Physics 6311: Statistical Mechanics - Homework Solutions 9

Problem 9.1

Radiation of Betelgeuse

a) energy density $u(\varepsilon)$

$$u(\varepsilon) = \frac{1}{\pi^2 c^3 \hbar^3} \frac{\varepsilon^3}{e^{\beta\varepsilon} - 1}$$

maximum at $\frac{du}{d\varepsilon} = 0$

$$\frac{d}{d\varepsilon} \frac{\varepsilon^3}{e^{\beta\varepsilon}-1} = 0 \quad \Rightarrow \quad \frac{d}{dx} \frac{x^3}{e^x - 1} = 0$$

$$\frac{3x^2}{e^x - 1} - \frac{x^3 e^x}{(e^x-1)^2} = 0 \implies 3x^2(e^x - 1) = x^3 e^x$$

$(3-x)e^x = 3$ numerical solution $x \approx 2.82$

$\beta \varepsilon_{max} = 2.82$ $\quad T_{Bet} = \frac{\varepsilon_{max}}{2.82 k_B} \approx 3290\ K$

b) $P_{Bet} \approx 10^4 P_{sun} = 10^4 \times 3.85 \times 10^{26} W = 3.85 \times 10^{30} W$

Stefan - Boltzmann:

$$P_{Bet} = 4\pi r_{Bet}^2 \times \frac{\pi^2}{60} \frac{(k_B T)^4}{\hbar^3 c^2}$$

$$r_{Bet} = \sqrt{\frac{15}{\pi^3} \frac{\hbar^3 c^2}{(k_B T)^4} P_{Bet}} = 2.15 \times 10^{11}\ m$$

c) maximum of spectrum is in the red color range

radius much larger than sun

Problem 9.2: Phonons in liquid ^{4}Helium

a)

$$\Theta_D = \frac{\hbar\omega_D}{k_B} = \frac{\hbar}{k_B}\left(\frac{6\pi^2 c^3 N}{V}\right)^{1/3} = \frac{\hbar}{k_B}\left(\frac{6\pi^2 c^3 \rho}{m}\right)^{1/3} = 19.8\text{K}$$

with $m = 4 * 1.67*10^{-27}$kg (mass of Helium atom).

b) no transversal phonons, degeneracy factor is 1

$$\begin{aligned}
\frac{C_V}{N} &= \frac{4}{5}\pi^4 k_B \left(\frac{T}{\Theta_D}\right)^3 \\
\frac{C_V}{mN} &= \frac{4}{5}\pi^4 \frac{k_B}{m} \left(\frac{T}{\Theta_D}\right)^3 \\
\frac{C_V}{mN} &= 0.021\ (T/K)^3\ J/(gK)
\end{aligned}$$

Problem 9.3: Thermodynamic of magnons

a)

$$N = \underset{\text{degeneracy}}{f} \sum_{\vec{k}} \frac{1}{e^{\beta \varepsilon_k} - 1}$$

$$= f \frac{V}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta\varepsilon_k} - 1} = f \frac{V}{2\pi^2} \int dk\, k^2 \frac{1}{e^{\beta\varepsilon_k} - 1}$$

$$\varepsilon = \hbar D k^2 \qquad k = \sqrt{\frac{1}{\hbar D}\varepsilon} \qquad dk = \frac{1}{2}\frac{1}{\sqrt{\hbar D}}\frac{1}{\sqrt{\varepsilon}}$$

$$N = f \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} \frac{1}{2} \int d\varepsilon \sqrt{\varepsilon} \frac{1}{e^{\beta\varepsilon} - 1}$$

$$g(\varepsilon) = \frac{f}{2} \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} \sqrt{\varepsilon}$$

b)

$$E = \int_0^\infty d\varepsilon\, g(\varepsilon)\, \varepsilon \frac{1}{e^{\beta\varepsilon} - 1}$$

$$= \frac{f}{2} \frac{V}{2\pi^2} \frac{1}{(\hbar D)^{3/2}} \int d\varepsilon\, \varepsilon^{3/2} \frac{1}{e^{\beta\varepsilon} - 1}$$

$$= \frac{f}{2} \frac{V}{(2\pi)^2} \frac{1}{(\hbar D)^{3/2}} (k_B T)^{5/2} \underbrace{\int dx\, x^{3/2} \frac{1}{e^x - 1}}_{\text{const}}$$

$$E \sim (k_B T)^{5/2}$$

$$C_v = \frac{5}{2} \frac{E}{T} \sim (k_B T)^{3/2}$$

Problem 9.4: Phonons in 1D chain

a)

$$H = \sum_{i=j}^{N} \frac{p_j^2}{2m} + \frac{A}{2} \sum_{j=1}^{N} (x_j - x_{j+1})^2 \ .$$

discrete Fourier transformation: $x_j = \frac{1}{\sqrt{N}} \sum_k e^{ikj} \tilde{x}_k$ with $k = -\pi, -\pi + \frac{2\pi}{L}, ..., \pi$ and analogously for p_j

$$\begin{aligned} H &= \sum_{j=1}^{N} \frac{p_j^2}{2m} + \frac{A}{2} \sum_{j=1}^{N} \left(x_j^2 + x_{j+1}^2 - 2x_j x_{j+1} \right) \\ &= \sum_k \frac{\tilde{p}_k^2}{2m} + \frac{A}{2} \sum_k \left(2|\tilde{x}_k|^2 - 2e^{-ik}|\tilde{x}_k|^2 \right) \\ &= \sum_k \frac{\tilde{p}_k^2}{2m} + \frac{A}{2} \sum_k (2 - 2\cos k)|\tilde{x}_k|^2 \end{aligned}$$

Comparison with harmonic oscillator: $\omega_k = (\frac{A}{m}(2 - 2\cos k))^{1/2}$, for small k: $\omega_k = (\frac{A}{m})^{1/2}|k|$

b)

transform k-sum into integral,

$$U = \frac{L}{2\pi} \int_{-\pi/2}^{\pi/2} dk \ \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k} - 1}$$

substitute $x = \beta\hbar\omega_k$, for low T, the x-integral can be extended to infinity.

$$\begin{aligned} U &= \frac{L}{2\pi\beta} \frac{2}{\beta\hbar(A/m)^{1/2}} \int_0^\infty dx \ \frac{x}{e^x - 1} \\ &= \text{const} * T^2 \\ C_V &= 2 * \text{const} * T \end{aligned}$$