Physics 6311: Statistical Mechanics - Homework Solutions 13

Problem 1: Probability distributions (8 points)

a)

$$\begin{aligned} P_Z(z) &= \int dx \int dy\ P_X(x)P_Y(y)\ \delta(z - x^2 - y^2) \\ f_Z(k) &= \int dx \int dy\ P_X(x)P_Y(y)\ \exp(ikx^2 + iky^2) \\ &= \frac{1}{2\pi}\left[\int dx \exp(-x^2/2 + ikx^2)\right]^2 \\ &= 1/(1 - 2ik) \end{aligned}$$

b) Expand characteristic function: $f_Z(k) = 1 + 2ik + (2ik)^2 + (2ik)^3 + ...$
The coefficients give the moments: $\langle z \rangle = 2,\ \langle z^2 \rangle = 8,\ \langle z^3 \rangle = 48$

To get the cumulants, expand $\ln f_Z(k) = -\ln(1 - 2ik) = 2ik + \frac{1}{2}(2ik)^2 + \frac{1}{3}(2ik)^3 + ...$
The coefficients give the cumulants: $C_1 = 2, C_2 = 4, C_3 = 16$.

c) back transformation

$$P_Z(z) = \frac{1}{2}\exp(-z/2)\Theta(z)$$

Problem 2: Ideal gas in a linear potential well (8 points)

a)

$$\begin{aligned} Q_N &= Q_1^N/N! \\ Q_1 &= \frac{1}{h^3}\int d^3r d^3p\ e^{-\beta(\vec{p}^2/(2m) + B|\vec{r}|)} \\ &= \frac{1}{h^3}(2\pi m k_B T)^{3/2} 4\pi \int_0^\infty dr\ r^2\ e^{-\beta B|\vec{r}|} \\ &= \frac{1}{h^3}(2\pi m k_B T)^{3/2} 8\pi (k_B T/B)^3 \\ A &= -k_B T \ln Q_N = -N k_B T[\ln Q_1 - \ln N + 1] \end{aligned}$$

b)

$$\begin{aligned} \langle E \rangle &= -\frac{\partial}{\partial \beta}\ln Q_N = N\left(\frac{3}{2}k_B T + 3k_B T\right) = \frac{9}{2}N k_B T \\ C_V &= \partial\langle E \rangle/\partial T = \frac{9}{2}N k_B \qquad \delta W = 0, \text{thus } \delta Q = dE \end{aligned}$$

linear potential makes larger contribution to C_V than quadratic degree of freedom.

c) particle density is N times the probability density for finding a single particle at $\vec{r}$

$$\begin{aligned} n(\vec{r}) &= \frac{N}{Q_1 h^3} \int d^3p \; e^{-\beta(\vec{p}^2/(2m)+B|\vec{r}|)} \\ &= N \frac{(\beta B)^3}{8\pi} e^{-\beta B|\vec{r}|} \end{aligned}$$

Problem 3: Ensemble of harmonic oscillators (8 points)

single harmonic oscillator at frequency ω (without zero-point motion):

$$\begin{aligned} Z &= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} = \frac{1}{1-e^{-\beta\hbar\omega}} \\ U &= -\frac{\partial \ln Z}{\partial \beta} = \frac{\partial}{\partial \beta} \ln(1-e^{-\beta\hbar\omega}) \\ &= \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \end{aligned}$$

ensemble of oscillators:

$$\begin{aligned} U &= \int d\omega \; \tilde{a}(\omega) \; \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \\ &= c\hbar \int d\omega \; \frac{\omega^{n+1}}{e^{\beta\hbar\omega}-1} \\ &= c\hbar^{-(n+1)}(k_B T)^{n+2} \int dx \frac{x^{n+1}}{e^x - 1} \\ &\sim T^{n+2} \\ C_V &= (n+2)U/T \end{aligned}$$

$n = 2$ corresponds to blackbody radiation.

