Physics 413: Statistical Mechanics - Midsemester test

Problem 1: Random window panes (50 points)

A machine in a factory making glass window panes is malfunctioning. As a result, it is producing rectangular windows of random size. Specifically, the horizontal and vertical sizes of the window are independent random quantities. They can take values between 0 and 2 m with a constant probability density.

a) Calculate the average area $\langle A \rangle$ of the produced windows and its standard deviation. (15 pts)

b) Derive the probability density of A. (Hint: Be careful with the integration bounds when transforming and integrating over the δ-function) (30 pts)

c) What is the most likely area? (5 pts)

Problem 2: Ultra-relativistic classical ideal gas (75 points + 10 BONUS)

Consider a gas of N non-interacting, indistinguishable, classical particles in a cubic box of linear size L. The energy-momentum relation is ultra-relativistic, i.e., the classical Hamiltonian is $H = \sum_i \mathrm{c}|\vec{\mathrm{p}}_\mathrm{i}|$ (c is the speed of light).

a) Calculate the canonical partition function and the Helmholtz free energy. (25 pts)

b) Calculate the thermodynamic equation of state (relation between p, V, T). (25 pts)

c) Calculate the caloric equation of state (energy-temperature relation) and the specific heat C_V at constant volume. (25 pts)

d) (10 BONUS pts) Determine the ratio of the specific heats C_p/C_v and compare to the conventional case with a quadratic energy-momentum relation.

Problem 3: System of 3-level atoms (75 points + 10 BONUS)

Consider a system of N atoms coupled to a heat bath at temperature T. Each atom has three states $|1\rangle, |2\rangle, |3\rangle$ with energies $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = \epsilon > 0$.

a) Use the canonical ensemble to calculate the partition function and the Helmholtz free energy. (25 pts)

b) Calculate the energy and the heat capacity as functions of temperature. (25 pts)

c) Calculate the entropy as function of temperature. What is the zero-temperature limit of the entropy? (25 pts)

d) (10 BONUS pts) Where does the zero-temperature entropy come from? (Count the available states at $T = 0$.)

$\int dx\, \ln(x) = x\ln(x) - x, \quad \int dx\, x\ln(x) = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2, \quad \int dx\, x^2\ln(x) = \frac{1}{3}x^3\ln(x) - \frac{1}{9}x^3$

$\int_0^\infty dx\, e^{-ax} = \frac{1}{a}, \quad \int_0^\infty dx\, xe^{-ax} = \frac{1}{a^2}, \quad \int_0^\infty dx\, x^2e^{-ax} = \frac{2}{a^3},$