Physics 6311: Statistical Mechanics - Homework Solutions 3

Problem 1:

Power-law distribution

$$P_J(j) = \begin{cases} A j^{\gamma} & 0 \leq j \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

a) $$1 = \int_{-\infty}^{\infty} P_J(j) = \int_0^1 A j^{\gamma} = \frac{A}{\gamma+1} j^{\gamma+1}\Big|_0^1$$

$A = \gamma+1$ integral converges for $\underline{\gamma > -1}$

b) $$\langle J \rangle = \int dj\, j\, P_J(j) = (\gamma+1)\int_0^1 j^{\gamma+1} = \frac{\gamma+1}{\gamma+2}$$

$$\ln J_{geo} = \int dj\, P_J(j) \ln j = \int_0^1 (\gamma+1) j^{\gamma} \ln j$$

$$= j^{\gamma+1} \ln j - \frac{1}{\gamma+1} j^{\gamma+1}\Big|_0^1 = -\frac{1}{\gamma+1}$$

$$J_{geo} = \exp\left(-\frac{1}{\gamma+1}\right)$$

Problem 2:

a)

$$f_{\underline{x}}(k) = \int dx\, P_{\underline{x}}(x)\, e^{ikx} = \frac{1}{2}\int_{-1}^{1} dx\, \cos kx$$

$$= \frac{1}{2k} \sin kx \Big|_{-1}^{1} = \frac{1}{k} \sin k$$

$$f_{\underline{y}}(k) = f_{\underline{x}}(k) = \frac{1}{k} \sin k$$

b)

$$P_{\underline{z}}(z) = \int dx\, dy\, P_{\underline{x}}(x)\, P_{\underline{y}}(y)\, \delta(z-(x+y))$$

$$f_{\underline{z}}(k) = \int dx\, dy\, P_{\underline{x}}(x)\, P_{\underline{y}}(y)\, e^{ik(x+y)}$$

$$= f_{\underline{x}}(x) \cdot f_{\underline{y}}(y)$$

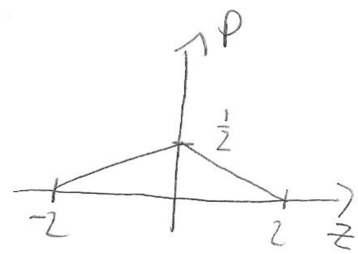
$$f_{\underline{z}}(k) = \frac{1}{k^2} \sin^2 k = \frac{1}{k^2}\left(k - \frac{k^3}{3!}\right)^2 = \left(1 - \frac{k^2}{3!}\right)^2$$

$$= 1 - \frac{k^2}{3} + \dots$$

c) Fourier back transformation

$$P_{\underline{z}}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikz}\, \frac{1}{k^2} \sin^2 k$$

$$= \begin{cases} \frac{1}{2} - \frac{|z|}{4} & |z| < 2 \\ 0 & |z| > 2 \end{cases}$$



d)

$$\langle z \rangle = 0, \quad \langle z^3 \rangle = 0, \quad C_1 = 0, \quad C_3 = 0$$

$$\langle z^2 \rangle = C_2 = -\frac{d^2 f_z(k)}{dk^2}\Big|_{k=0} = \frac{2}{3}$$

Problem 3: Distribution of the minimum of several random variables

a) The minimum of the x_i is *larger* than m if and only if all x_i are larger than m. The probability for this to happen is $[(1-m)/2]^N$. The probability for the minimum to be *smaller* than y is thus

$$F_M(m) = 1 - [(1-m)/2]^N \qquad\qquad (-1 < m < 1)$$

. Taking the derivative

$$P_M(y) = \frac{d}{dm}\left\{1 - [(1-m)/2]^N\right\} = (N/2)\ [(1-m)/2]^{N-1}$$

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b) $\langle m \rangle = (1-N)/(1+N) \ \rightarrow \ -1$ with $N \rightarrow \infty$
$\langle m^2 \rangle = (N^2 - N + 2)/(N^2 + 3N + 2)$
$\sigma_m^2 = \langle m^2 \rangle - \langle m \rangle^2 = 4N/[(N+1)^2(N+2)] \ \rightarrow \ 0$ with $N \rightarrow \infty$

Problem 4: Diode

Diode

$$I = I_0 \left(e^{eV/k_B T} - 1 \right)$$

I between $-I_0$ (for $V \to -\infty$) and ∞ (for $V \to \infty$)

$$P_I(I) = \int dV \, P_V(V) \, \delta\left[I - I_0 \left(e^{eV/k_B T} - 1 \right) \right]$$

$$P_V(V) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{V^2}{\sigma^2}}$$

$$\delta[f(V)] = \frac{1}{|f'(V_0)|} \delta(V - V_0)$$

only one solution $I(V)$ monotonous

$$e^{eV/k_B} = \frac{I}{I_0} + 1$$

$$\frac{eV}{k_B T} = \ln\left(1 + \frac{I}{I_0}\right) \qquad V_0 = \frac{k_B T}{e} \ln\left(1 + \frac{I}{I_0}\right)$$

$$|f'(V)| = I_0 \, e^{eV/k_B T} \frac{e}{k_B T} \underset{\text{using } \delta\text{-function}}{=\!=} \frac{e}{k_B T}(I + I_0)$$

$$\boxed{P_I(I) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{k_B T}{e} \frac{1}{I + I_0} e^{-\frac{1}{2}\left(\frac{k_B T}{e\sigma}\right)^2 \ln^2\left(1 + \frac{I}{I_0}\right)}}$$

$$\langle I \rangle = \int dI \; P_I(I) \; I$$

$$= I_0 \left[e^{\frac{1}{2}\left(\frac{e\sigma}{k_B T}\right)^2} - 1 \right]$$

<u>most probable I</u>

$$\frac{\partial}{\partial I} P(I) = 0 \qquad \text{use variable } x = \frac{I + I_0}{I_0}$$

$$\frac{\partial}{\partial x} \bar{P}(x) = 0$$

$$-\frac{1}{x^2} e^{-\frac{1}{2} A^2 \ln^2(x)} + \frac{1}{x} e^{-\frac{1}{2} A^2 \ln^2(x)} \left(-\frac{1}{2} A^2 \, 2 \ln(x)\right) \frac{1}{x} = 0$$

$$1 = -A^2 \ln(x) \qquad x = e^{-\frac{1}{A^2}}$$

$$I_p = I_0 (x_p - 1) = I_0 \left(e^{-\left(\frac{e\sigma}{k_B T}\right)^2} - 1 \right)$$