Problem 4: Fermions on a lattice (6 points)

13.4

$$\langle N \rangle = 2 \sum_{\vec{k}} \langle n_{\vec{k}} \rangle = 2 \sum_{\vec{k}} \frac{1}{e^{\beta(\varepsilon_{\vec{k}} - \mu)}+1}$$

$$\varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2m} \qquad d\varepsilon = \frac{\hbar^2}{2m} 2k\,dk = \frac{\hbar^2}{m} k\,dk$$

$$\langle N \rangle = 2 \frac{A}{(2\pi)^2} \int d^2k \frac{1}{e^{\beta(\varepsilon_{\vec{k}} - \mu)}+1}$$

$$= \frac{2A}{2\pi} \int dk\, k \frac{1}{e^{\beta(\varepsilon_k - \mu)}+1}$$

$$= \frac{A}{\pi} \frac{m}{\hbar^2} \int_0^\infty d\varepsilon \frac{1}{e^{\beta(\varepsilon - \mu)}+1} \qquad \boxed{g(\varepsilon) = \frac{Am}{\pi \hbar^2}}$$

$$= \frac{Am}{\pi \hbar^2} k_B T \underbrace{\int_0^\infty dx \frac{1}{e^{x-\beta\mu}+1}}_{\ln(1+e^{\beta\mu})}$$

$$\langle N \rangle = \frac{Am}{\pi \hbar^2} k_B T \ln\left(1 + e^{\beta\mu}\right)$$

$$\exp\left[\frac{\pi \hbar^2 \langle N \rangle}{A m k_B T}\right] = 1 + e^{\beta\mu}$$

$\underline{T \to 0}$ $$\frac{\pi \hbar^2 \langle N \rangle}{Am} = \mu$$

$\underline{T \to \infty}$ $$\cancel{1} + \frac{\pi \hbar^2 \langle N \rangle}{A m k_B T} = \cancel{1} + e^{\beta\mu}$$

$$\mu = k_B T \ln\left(\frac{\pi \hbar^2 \langle N \rangle}{A m k_B T}\right) \to -\infty$$

Problem 5: Debye phonons in two dimensions (10 points)

Debye phonons in 2D

a) $\langle N_p \rangle = 3 \sum_{\vec{k}} \frac{1}{e^{\beta \varepsilon_{\vec{k}}} - 1}$ $\qquad \varepsilon_{\vec{k}} = \hbar |\vec{k}| c$

$k_i = \frac{2\pi}{L} n_i \,, \; n_i = 0, \pm 1, ..$

$$\langle N_p \rangle = \frac{3L^2}{4\pi^2} \int d^2k \, \frac{1}{e^{\beta \varepsilon_{\vec{k}}} - 1} = \frac{3L^2}{2\pi} \int_0^\infty k\, dk \, \frac{1}{e^{\beta \varepsilon_{\vec{k}}} - 1}$$

$$k = \frac{\varepsilon}{\hbar c} \,, \quad dk = \frac{d\varepsilon}{\hbar c}$$

$$\langle N_p \rangle = \frac{3L^2}{2\pi \hbar^2 c^2} \int_0^\infty d\varepsilon \, \frac{\varepsilon}{e^{\beta \varepsilon} - 1} \qquad \boxed{g(\varepsilon) = \frac{3L^2 \varepsilon}{2\pi \hbar^2 c^2}}$$

b) total number of modes is $3N$

$$\int_0^{\hbar \Omega_D} d\varepsilon \, g(\varepsilon) = \frac{3L^2}{2\pi \hbar^2 c^2} \int_0^{\hbar \Omega_D} d\varepsilon \, \varepsilon = \frac{3 \, L^2 \hbar^2 \Omega_D^2}{4\pi \, \hbar^2 c^2} = 3N$$

$$L^2 \Omega_D^2 = 4\pi c^2 N \qquad \boxed{\Omega_D^2 = 4\pi c^2 \frac{N}{L^2}}$$

using $L^2/c^2 = \frac{4\pi N}{\Omega_D^2}$ $\Rightarrow$ $g(\varepsilon) = \frac{3\varepsilon}{2\pi \hbar^2} \frac{4\pi N}{\Omega_D^2} = \frac{6N\varepsilon}{\hbar^2 \Omega_D^2}$

c) $$\langle E_p \rangle = \int d\varepsilon \, \varepsilon \, g(\varepsilon) \frac{1}{e^{\beta\varepsilon} - 1} = \frac{6N}{\hbar^2 \Omega_D^2} \int_0^{\hbar \Omega_D} d\varepsilon \, \frac{\varepsilon^2}{e^{\beta \varepsilon} - 1}$$

use $x = \beta \varepsilon$

$$\langle E_p \rangle = \frac{6N}{\hbar^2 \Omega_D^2} (k_B T)^3 \int_0^{\hbar \Omega_D / k_B T} dx \, \frac{x^2}{e^x - 1}$$

$k_B T \ll \hbar \Omega_D$ extend integration to ∞

$$\langle E_p \rangle = \frac{6N}{\hbar^2 \Omega_D^2} (k_B T)^3 \, \zeta(3) \leftarrow \approx 1.202$$