Physics 6311: Statistical Mechanics - Homework 5

due date: Tuesday, Feb 26, 2019

Problem 1: Comparison of the microcanonical and canonical ensembles: system of two-level atoms (15 points)

A system consists of $N \gg 1$ non-interacting, distinguishable two-level atoms. Each atom can be in one of two states 0 and 1 with energies $E_0 = 0$ and $E_1 = \epsilon$.

a) Microcanonical ensemble: The total internal energy $E = N_0 E_0 + N_1 E_1$ is fixed. (N_0 is the number of atoms in state 0, N_1 is the number of atoms in state 1.)

   (i) Calculate the entropy as a function of E. When is the entropy minimal? When is it maximal?

   (ii) Calculate the temperature as a function of the energy and discuss its sign. Calculate the specific heat.

b) Canonical ensemble: The system is in equilibrium with the heat bath at temperature T.

   (i) Find the partition function and the Helmholtz free energy.

   (ii) Calculate the internal energy, the entropy, and the heat capacity as functions of temperature.

   (iii) Compare the results of the canonical and microcanonical ensembles. Show that both lead to the same thermodynamic relations.

Problem 2: Two interacting magnetic moments in the canonical ensemble (10 points)

Consider two classical magnetic moments characterized by three-dimensional unit vectors $\mathbf{S}_1$ and $\mathbf{S}_2$. They interact via an exchange interaction with the Hamiltonian $H = -J\ \mathbf{S}_1 \cdot \mathbf{S}_2$ where J is a positive constant.

a) In the ground state, what do you expect the relative orientation of the two moments to be?

b) Use the canonical ensemble to calculate the partition function and the Helmholtz free energy.

c) Determine average energy and heat capacity as functions of temperature.

d) At low temperatures, what is the average angle between the moments, and what is its standard deviation?

Problem 3: Specific heat of an anharmonic oscillator (15 points)

An anharmonic *classical* oscillator has the Hamiltonian

$$H = \frac{p^2}{2m} + V_0 \cosh(x/x_0)$$

where the mass m as well as the potential parameters V_0 and x_0 are constants.

a) Calculate the specific heat as a power series in the temperature T to linear order in T. (To decide how to set up the expansion, think about where the particle will be at low and high temperatures, respectively. What does this mean for the potential?)

b) Also calculate the contribution of order T^2.