Physics 413: Statistical Mechanics - Final exam

Problem 1: Random boxes (50 points + 10 BONUS)

A machine in a factory making cylindrical boxes is malfunctioning. As a result, it is producing cylinders of random size. Specifically, the diameter and the height of the cylinder are independent random quantities. They can take values between 0 and 2 m with a constant probability density.

a) Write down the probability densities for diameter and height. (5 points)
b) What are the largest and smallest possible volumes of the boxes? (5 points)
c) Calculate the average volume $\langle V \rangle$ of the produced cylinders and its standard deviation. (10 pts)
d) Derive the probability density of V. (Hint: Be careful with the integration bounds when transforming and integrating over the δ-function) (25 pts)
e) What is the most likely volume? (5 pts)
f) (10 BONUS points) Check the results of part c) by integrating the probability density of V.

Problem 2: Blackbody radiation in two dimensions (75 points + 5 BONUS)
Consider photons in a two-dimensional square cavity of "volume" $V = L^2$. The Hamiltonian is $H = \sum_i c|p_i|$.

a) What are the single-particle states and the allowed wavevectors? (Use periodic boundary conditions.) (5 points)

b) Calculate the density of states $g(\epsilon)$. (20 points)

c) Calculate the total energy E and the specific heat C_V as functions of L and the temperature T. (25 points)

d) Calculate the entropy S, Helmholtz free energy A and the pressure p as functions of L and T. (25 points)

e) (5 BONUS points) Calculate and discuss the isothermal compressibility $\kappa = -(\partial V/\partial p)_T/V$.

Problem 3: Ising model with next-nearest neighbor interactions (75 points + 10 BONUS)

Consider an Ising model ($S_i = \pm 1$) on a square lattice. There are interactions $J > 0$ and $K > 0$ between nearest neighbors and next-nearest neighbors as shown in the picture.

$$H = -J \sum_{\langle ij \rangle} S_i S_j - K \sum_{\{ij\}} S_i S_j - \mu_B B \sum_i S_i$$

where the first sum runs over all pairs of nearest neighbors and the second sum runs over all pairs of next-nearest neighbors.

a) What is the ground state in the absence of a field ($B = 0$)? What is the ground state for large field B? (10 points)

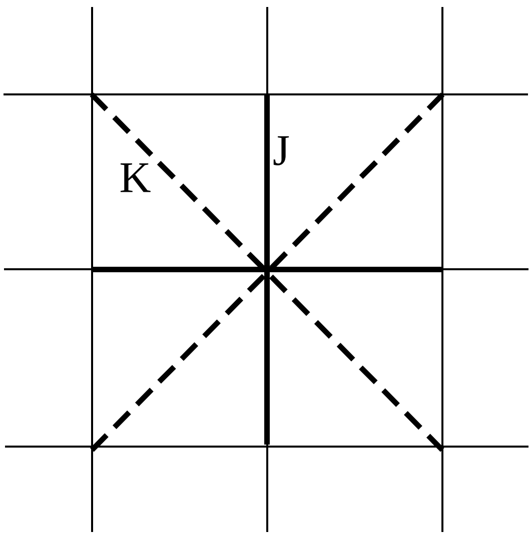
b) Determine the mean-field approximation for this Hamiltonian. (15 points)

c) Solve the mean-field Hamiltonian and derive the mean-field equation. (20 points)

d) Solve the mean-field equation for $B = 0$ and find the critical temperature. (10 points)

e) Calculate the critical exponents β, γ, and δ for the magnetization, susceptibility, and the critical isotherm, respectively. (20 points)

f) (10 BONUS points) Determine the ground state for $J < 0, K > 0$. Discuss qualitatively what happens for $J > 0, K < 0$ or $J < 0, K < 0$. (The field $B = 0$ in all cases.)



$\int_0^\infty dx\, e^{-ax} = \frac{1}{a}$, $\int_0^\infty dx\, xe^{-ax} = \frac{1}{a^2}$, $\int_0^\infty dx\, x^2 e^{-ax} = \frac{2}{a^3}$, $\int_0^\infty dx\, x^2/(e^x - 1) = 2\zeta(3)$

$\tanh(x) = x - \frac{1}{3}x^3 + ...$, $\mathrm{Artanh(x) = x + \frac{1}{3}x^3 + ...}$