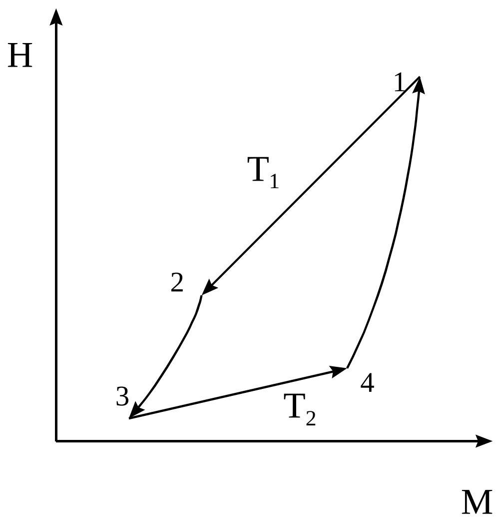
Physics 6311: Statistical Mechanics - Homework Solutions 2

Problem 1: Carnot process for paramagnetic substance

isotherms are straight lines, form of adiabats follows from below: $CD\ln(T/T_0) = (M^2 - M_0^2)/2$. Substitute for T: $H/H_0 = (M/M_0)\exp[(M^2 - M_0^2)/(2CD)]$.



1—2 isothermal at T_1: $dU = 0$

$$\Delta Q_{12} = -\Delta W = -\int_{M_1}^{M_2} H dM = \int_{M_1}^{M_2} \frac{T_1 M}{D} dM = -\frac{T_1}{2D}(M_2^2 - M_1^2)$$

3—4 isothermal at T_2, analogously: $\Delta Q_{34} = -\frac{T_2}{2D}(M_4^2 - M_3^2)$

2—3 adiabatic, $\delta Q = 0 = dU - HdM = CdT - HdM$

Use equation of state to substitute H:
$CDdT/T = MDM$
$CD\ln(T_2/T_1) = (M_3^2 - M_2^2)/2$

4—1 analogously $CD\ln(T_1/T_2) = (M_1^2 - M_4^2)/2$

Combining the two adiabatic parts gives $M_3^2 - M_2^2 = -(M_1^2 - M_4^2)$.

$$\eta = 1 + \frac{\Delta Q_{34}}{\Delta Q_{12}} = 1 + \frac{T_2(M_3^2 - M_4^2)}{T_1(M_1^2 - M_2^2)} = 1 - \frac{T_2}{T_1}$$

Problem 2: Entropy of the ideal gas

$pV = Nk_BT$, $U = (3/2)Nk_BT$

$dU = TdS - pdV$

$dS = dU/T + pdV/T = (3/2)Nk_B dT/T + Nk_B dV/V$

$S = (3/2)Nk_B \log(T/T_0) + Nk_B \log(V/V_0) + S_0$

S diverges for $T \to 0$: violation of the third law. Ideal gas ill defined for $T \to 0$.

Problem 3: Maxwell relations

$dU = T\ dS + E\ dP$: $\quad (\partial T/\partial P)_S = (\partial E/\partial S)_P$

$dH = T\ dS - P\ dE$: $\quad (\partial T/\partial E)_S = -(\partial P/\partial S)_E$

$dA = -S\ dT + E\ dP$: $\quad (\partial S/\partial P)_T = -(\partial E/\partial T)_P$

$dG = -S\ dT - P\ dE$: $\quad (\partial S/\partial E)_T = (\partial P/\partial T)_E$

Problem 4: Rubber elasticity

a) $dW = F\, dL$

$$= aT\left[\frac{L}{L_0} - \frac{L_0^2}{L^2}\right] dL$$

$$\Delta W = aT\left[\frac{1}{2}\frac{L^2}{L_0} + \frac{L_0^2}{L}\right]_{L_1}^{L_2}$$

$$\Delta W = aT\left[\frac{1}{2}\frac{L_2^2}{L_0} - \frac{1}{2}\frac{L_1^2}{L_0} + \frac{L_0^2}{L_2} - \frac{L_0^2}{L_1}\right]$$

b) $L_0 = L_1 = 20\,cm$

$L_2 = 50\,cm$

$T = 293\,K$

$a = 1.33 \times 10^{-2}\; N/K$

$$\Delta W = aT\,[0.405\; m]$$

$$= 1.578\; Nm$$

c) • $F = const$ implies that

$[\;]$ decreases as T increases

• $\frac{d}{dL}[\;] = \frac{1}{L_0} + 2\frac{L_0^2}{L^3} > 0$

$\Rightarrow$ rubber band shrinks as T increases