Physics 6311: Statistical Mechanics - Homework 11

due date: Tuesday, April 23, 2019

Problem 1: Ferromagnets and antiferromagnets (8 points)

Consider classical Ising models ($S_i = \pm 1$) described by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} S_i S_j$$

on four different lattices:

a) ferromagnet ($J > 0$) on a square lattice
b) antiferromagnet ($J < 0$) on a square lattice
c) ferromagnet ($J > 0$) on a triangular lattice
d) antiferromagnet ($J < 0$) on a triangular lattice

Qualitatively describe and compare the ground states ($T = 0$) of these systems. Calculate their ground state energies (per spin).

Problem 2: Mean-field theory of the classical Heisenberg ferromagnet (16 points)

The classical Heisenberg model of ferromagnetism is given by a Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j - \mu_B \sum_i \vec{B} \cdot \vec{S}_i \ .$$

Here, $\vec{S}_i$ is a three-dimensional unit vector, $\vec{B}$ is the external magnetic field, and $J > 0$ is the exchange interaction.

a) Derive the mean-field approximation H_{MF} of this Hamiltonian.
b) Solve H_{MF} and derive the mean-field equation for the magnetization.
c) Find the critical temperature and compare it to that of an Ising model.
d) Calculate the critical exponents β, γ, and δ.

Problem 3: Exact solution of the one-dimensional Ising model (16 points)

Consider a chain of N Ising spins ($S_i = \pm 1$). The Hamiltonian reads

$$H = -J \sum_{i=1}^{N} S_i S_{i+1} - \mu_B B \sum_{i=1}^{N} S_i$$

with periodic boundary conditions $S_{N+1} \equiv S_1$.

a) Write the partition function $Q = \sum_{\{S_i\}} \exp(-\beta H)$ as a product of identical 2×2 matrices (one for each pair S_i, S_{i+1}).

b) Diagonalize the matrices, and write the partition function in terms of the eigenvalues.

c) Find the Helmholtz free energy, and compute the magnetization m. Study the limits of high and low temperatures. Does the system show a spontaneous magnetization?