Physics 6311: Statistical Mechanics - Homework Solutions 6

Problem 6.1

$$Q = \int d^{3N}p \, d^{3N}q \quad e^{-\beta H}$$

$$= \prod_i \left[\int_{-\infty}^{\infty} dp_i \, e^{-\beta B_i p_i^2/2} \right] \prod_i \left[\int_{-\infty}^{\infty} dq_i \, e^{-\beta A_i |q_i|^n/2} \right]$$

Use $\int_0^{\infty} dx \, e^{-ax^n} = \frac{1}{n} \Gamma\left(\frac{1}{n}\right) a^{-\frac{1}{n}}$

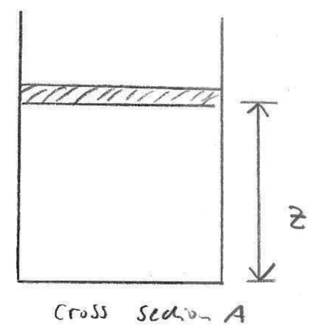
$$Q = \prod_i \left[2 \tfrac{1}{2} \Gamma\left(\tfrac{1}{2}\right) \left[\beta B_i/2\right]^{-\frac{1}{2}} \right] \prod_i \left[2 \tfrac{1}{n} \Gamma\left(\tfrac{1}{n}\right) \left[\beta A_i/2\right]^{-\frac{1}{n}} \right].$$

$$Q = \text{Const} \prod_{i=1}^{3N} \left[\beta^{-\frac{1}{2}} \beta^{-\frac{1}{n}} \right] = \text{Const} \; \beta^{-3N\left(\frac{1}{2}+\frac{1}{n}\right)}$$

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Q = -\frac{\partial}{\partial \beta} \left[-3N\left(\tfrac{1}{2}+\tfrac{1}{n}\right) \ln \beta \right]$$

$$\boxed{\begin{aligned} \langle E \rangle &= 3N\left(\tfrac{1}{2}+\tfrac{1}{n}\right) k_B T \\ C_V &= \frac{\partial}{\partial T} \langle E \rangle = 3N\left(\tfrac{1}{2}+\tfrac{1}{n}\right) k_B \end{aligned}}$$

Problem 6.2



$$H = \sum_{i=1}^{3N} \frac{p_i^2}{2m} + \underbrace{\frac{p_p^2}{2M} + Mgz}_{\text{piston}}$$

a)

$$Q = \int d^{3N}q \, d^{3N}p \int dp_p \, dz \quad e^{-\beta H}$$

$$= \sqrt{2\pi m k_B T}^{3N} \sqrt{2\pi M k_B T} \, A^N \int_0^\infty dz \, z^N \, e^{-\beta Mgz}$$

$x = \beta Mgz$

$z = x/(\beta Mg)$

$$Q = \sqrt{2\pi m k_B T}^{3N} \sqrt{2\pi M k_B T} \; A^N \frac{1}{(\beta Mg)^{N+1}} \underbrace{\int_0^\infty dx \, x^N e^{-x}}_{\text{const}}$$

b) • pressure given by piston

$$p = \frac{Mg}{A}$$

• average volume: $\langle V \rangle = A \langle z \rangle$

$$\langle z \rangle = \frac{\int_0^\infty dz \; z^{N+1} \, e^{-\beta Mgz}}{\int_0^\infty dz \; z^N \, e^{-\beta Mgz}} = -\frac{\partial}{\partial(\beta Mg)} \ln Q$$

$$\langle z \rangle = -\frac{\partial}{\partial(\beta Mg)} \ln\left(\frac{1}{(\beta Mg)^{N+1}}\right) = N+1 \; \frac{1}{\beta Mg}$$

$$pV = \frac{\cancel{Mg}}{\cancel{A}} \, \cancel{A} \, (N+1) \frac{1}{\beta \cancel{Mg}} = (N+1) \, k_B T$$

For $N \gg 1$ $\quad \boxed{pV = N k_B T}$

<u>energy:</u>

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \ln Q$$

$$= \frac{3}{2} N k_B T + \frac{1}{2} k_B T + (N+1) k_B T$$

$$C = \frac{\partial \langle E \rangle}{\partial T} = \left(\frac{3}{2} + 1\right) N k_B + \frac{1}{2} k_B + k_B$$

For $N \gg 1$

$$\boxed{C = 5/2 \, N k_B}$$

c) heat capacity at constant pressure because piston realizes ensemble with $p = \frac{Mg}{A} = \text{const}$, V fluctuating

Problem 6.3: Broadening of spectral lines

$$\begin{aligned}
\rho(\nu) &= \frac{\int d^3v\ e^{-\beta m v^2/2}\delta[\nu-\nu_0(1-v\cos\Theta/c)]}{\int d^3v\ e^{-\beta m v^2/2}} \\
&= \left(\frac{m\beta}{2\pi}\right)^{3/2} 2\pi \int d\Theta \sin\Theta v^2 dv\ e^{-\beta m v^2/2}\delta[\nu-\nu_0(1-v\cos\Theta/c)] \\
&= \left(\frac{m\beta}{2\pi}\right)^{3/2} \frac{2\pi c}{\nu_0} \int d\eta v dv\ e^{-\beta m v^2/2}\delta[\eta + c(\nu-\nu_0)/(\nu_0 v)]
\end{aligned}$$

The η integral gives a contribution if $|c(\nu-\nu_0)/(\nu_0 v)| < 1$.

$$\begin{aligned}
\rho(\nu) &= \left(\frac{m\beta}{2\pi}\right)^{3/2} \frac{2\pi c}{\nu_0} \int_{c(\nu-\nu_0)/\nu_0}^{\infty} dv v e^{-\beta m v^2/2} \\
&= \sqrt{\frac{\beta m c^2}{2\pi\nu_0^2}}\ e^{-\beta m c^2(\nu-\nu_0)^2/(2\nu_0^2)}
\end{aligned}$$

The width of the distribution is $\sqrt{\langle\Delta\nu^2\rangle} = \sqrt{\nu_0^2/(\beta m c^2)}$.