# Physics 6311: Statistical Mechanics - Test Preparation Homework 13

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due date: Monday, May 13, 2019

**Problem 1: Probability distributions** (8 points)

Consider two statistically independent random variables $X$ and $Y$. Their probability densities are Gaussian with the first and second moments being $\langle x \rangle = \langle y \rangle = 0$ and $\langle x^2 \rangle = \langle y^2 \rangle = 1$.

a) Calculate the characteristic function of the random variable $Z = X^2 + Y^2$.

b) Calculate the moments $\langle z \rangle, \langle z^2 \rangle$, and $\langle z^3 \rangle$ and the three cumulants $C_1, C_2$, and $C_3$ of $Z$.

c) Calculate the probability density $P_Z(z)$.

Hint:

$$\int_{-\infty}^{\infty} dx \frac{x \sin(ax)}{b^2 + x^2} = \pi e^{-ab}, \quad \int_{-\infty}^{\infty} dx \frac{\cos(ax)}{b^2 + x^2} = \frac{\pi}{b} e^{-ab} \quad (a > 0)$$

**Problem 2: Ideal gas in a linear potential well** (8 points)

Consider a non-relativistic classical ideal gas of $N$ indistinguishable particles of mass $m$ at temperature $T$. It is located in a three-dimensional potential well of the form $E_{pot}(\vec{r}) = B|\vec{r}|$ ($B$ is a constant).

a) Calculate the partition function and the Helmholtz free energy of the gas.

b) Determine the internal energy and the specific heat. Compare with the equipartition theorem.

c) Calculate how the particle density $n(\vec{r})$ changes with $\vec{r}$.

**Problem 3: Ensemble of harmonic oscillators** (8 points)

Consider an ensemble of harmonic oscillators with different frequencies $\omega$. The density of states in frequency space is

$$\tilde{g}(\omega) = \begin{cases} c * \omega^n & \omega > 0 \\ 0 & \omega < 0 \end{cases}$$

where $n$ is a positive integer. Calculate the internal energy and the specific heat as functions of $T$ and $n$. (Hint: Neglect the zero-point motion). Compare with the result for the blackbody radiation.

**Problem 4: Fermions on a surface** (6 points)

Consider an ideal gas of $N$ spin-1/2 fermions of mass $m$ on a planar surface of area $A$. Derive a closed form expression for the chemical potential as a function of temperature $T$ (valid for all temperatures). Discuss the limits $T \to 0$ and $T \to \infty$.

**Problem 5: Debye phonons in two dimension** (10 points)

Consider a thin film (two-dimensional solid) of $N$ atoms and linear size $L$. This solid has 3N phonon modes. Within the Debye model, the phonon frequencies are $\omega_{\vec{k}} = c|k|$ for $0 \leq \omega_{\vec{k}} < \Omega_D$. Here, $c$ is the speed of sound.

a) Calculate the density of states $g(\epsilon)$.

b) Determine the Debye frequency $\Omega_D$.

c) Calculate the internal energy and the specific heat for low temperatures ($k_B T \ll \hbar\Omega_D$) and for high temperatures ($k_B T \gg \hbar\Omega_D$).