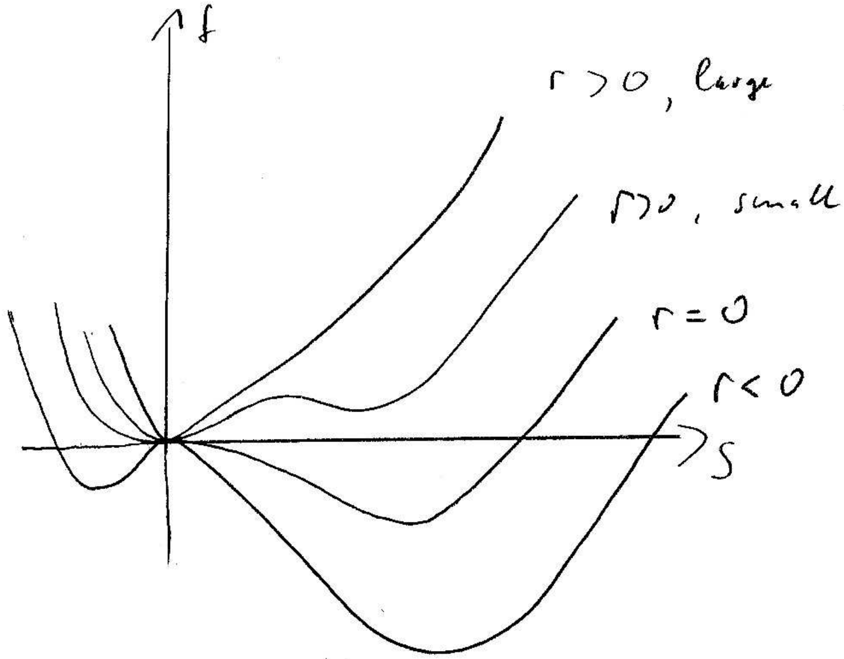
$$= \frac{1}{2} \left(-\frac{2}{15} + 2c \right) \quad \Rightarrow \quad c = \frac{1}{15}$$

$$\boxed{S' = \left\langle \cos(4\theta) + \frac{1}{15} \right\rangle}$$

c.f. for nematic case $S' = \langle \cos(2\theta) + \frac{1}{3} \rangle$

Problem 4.2.

a) set $u = w = 1$



b) $$\frac{\partial f}{\partial s} = r s - 3 w s^2 + 4 u s^3 = s\left(r - 3 w s + 4 u s^2\right)$$

$s_1 = 0$

$s_{2,3}$: $$s^2 - \frac{3w}{4u} s + \frac{r}{4u} = 0$$

$$s_{2,3} = \frac{3}{8}\frac{w}{u} \pm \sqrt{\frac{9}{64}\frac{w^2}{u^2} - \frac{r}{4u}}$$

c) Nematic state:

$$S = \frac{3}{8}\frac{w}{u} + \sqrt{\frac{9}{64}\frac{w^2}{u^2} - \frac{r}{4u}}$$

<u>To find r_c:</u>

$$f = 0 \qquad \frac{1}{2}r - wS + uS^2 = 0 \qquad \mathrm{I}$$

$$\frac{\partial f}{\partial S} = 0 \qquad r - 3wS + 4uS^2 = 0 \qquad \mathrm{II}$$

$$4\mathrm{I} - \mathrm{II} = r - wS = 0 \qquad r = wS$$

$$2\mathrm{I} - \mathrm{II} = wS - 2uS^2 = 0 \qquad w = 2uS$$

$$\boxed{S_c = \frac{w}{2u}, \quad r_c = \frac{w^2}{2u}}$$

d) isotropic phase locally stable for $r > 0$

nematic phase locally stable for

$$\frac{r}{4u} < \frac{9}{64}\frac{w^2}{u^2}$$

$$r^* = \frac{9}{16}\frac{w^2}{u}$$