# Physics 481: Condensed Matter Physics - Homework Solutions 8

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Problem 8.1 - Damped Oscillations

$$M \ddot{u}_j = -\Gamma \dot{u}_j + K(u_{j+1} - 2u_j + u_{j-1})$$

$$u = u_0 e^{ikx - i\omega t}$$

$$-M\omega^2 = i\omega\Gamma + K(e^{ika} - 2 + e^{-ika})$$

$$\omega^2 + i\underbrace{\frac{\Gamma}{M}}_{2\gamma}\omega - \frac{K}{M}\underbrace{(2 - 2\cos ka)}_{4\sin^2\frac{ka}{2}} = 0$$

$$\omega_{1,2} = -i\gamma \pm \sqrt{\frac{4K}{M}\sin^2\frac{ka}{2} - \gamma^2}$$

- frequency is reduced by $-\gamma^2$ term
- frequency has imaginary part $-i\gamma$ → decay

  relaxation time $\tau = \frac{1}{\gamma} = \frac{2M}{\Gamma}$

$\gamma^2 \ll \frac{K}{M}$ ⇒

$k \to 0$ $\quad \omega_{1,2} = -i\gamma \pm i\gamma$ $\quad$ Overdamped

$k \to \frac{\pi}{a}$ $\quad \omega_{1,2} = -i\gamma \pm \sqrt{\frac{4K}{M} - \gamma^2}$ $\quad$ damped oscillations

Problem 8.1.

a) $$[a, a^\dagger] = a a^\dagger - a^\dagger a = \left(\sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{ip}{\sqrt{2m\omega\hbar}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{ip}{\sqrt{2m\omega\hbar}}\right)$$
$$- \left(\sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{ip}{\sqrt{2m\omega\hbar}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}} + \frac{ip}{\sqrt{2m\omega\hbar}}\right)$$
$$= \frac{1}{2\hbar}\left[ipx - ixp - ixp + ipx\right] = \frac{i}{\hbar}(px - xp) = \frac{i}{\hbar}\frac{\hbar}{i} = 1$$

b) $$\hbar\omega(a^\dagger a + \tfrac{1}{2}) = \hbar\omega\left\{\left(\sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{ip}{\sqrt{2m\omega\hbar}}\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\, x + \frac{ip}{\sqrt{2m\omega\hbar}}\right) + \frac{1}{2}\right\}$$
$$= \hbar\omega\left\{\frac{m\omega}{2\hbar}x^2 + \frac{p^2}{2m\omega\hbar} + i(xp - px)\frac{1}{2\hbar} + \frac{1}{2}\right\}$$
$$= \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = H$$

c) norm of $a^\dagger|n\rangle$

$$\langle a^\dagger n | a^\dagger n\rangle = \langle n | a a^\dagger | n\rangle = \langle n | 1 + a^\dagger a | n\rangle$$
$$= (n+1)\langle n|n\rangle = n+1 \qquad \Rightarrow a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

norm of $a|n\rangle$

$$\langle an | an\rangle = \langle n | a^\dagger a | n\rangle = n\langle n|n\rangle = n$$
$$\Rightarrow \quad a|n\rangle = \sqrt{n}\,|n-1\rangle$$

Problem 8.3.

number of polarization directions

a) $$D(\omega) = \frac{n_p}{(2\pi)^d} \int d^d q \; \delta(\omega - c|\vec{q}|^\nu)$$

$$= \frac{n_p \Omega_d}{(2\pi)^d} \int_0^{(\omega_D/c)^{\frac{1}{\nu}}} dq\, q^{d-1} \; \delta(\omega - c q^\nu)$$

$$x = c q^\nu \qquad q = (x/c)^{\frac{1}{\nu}} \qquad dq = \frac{1}{\nu}\left(\frac{x}{c}\right)^{\frac{1}{\nu}-1} \frac{dx}{c}$$

$$D(\omega) = \frac{n_p \Omega_d}{(2\pi)^d} \int_0^{\omega_D} dx \; \frac{1}{\nu}\left(\frac{x}{c}\right)^{\frac{1}{\nu}-1} \frac{1}{c} \left(\frac{x}{c}\right)^{\frac{1}{\nu}(d-1)} \; \delta(\omega - x)$$

$$D(\omega) = \frac{n_p \Omega_d}{(2\pi)^d} \frac{1}{\nu} \frac{\omega^{d/\nu - 1}}{c^{d/\nu}} \qquad (\omega < \omega_D)$$

Debye frequency:

$$\int_0^{\omega_D} d\omega = \frac{\cancel{n_p} N}{V} = \frac{\cancel{n_p} \Omega_d}{(2\pi)^d} \frac{1}{\cancel{\nu}} \frac{1}{d/\cancel{\nu}} \left(\frac{\omega_D}{c}\right)^{d/\nu}$$

$$\omega_D^{d/\nu} = c^{d/\nu} \frac{(2\pi)^d d}{\Omega_d} \frac{N}{V}$$

b) $$\frac{C}{V} = \int d\omega\, D(\omega) \quad C_{osc}\left(\frac{\hbar\omega}{k_B T}\right) \qquad x = \frac{\hbar\omega}{k_B T}$$

$$\sim \int_0^{\omega_D} d\omega\, \omega^{d/\nu - 1} \quad C_{osc}\left(\frac{\hbar\omega}{k_B T}\right)$$

$$\sim T^{d/\nu} \int_0^\infty dx\, x^{d/\nu - 1} \quad C_{osc}(x)$$

$$\frac{C}{V} \sim T^{d/\nu}$$

c) general case already